On Regular Graphs with Complete Tripartite Star Complements

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Abstract

Let G be a graph of order n and let μ be an eigenvalue of multiplicity m. A star complement for μ in G is an induced subgraph of G of order $n-m$ with no eigenvalue μ. Some general observations concerning graphs with the complete tripartite graph $K_{r,s,t}$ as a star complement. We study the maximal regular graphs which have $K_{n,n,n}$ as a star complement for eigenvalue μ. The results include a complete analysis of the regular graphs which have $K_{n,n,n}$ as a star complement for 1. It turns out that some well known strongly regular graphs are uniquely determined by such a star complement.

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1 Introduction

Let G be a finite graph of order n with an eigenvalue μ of multiplicity m. (Thus the corresponding eigenspace of a (0, 1)-adjacency matrix of G has dimension m. For more details on graph spectra, see [1, 5]) If μ is an eigenvalue of G of multiplicity m, then a *star set* for μ in G is a set X of m vertices taken from G such that μ is not an eigenvalue of $G-X$. The graph $H=G-X$ is then called a *star complement* for μ in G. Star sets and star complements exist for any eigenvalue in a graph and they are not necessarily unique. For the background and results on star sets and star complements, one may consult [2, 7, 8, 9, 10, 11, 12].

The graphs (not necessarily regular) with the complete bipartite graph $K_{r,s}$ $(r+s>2)$ as a star complement were discussed in [9]. Here we discuss graphs with the complete tripartite graph $K_{r,s,t}$ as a star complement. We give a general description of the regular graphs which have $K_{n,n,n}$ as a star complement for μ. The results include a complete analysis of the case $r=s=t=n$ and $\mu=1$. This enables us to characterize the Kneser graph $KG(9,2)$ as the unique maximal graph G having $K_{3,3,3}$ as a star complement corresponding to eigenvalue 1. Here the value -1 is the sole exception because 0 is an eigenvalue of $K_{r,s,t}$ $(r+s+t>3)$.

We now fix some notation and terminology. We write $H+u=H(a,b,c)$ for a generic graph obtained from H by introducing a new vertex u and joining it to a vertices of R, b vertices of S and c vertices of T, then we say that u is of type (a,b,c). Next we consider a graph $H+u+v$, obtained from H by adding vertices u, v of type (a,b,c), (α,β,γ), respectively. If μ is an eigenvalue of $H+u+v$ of multiplicity 2, but not an eigenvalue of H, by [6, Theorems 7.4.1 and 7.4.4], the matrix $[b_{u_1}|b_{u_2}]$ determines whether or not there is an edge between u and v: we write $a_{uv}=1$ if $u\sim v$, $a_{uv}=0$ otherwise.

In Section 2, we discuss the addition of vertices to $K_{r,s,t}$ to obtain μ as a eigenvalue. In Section 3 we discuss special case $K_{n,n,n}$; in particular, we investigate conditions under which two vertices can be added to $K_{n,n,n}$ to obtain a regular graph with a double eigenvalue μ and in Section 4 we analyze the case $r=s=t=n$, $\mu=1$.

2 The general case

Here we suppose that H is the complete tripartite graph with three part R, S and T, where $|R|=r$, $|S|=s$ and $|T|=t$, $r\leq s\leq t$. We take

$$\mathbf{C}=\begin{pmatrix} 0 & J_{r\times s} & J_{r\times t}\\ J_{s\times r} & 0 & J_{s\times t}\\ J_{t\times r} & J_{t\times s} & 0\end{pmatrix}$$

where C is the adjacency matrix of H and $J_{r\times s}$ denotes the all-1 matrix of size $r\times s$. Note that C has characteristic polynomial $x^{r+s+t-3}(x^3-(rs+st+rt)x-2rst)$ and that if μ is not an eigenvalue of C, by proposition 0.2 of [?], if $r=s=t=n$ then

$$\mu(\mu^2-n\mu-2n^2)(\mu I-C)^{-1}=C^2+(\mu-n)C+(\mu^2-n\mu-2n^2)I \tag{2.1}$$

else

$$\begin{aligned}\mu(\mu^3-(rs+st+rt)\mu-2rst)(\mu I-C)^{-1}=&(\mu^3-(rs+st+rt)\mu-2rst)I\\&+(\mu^2-(rs+st+rt))C+\mu C^2\\&+C^3.\end{aligned} \tag{2.2}$$

Consider a graph $H+u+v$, obtained from H by adding vertices u, v of type (a,b,c), (α,β,γ), respectively. We denote by ρ the number of vertices in H which are common neighbours of u and v. On equating entries in Reconstruction Theorem [6, Theorems 7.4.1 and 7.4.4], with $B=[b_u|b_v]$, we find that this matrix equation is equivalent to the three simultaneous polynomial equations that if $r = s = t$, then $f_1(\mu) = f_2(\mu) = f_3(\mu) = 0$, otherwise $g_1(\mu) = g_2(\mu) = g_3(\mu) = 0$. Where

$$f_1(x) = (x^2-(a+b+c))(x^2-nx-2n^2)-2n(a^2+b^2+c^2)-2(ab+bc+ac)x \quad (2.3)$$

$$f_2(x) = (x^2 - (\alpha + \beta + \gamma))(x^2 - nx - 2n^2) - 2(\alpha\beta x + \beta\gamma + \alpha\gamma)x - 2n\alpha^2 - 2n\beta^2 - 2n\gamma^2 \quad (2.4)$$

$$f_3(x) = (a_{uv}x + \rho)(x^2 - nx - 2n^2) + 2n(a\alpha + b\beta + c\gamma) + \alpha(b + c)x + \beta ax + \beta cx + \gamma(a + b)x \quad (2.5)$$

and

$$g_1(x) = (x^2 - (a + b + c))(x^3 - (rs + st + rt)x - 2rst) - a^2(2ts + (t + s)x) - b^2(2tr + (t + r)x) - c^2(2rs + (r + s)x) - 2abx(x + t) - 2acx(x + s) - 2cbx(x + r) \quad (2.6)$$

$$g_2(x) = (x^2 - \alpha - \beta - \gamma)(x^3 - (rs + st + rt)x - 2rst) - \alpha^2(2ts + (t + s)x) - \beta^2(2tr + (t + r)x) - \gamma^2(2rs + (r + s)x) - 2\alpha\beta x(x + t) - 2\alpha\gamma x(x + s) - 2\gamma\beta x(x + r) \quad (2.7)$$

$$g_3(x) = (a_{uv}x + \rho)(x^3 - (rs + st + rt)x - 2rst) + a\alpha(2ts + (t + s)x) + 2b\beta tr + b\beta(t + r)x + c\gamma(2rs + (r + s)x) + (a\beta + \alpha b)x(x + t) + (a\gamma + \alpha c)x(x + s) + (b\gamma + \beta c)x(x + r) \quad (2.8)$$

In [9], it is shown that if H is a complete bipartite graph and u and v are of the same type (a, b), then $\mu^2 + a_{uv}\mu = a + b - \rho$. We also show that, in the following lemmas, if H is a complete tripartite graph and u and v are of the same type (a, b, c), then $\mu^2 + a_{uv}\mu = a + b + c - \rho$.

Lemma 2.1 *Suppose that u and v are of the same type (a, b, c) and that u is not adjacent to v. Then $\mu^2 = a + b + c - \rho$.*

Proof. Taking $(a, b, c) = (\alpha, \beta, \gamma)$ in equations (2.5) and (2.8), we find that $f_1(x) + f_3(x) = (x^2 - nx - 2n^2)g(x)$ and $g_1(x) + g_3(x) = (x^3 - (rs + st + rt)x - 2rst)g(x)$, where $g(x) = x^2 + \rho - (a + b + c)$. Hence $g(\mu) = 0$. □

Lemma 2.2 *Suppose that u, v are of the same type (a, b, c) and that u adjacent to v. Then $\mu^2 + \mu = a + b + c - \rho$.*

Proof. Here we have $f_1(x) + f_3(x) = (x^2 - nx - 2n^2)g(x)$ and $g_1(x) + g_3(x) = (x^3 - (rs + st + rt)x - 2rst)g(x)$, where $g(x) = x^2 + x + \rho - (a + b + c)$. Hence $g(\mu) = 0$. □

We now suppose that G is a r-regular extension of H with star set X, where H is $K_{n,n,n}$.

3 The case $H \simeq K_{n,n,n}$

Here we suppose that H is the complete tripartite graph with three part R, S and T, where $|R| = |S| = |T| = n$. Note that adjacency matrix of H has minimal polynomial $x(x-2n)(x+n)$. Suppose that G is a r-regular extension of H with star set X. Let $u \in X$. Then it is well known that $\langle b_u, j \rangle = -1$. Therefore by (2.1) we have

$$A = 2n - \mu \qquad C = 2n^2 - \mu^3 - \mu^2 n - \mu n \tag{3.1}$$

where $A = a+b+c$ and $C = a^2+b^2+c^2$. Since A and C are positive number, so $-n < \mu < n$. With no loss of generality we assume that $a \le b \le c$, if we find a solution such that $a > b$, $b > c$ or $a > c$, we should interchange the roles of a, b and c.

Proposition 3.1 *If μ is an eigenvalue of $H+u$, but not an eigenvalue of H, then $\frac{-3-\sqrt{9+32n}}{4} < \mu < \frac{-1+\sqrt{1+8n}}{2}$.*

Proof. We know from [6, Theorems 7.4.1 and 7.4.4] that

$$\mu^2(\mu^2 - n\mu - 2n^2) = b_u^T(C^2 + (\mu - n)C + (\mu^2 - n\mu - 2n^2)I)b_u.$$

Now suppose that u, v are of type (a,b,c) and (α,β,γ), respectively. We know that $f_1(\mu) + f_2(\mu) + f_3(\mu) = 0$, so

$$\begin{aligned}(2\mu^2 + a_{uv}\mu + \rho - (a+b+c) - (\alpha+\beta+\gamma))(\mu-2n)(\mu+n) = 2n(a\alpha + b\beta + c\gamma)\\ + 2n((\alpha-a)^2 + (\beta-b)^2 + (\gamma-c)^2) - (\alpha(b+c) + \beta(a+c) + \gamma(a+b))\mu +\\ + (2ab + 2ac + 2bc + 2\gamma\alpha + 2\beta\gamma + 2\beta\alpha)\mu \end{aligned} \tag{3.2}$$

By inserting $b+c = 2n-\mu-a$, $a+c = 2n-\mu-b$, $a+b = 2n-\mu$ in (3.2),we have

$$\begin{aligned}2(2\mu^2 + a_{uv}\mu + \rho - (a+b+c) - (\alpha+\beta+\gamma))(\mu-2n)(\mu+n) =\\ (4n-\mu)((\alpha-a)^2 + (\beta-b)^2 + (\gamma-c)^2) + (2ab+2ac+2bc)\mu\\ + 4n(a\alpha + b\beta + c\gamma) + (2\gamma\alpha + 2\beta\gamma + 2\beta\alpha)\mu \end{aligned} \tag{3.3}$$

Since $-n < \mu < n$, the right hand side of (3.3) is nonnegative. Thus $2\mu^2 + a_{uv}\mu + \rho - (a+b+c) - (\alpha+\beta+\gamma) < 0$. Note that $a+b+c = \alpha+\beta+\gamma = 2n-\mu$, so we have

$$\frac{-(a_{uv}+2) - \sqrt{(a_{uv}+2)^2 - 8\rho + 32n}}{4} < \mu < \frac{-(a_{uv}+2) + \sqrt{(a_{uv}+2)^2 - 8\rho + 32n}}{4}$$

Moreover $a_{uv} \in \{1, 0\}$ and $0 < \rho < 2n - \mu$. So $\frac{-3-\sqrt{9+32n}}{4} < \mu < \frac{-1+\sqrt{1+8n}}{2}$.

Proposition 3.2 *Suppose that u is of type (a,b,c) where $a=b=c$, then $a=\mu^2$ and $n=\frac{3\mu^2+\mu}{2}$.*

Proof. Taking $a=b=c$ in equations (3.1), we have $A=3a=2n-\mu$. Since $3a^2=2n^2-\mu(\mu+1)n-\mu^3$, we have $n=\frac{3\mu^2+\mu}{2}$. So $a=\mu^2$. □

Proposition 3.3 *Suppose that u is of type (a,b,c) where $a=b<c$ (or $a<b=c$), then $\frac{3\mu^2+\mu}{2}<n\leq 2\mu^2+\mu$ and $a<\frac{4\mu^2+\mu}{3}$. Furthermore if $n=2\mu^2+\mu$, then $(a,a,c)=(\mu^2,\mu^2,2\mu^2+\mu)$ and $(a,c,c)=(\frac{2\mu^2-\mu}{3},\frac{5\mu^2+2\mu}{3},\frac{5\mu^2+2\mu}{3})$.*

Proof. Taking $a=b$ in (3.1), then we have

$$2a+c=2n-\mu$$
$$2a^2+c^2=2n^2-\mu^3-\mu^2n-\mu n$$

so $a=\frac{2(2n-\mu)\pm\sqrt{\Delta}}{6}$ and $c=\frac{2n-\mu\mp\sqrt{\Delta}}{3}$, where $\Delta=4(n+\mu)(n-\frac{3\mu^2+\mu}{2})$. Since $\Delta>0$ and $c\leq n$, we have $\frac{3\mu^2+\mu}{2}<n\leq 2\mu^2+\mu$. Therefore $3a<4\mu^2+\mu$, the result follows. □

4 $H\simeq K_{n,n,n}$ and $\mu=1$

Suppose that $H=K_{n,n,n}$ is a star complement for eigenvalue $\mu=1$ in G. Note that G has 1 as the second largest eigenvalue (by Interlacing Theorem [5, p. 19]). If $\mu=1$, then by Reconstruction Theorem [6, Theorems 7.4.1 and 7.4.4], we have

$$2(A-1)n^2+(4B-2A^2+A-1)n-A-2B+1=0 \qquad (4.1)$$

where $A=a+b+c$ and $B=ab+bc+ac$.

Theorem 4.1 *If $K_{n,n,n}$ be a star complement for $\lambda_2=1$ with regular extension, then $n=2,3$.*

Proof. Suppose that G is a regular extension of $K_{n,n,n}$. Then by (3.1) we have $A=2n-1$. Thus

$$a^2+b^2+c^2-2ab-2ac-2bc+3=0$$

so $c=b+a\pm\sqrt{4ab-3}$. Note that $a>0$, because if $a=0$ we have $(b-c)^2+3=0$, this is a contradiction. First suppose that $c=b+a+\sqrt{4ab-3}$. Since $\sqrt{4ab-3}\geq 1$ then $c\geq n=\frac{a+b+c+1}{2}$, but we know that $c\leq n$ so $c=n$. If $c=n$ we have $a=b=1$ and $n=3$.
Now suppose that $c=b+a-\sqrt{4ab-3}$, since $a\leq b\leq c=b+a-\sqrt{4ab-3}$ we have

$$4a^2-3\leq 4ab-3\leq a^2$$

so we find $a=1$, which gives the solution $a=b=c=1$ and $n=2$. □

Theorem 4.2 *There is a unique regular extension with star complement $K_{3,3,3}$ for $\lambda_2 = 1$.*

Proof. Due to Theorem 4.1, here we have 27 vertices of (a, b, c)-type. Denote them by $U_1, U_2, \ldots, U_{27}$, and the corresponding vertices by $u_1, u_2, \ldots, u_{27}$, respectively. By Lemmas 2.1 and 2.2, we find that all vertices which have same type are in star set. We have exactly one case for the intersection of any pair of $U_i, U_j, 1 \leq i < j \leq 27$, where u_i and u_j don't have same type; $|U_i \cap U_j| = 3$, u_i and u_j are in star set if they are non-adjacent.
Therefore, each pair of $U_i, U_j, 1 \leq i < j \leq 27$ correspond to two vertices which are in star set. This leads us to the unique maximal graphs whose star set contains each of 27 vertices $u_1, u_2, \ldots, u_{27}$. The proof is complete.□

Remark 4.3 *The maximal graph from the previous Theorem is Kneser graph $KG(9, 2)$. Its spectrum is $[-6^8, 1^{27}, 21^1]$.* (see [3, 4])

Theorem 4.4 *There are exactly two isomorphic maximal graphs with star complement $K_{2,2,2}$ for $\lambda_2 = 1$.*

Proof. Due to Theorem 4.1, we have 8 vertices of $(1, 1, 1)$-type can have between zero, one and two vertices in common, and by inspecting these situations we find that they are in one star set in two cases: if they have precisely two vertices in common (then the vertices in the star set are non-adjacent), and if they have precisely one vertices in common (then the corresponding vertices are adjacent). Therefore we have two star sets which have 4 vertices of $(1, 1, 1)$-types, and two obtained maximal graphs are isomorphism. □

Remark 4.5 *The maximal graph from the previous theorem is complement of Petersen graph. Its spectrum is $[-2^5, 1^4, 6^1]$.* (see [3, 4])

Using the SCL - star complement library, one can compute the maximal extensions for this family of star complements. So Theorems 4.2 and 4.6 can be obtaind by using the facilities of SCL(For more details, see [12]). We summarize the above in the following theorem.

Theorem 4.6 *$K_{n,n,n}$ is a star complement for $\lambda_2 = 1$ with strongly regular extension if and only if $n = 2, 3$.*

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