ON A p-ADIC INTERPOLATING FUNCTION FOR THE GENERALIZED GENOCCHI NUMBERS AND ITS BEHAVIOUR AT $s = 0$

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ABSTRACT. In this work, we consider the generalized Genocchi numbers and polynomials. However, we introduce analytic interpolating function for the generalized Genocchi numbers attached to χ at negative integers in complex plane and also we define the Genocchi p-adic L-function. As a result, we derive the value of the partial derivative of the Genocchi p-adic l-function at $s = 0$.

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1. Preliminaries

The p-adic numbers were invented by German Mathematician Kurt Hensel around the end of the nineteenth century. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community. The p-adic integral was used in mathematical physics, for instance, the functional equation of the q-zeta function, q-stirling numbers and q-Mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L functions. Furthermore, the p-adic interpolation functions of the Bernoulli and Euler polynomials have been treated by Tsumura [33] and Young [34]. T. Kim [7]-[23] also studied on p-adic interpolation functions of these numbers and polynomials. In [35], Carlitz originally constructed q-Bernoulli numbers and polynomials. These numbers and polynomials are studied by many authors (see cf. [8]-[27], [38]). In the last decade, a surprising number of papers appeared proposing new generalizations of the Bernoulli, Euler and Genocchi polynomials to real and complex variables (see [1-44]).

In [7]-[25], Kim studied some families of the Bernoulli, Euler and Genocchi numbers and polynomials. By using the fermionic p-adic invariant integral on $\mathbb{Z}_p$, he constructed p-adic Bernoulli, p-adic Euler and p-adic Genocchi numbers and polynomials of higher order. In this paper, by using Kim's way, we derive arithmetic properties for the generalized Genocchi numbers and polynomials attached to χ.

The famous Genocchi numbers are given in the complex plane by the following exponential generating function:

$$E(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \; |t| < \pi. \tag{1}$$

It follows from the definition that $G_0 = 0$, $G_1 = 1$, $G_2 = -1$, $G_3 = 0$, $G_4 = 1$, $G_5 = 0, \cdots$, and $G_{2k+1} = 0$ for $k = 1, 2, 3, \cdots$.

The Genocchi polynomials are also introduced by the rule:

$$E(t, x) = e^{tG(x)} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt}, \tag{2}$$

with the usual convention of replacing $G^n(x) := G_n(x)$ (see [8], [9] and [11]).

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, we assume that χ is a primitive Dirichlet's charachter with conductor f. The Genocchi numbers associated with χ, $G_{n,\chi}$, may be defined as follows:

$$E_\chi(t) = 2t \sum_{\xi=1}^{f} \frac{\chi(\xi)(-1)^\xi e^{\xi t}}{e^{ft} + 1} = \sum_{n=0}^{\infty} G_{n,\chi} \frac{t^n}{n!}, \; |t| < \frac{\pi}{f}. \tag{3}$$

In this paper, we contemplate the definition of the generating functions of the generalized Genocchi numbers attached to χ in complex plane. From this definition, we introduce an analytic interpolating function for the multiple generalized Genocchi numbers attached to χ. Finally, we investigate behaviour of analytic function at $s = 0$.

2. ON AN ANALYTIC FUNCTION RELATED TO GENERALIZED GENOCCHI NUMBERS

In this part, we consider equation (3):

$$E_\chi(t) = \sum_{n=0}^{\infty} G_{n,\chi} \frac{t^n}{n!} = 2t \sum_{a=1}^{f} \frac{(-1)^a \chi(a) e^{ta}}{e^{ft} + 1}. \tag{4}$$

Because of (2) and (4), we readily see that

$$G_{n,\chi} = f^{n-1} \sum_{a=1}^{f} (-1)^a \chi(a) G_n\left(\frac{a}{f}\right). \tag{5}$$

For $s \in \mathbb{C}$, we have

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \{-E(-t, x)\} dt = 2 \sum_{n \geq 0} \frac{(-1)^n}{(x+n)^s}, \; x \neq 0, -1, -2, \cdots. \tag{6}$$

where $\Gamma(s)$ is Euler-Gamma function, which is defined by the rule

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Via the (6), we give the Genocchi-zeta function as follows: for $s \in \mathbb{C}$ and $x \neq 0, -1, -2, \cdots$,

$$\zeta_G(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(x+n)^s}. \tag{7}$$

By (2) and (6), we derive

$$\zeta_G(-n,x) = \frac{G_{n+1}(x)}{n+1} \text{ for } n \in \mathbb{N}.$$

By utilizing from complex integral and (4), we obtain the following equation: for $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-2} \{-E_\chi(-t)\} dt = 2\sum_{n=1}^{\infty} \frac{\chi(a)(-1)^n}{n^s} \tag{8}$$

where χ is the primitive Dirichlet's character with conductor

$$f \in \mathbb{N} \text{ and } f \equiv 1 \pmod 2.$$

Thanks to (8), we give the definition Dirichlet's type of the Genocchi L-function in complex plane as follows:

$$L(s \mid \chi) = 2\sum_{n=1}^{\infty} \frac{\chi(a)(-1)^n}{n^s}. \tag{9}$$

Via the (4) and (9), we state the following

$$L(-n \mid \chi) = \frac{G_{n+1,\chi}}{n+1} \text{ for } n \in \mathbb{N}. \tag{10}$$

Let s be a complex variable, and let a and b be integer with $0 < a < F$ and $F \equiv 1 \pmod 2$.

Thus, we can consider the partial zeta function $S(s; a \mid F)$ as follows:

$$S(s; a \mid F) = 2 \sum_{\substack{m>0 \\ m \equiv a \pmod F}} \frac{(-1)^m}{m^s} = (-1)^a F^{-s} \zeta_G\left(s, \frac{a}{F}\right). \tag{11}$$

Then Dirichlet's type of L-function can be expressed as the sum: for $s \in \mathbb{C}$

$$L(s \mid \chi) = \sum_{a=1}^{F} \chi(a) S(s; a \mid F). \tag{12}$$

Substituting $s = -n$ into (11), we readily derive the following: for $n \in \mathbb{N}$

$$(n+1) S(1-n; a \mid F) = (-1)^a F^{n-1} G_n\left(\frac{a}{F}\right). \tag{13}$$

By (2), it is easy to show the following

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_k = \sum_{k=0}^{n} \binom{n}{k} x^k G_{n-k}. \tag{14}$$

Thanks to (11), (13) and (14), we discover the following

$$-sS(s+1; a \mid F) = (-1)^a F^{-1} a^{-s} \sum_{k \geq 0} \binom{-s}{k} \left(\frac{F}{a}\right)^k G_k \tag{15}$$

From (12), (13) and (15), we obtain

$$-sL(s+1 \mid \chi) = \frac{1}{F} \sum_{1 \leq a \leq F} \chi(a) (-1)^a a^{-s} \sum_{k \geq 0} \binom{-s}{k} \left(\frac{F}{a}\right)^k G_k. \tag{16}$$

In the next section, we search a p-adic function that agrees $L(s \mid \chi)$ at negative integers.

3. CONCLUSION

In this final section, we consider p-adic interpolation function of the generalized Genocchi L-function, which interpolate Dirichlet's type of Genocchi numbers at negative integers. Firstly, Washington constructed p-adic l-function which interpolates generalized classical Bernoulli numbers.

Here, we use some the following notations, which will be useful in reminder of paper.

Let ω denote the *Teichmüller* character by the conductor $f_\omega = p$. For an arbitrary character χ, we set $\chi_n = \chi\omega^{-n}$, $n \in \mathbb{Z}$, in the sense of product of characters.

Let

$$\langle a \rangle = \omega^{-1}(a)\, a = \frac{a}{\omega(a)}.$$

So, we want to note that $\langle a \rangle \equiv 1 \,(\mathrm{mod}\, p\mathbb{Z}_p)$. Let

$$A_j(x) = \sum_{n=0}^{\infty} a_{n,j} x^n, \; a_{n,j} \in \mathbb{C}_p, \; j = 0, 1, 2, \ldots$$

be a sequence of power series, each convergent on a fixed subset

$$T = \left\{ s \in \mathbb{C}_p \mid |s|_p < p^{-\frac{2-p}{p-1}} \right\},$$

of $\mathbb{C}_p$ such that

(1) $a_{n,j} \to a_{n,0}$ as $j \to \infty$ for any n;

(2) for each $s \in T$ and $\epsilon > 0$, there exists an $n_0 = n_0(s, \epsilon)$ such that

$$\left| \sum_{n \geq n_0} a_{n,j} s^n \right|_p < \epsilon \text{ for } \forall j.$$

So,

$$\lim_{j\to\infty} A_j(s) = A_0(s)\,, \text{ for all } s \in T.$$

This was firstly introduced by Washington [40] to indicate that each functions $\omega^{-s}(a)\,a^s$ and

$$\sum_{l=0}^{\infty} \binom{s}{l} \left(\frac{F}{a}\right)^l B_l,$$

where F is multiple of p and f and B_l is the l-th Bernoulli numbers, is analytic on T (for more information, see [40]).

We assume that χ is a primitive Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. Then we think the Genocchi p-adic L-function, $L_p(s \mid \chi)$, which interpolates the generalized Genocchi numbers attached to χ at negative integers.

For $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$, let us assume that F is a positive integral multiple of p and $f = f_\chi$. We now give the definition of Genocchi p-adic L-function as follows:

$$(17)\quad -sL_p(s+1 \mid \chi) = \frac{1}{F} \sum_{1\le a\le F} \chi(a)(-1)^a \langle a\rangle^{-s} \sum_{k\ge 0} \binom{-s}{k} \left(\frac{F}{a}\right)^k G_k.$$

With the help of (17), we want to note that $L_p(s+1 \mid \chi)$ is an analytic function on $s \in T$.

For $n \in \mathbb{N}$, we have

$$(18)\qquad G_{n,\chi_n} = F^{n-1} \sum_{a=1}^{F} (-1)^a \chi_n(a)\, G_n\left(\frac{a}{F}\right).$$

If $\chi_n(p) \neq 0$, then $(p, f_{\chi_n}) = 1$, and so the ratio $\frac{F}{p}$ is a multiple of f_{χ_n}. Let

$$\rho = \left\{ \frac{a}{p} \mid a \equiv 0 (\operatorname{mod} p) \text{ for some } a \in \mathbb{Z} \text{ with } 0 \le a \le F \right\}.$$

Therefore we can write the following

$$(19)\quad F^{n-1} \sum_{\substack{a=1\\ p\mid a}}^{F} (-1)^a \chi_n(a)\, G_n\left(\frac{a}{F}\right)$$

$$= p^{n-1} \left(\frac{F}{p}\right)^{n-1} \chi_n(p) \sum_{\substack{a=1\\ \xi\in\rho}}^{\frac{F}{p}} (-1)^{\xi} \chi_n(\xi)\, G_n\left(\frac{\xi}{F/p}\right).$$

By (19), we define the second generalized Genocchi numbers attached to χ as follows:

$$G^*_{n,\chi_n} = \left(\frac{F}{p}\right)^{n-1} \sum_{\substack{a=1 \\ \xi\in\rho}}^{\frac{F}{p}} (-1)^{\xi} \chi_n(\xi) G_n\left(\frac{\xi}{F/p}\right). \tag{20}$$

On accounct of (18), (19) and (20), we attain the following

$$G_{n,\chi_n} - p^{n-1}\chi_n(p) G^*_{n,\chi_n} = F^{n-1} \sum_{\substack{a=1 \\ p\nmid a}}^{F} (-1)^a \chi_n(a) G_n\left(\frac{a}{F}\right). \tag{21}$$

By the definition of the familiar Genocchi polynomials, we can state the following

$$G_n\left(\frac{a}{F}\right) = F^{-n} a^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a}\right)^k G_k. \tag{22}$$

By (21) and (22), we have

$$\begin{aligned} & G_{n,\chi_n} - p^{n-1}\chi_n(p) G^*_{n,\chi_n} \\ &= \frac{1}{F} \sum_{\substack{a=1 \\ p\nmid a}}^{F} (-1)^a \chi_n(a) a^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a}\right)^k G_k \end{aligned} \tag{23}$$

By (17) and (23), we readily see that

$$\begin{aligned} nL_p(1-n \mid \chi) &= \frac{1}{F} \sum_{\substack{a=1 \\ p\nmid a}}^{F} (-1)^a \chi_n(a) a^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{F}{a}\right)^k G_k \\ &= G_{n,\chi_n} - p^{n-1}\chi_n(p) G^*_{n,\chi_n} \end{aligned} \tag{24}$$

Therefore, we obtain the following theorem.

Theorem 3.1. *The following identity holds true:*

$$-sL_p(s+1 \mid \chi) = \frac{1}{F} \sum_{a=1}^{F} \chi(a) (-1)^a \langle a \rangle^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} \left(\frac{F}{a}\right)^k G_k.$$

Thus $L_p(s+1 \mid \chi)$ is an analytic function on T. Moreover, for any $n \in \mathbb{N}$, we get the following:

$$L_p(1-n \mid \chi) = \frac{1}{n}\left(G_{n,\chi_n} - p^{n-1}\chi_n(p) G^*_{n,\chi_n}\right).$$

Using Taylor expansion at $s=0$, we have

$$\binom{-s}{k} = \frac{(-1)^k}{k} s + \cdots \text{ if } k \geq 1. \tag{25}$$

Differentiating on both sides in (17), with respect to s at $s = 0$, we derive the following theorem which is an importan in p-adic analysis and Analytic numbers theory.

Theorem 3.2. *Let F be a positive integral multiple of p and f. Then we have*

$$\frac{\partial}{\partial s} L_p (s+1 \mid \chi) \mid_{s=0} = L_p (1 \mid \chi)$$

$$+ \frac{1}{F} \left(\sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) (-1)^a \left((1 - \log_p a) + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{F}{a} \right)^k G_k \right) \right)$$

where $\log_p x$ is denoted by the p-adic logarithm.

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