On $L(d,1)$-Labelings of the Cartesian Product of Two Cycles

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Abstract

A k-$L(d,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to $\{0,1,\cdots,k\}$ such that $|f(u)-f(v)| \geq 1$ if $d(u,v)=2$ and $|f(u)-f(v)| \geq d$ if $d(u,v)=1$. The *$L(d,1)$-labeling number* $\lambda_d(G)$ of G is the smallest number k such that G has a k-$L(d,1)$-labeling. In this paper, we show that $2d+2 \leq \lambda_d(C_m \Box C_n) \leq 2d+4$ if either m or n is odd. Futhermore, the following cases are determined. (a) $\lambda_d(C_3 \Box C_n)$ and $\lambda_d(C_4 \Box C_n)$ for $d \geq 3$, (b) $\lambda_3(C_m \Box C_n)$ for some m and n, (c) $\lambda_d(C_{2m} \Box C_{2n})$ for $d \geq 5$ when m and n are even.

Key words: $L(d,1)$-labeling, Cartesian product, cycle.

1 Introduction

Roberts [23] put forward a variation of the frequency assignment problem introduced by Hale [17] via a private communication with Griggs and then Griggs and Yeh [16] proposed the $L(2,1)$-labeling problem on graphs. Chang et al. [3] generalized the $L(2,1)$-labeling problem to the $L(d,1)$-labeling problem. Given a positive integer d, an *$L(d,1)$-labeling* of a graph G is an assignment f of non-negative integers to the vertices of G such that

$$|f(u)-f(v)| \geq \begin{cases} d, & \text{if } d(u,v)=1; \\ 1, & \text{if } d(u,v)=2. \end{cases}$$

For a nonnegative integer k, a *k-$L(d,1)$-labeling* is an $L(d,1)$-labeling such that no label is greater than k. The *$L(d,1)$-labeling number* $\lambda_d(G)$ of G is

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the smallest number k such that G has a k-$L(d,1)$-labeling. We often use $\lambda(G)$ to represent the $L(2,1)$-labeling number of G.

Griggs and Yeh [16] showed that the $L(2,1)$-labeling problem is NP-complete and conjectured that $\lambda(G) \leq \Delta^2(G)$ for any graph G with $\Delta(G) \geq 2$. Chang and Kuo [2] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G)$ and gave polynomial algorithms for the $L(2,1)$-labeling problem on cographs and trees. Chang et al. [3] proved that $\lambda_d(G) \leq \Delta^2(G)+(d-1)\Delta(G)$ when $\Delta(G) \geq 2$. Gonçalves [15] showed that $\lambda_d(G) \leq \Delta^2(G) + (d-1)\Delta(G) - 2$ when $\Delta(G) \geq 3$. Yeh [29] and Calamoneri [1] gave two good surveys on the $L(2,1)$-labeling and its generalizations, respectively. The purpose of this paper is to study the $L(d,1)$-labeling problem for the Cartesian product of two cycles. Given two graphs G and H, the *Cartesian product* of these two graphs, denoted by $G\Box H$, is defined by $V(G\Box H) = \{(u,v) | u \in V(G), v \in V(H)\}$ and $E(G\Box H) = \{(u,x)(v,y) | (u = v, xy \in E(H))$ or $(uv \in E(G), x = y)\}$. The $L(d,1)$-labeling number of Cartesian products of graphs was studied in [5, 12, 14, 18, 19, 20, 21, 25, 27]. We could find further studies on the $L(d,1)$-labelings in [3, 4, 6, 7, 8, 9, 11, 13, 22, 24, 28].

The $L(2,1)$-labeling of the Cartesian product of any two graphs were considered in [26] and the $L(2,1)$-labeling of the Cartesian product of two cycles were studied in [18, 21, 25].

Theorem 1 *[18, 21, 25] If $n \geq m \geq 3$, then*

$$\lambda(C_m \Box C_n) = \begin{cases} 6, & \text{if } m,n \equiv 0 (\text{mod } 7); \\ 8, & \text{if } (m,n) \in A; \\ 7, & \text{otherwise,} \end{cases}$$

where $A = \{(3,i) \mid i \geq 3, i \text{ is odd or } i = 4, 10\} \cup \{(5,i) \mid i = 5,6,9,10,13,17\} \cup \{(6,7),(6,11),(7,9),(9,10)\}$.

Jha, Klavzar, and Vesel [20] proposed $\lambda_d(C_m \Box C_n)$ for $d = 3,4$ and $4 \leq m,n \leq 11$. Their results are summarized in the following **Table 1**.

m	n	$\lambda_3(C_m\Box C_n)$	$\lambda_4(C_m\Box C_n)$
4	$n=4,8$	9	10
4	$n=5,6,7,9,10,11$	9	11
5	$n=5$	10	12
5	$n=7$	10	11
5	$n=6,8,9,10,11$	9	11
6	$n=9$	8	10
6	$n=6,7,8,10,11$	9	11
7	7,8,9,10,11	9	11
8	8	9	10
9	9	8	10
10	10	9	11

Table 1.[20] $\lambda_d(C_m\Box C_n), 10 \geq n \geq m \geq 4, d=3,4$

From now on, for convenience, we use $v_{i,j}$ to denote the vertices of $C_m\Box C_n$, where $i \in Z_m$ and $j \in Z_n$.

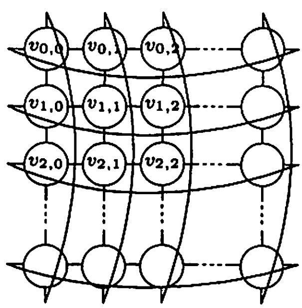


Figure 1. $C_m\Box C_n$

2 Upper bound and Lower bound

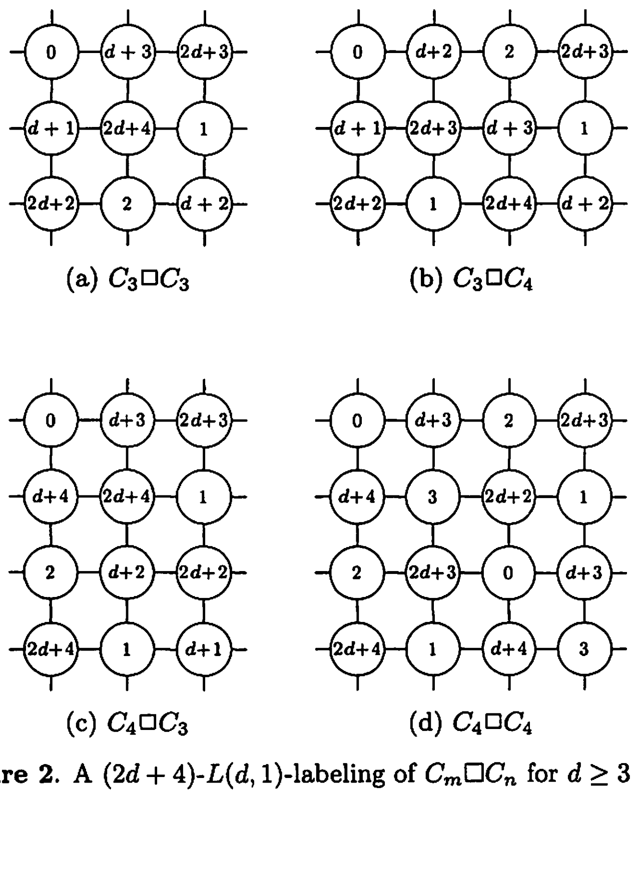
In this section, we give the upper bounds and lower bounds for $\lambda_d(C_m\Box C_n)$ under various parameters. Since $C_m\Box P_n$ is a subgraph of $C_m\Box C_n$, we have the following lemma from [5].

Lemma 2 *Suppose $d \geq 3$. Then*
(a) $\lambda_d(C_m\Box C_n) \geq d+6$ *if* $d \geq 4$.
(b) $\lambda_d(C_m\Box C_n) \geq 2d+2$ *if either* m *or* n *is odd.*
(c) $\lambda_d(C_4\Box C_n) \geq d+6$.
(d) $\lambda_3(C_m\Box C_n) \geq 8$.

By re-constructing and combining some labelings of smaller $C_m \Box C_n$, we got the upper bound of $\lambda_d(C_m \Box C_n)$.

Lemma 3 *Suppose* $d \geq 3$. *Then*
(a) $\lambda_d(C_m \Box C_n) \leq 2d + 4$.
(b) $\lambda_d(C_{2m} \Box C_{2n}) \leq d + 7$ *if* $m, n \neq 2, 5$.
(c) $\lambda_d(C_m \Box C_{4n}) \leq 2d + 3$.
(d) $\lambda_d(C_{4m} \Box C_{4n}) \leq d + 6$.

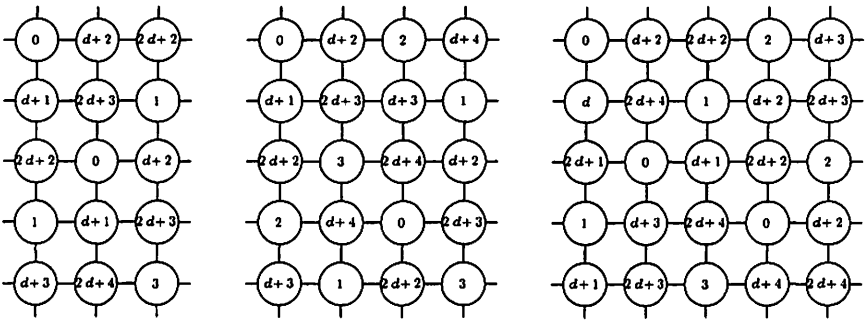
Proof. By combining (a), (b), (c), and (d) of Figure 2, we have a $(2d+4)$-$L(d,1)$-labeling of $C_m \Box C_n$ if $m, n \neq 5$. By combining (a) and (b) of Figure 3, we have a $(2d+4)$-$L(d,1)$-labeling of $C_5 \Box C_n$ for $n \neq 5$. Figure 3(c) gives a $(2d+4)$-$L(d,1)$-labeling of $C_5 \Box C_5$ for $d \geq 3$. Similarly, Figure 4 gives $\lambda_d(C_{2m} \Box C_{2n}) \leq d + 7$ for $m, n \neq 2, 5$, and Figure 5 gives $\lambda_d(C_m \Box C_{4n}) \leq 2d + 3$. Figure 6 gives $\lambda_d(C_{4m} \Box C_{4n}) \leq d + 6$. ■



(a) $C_3 \Box C_3$ (b) $C_3 \Box C_4$

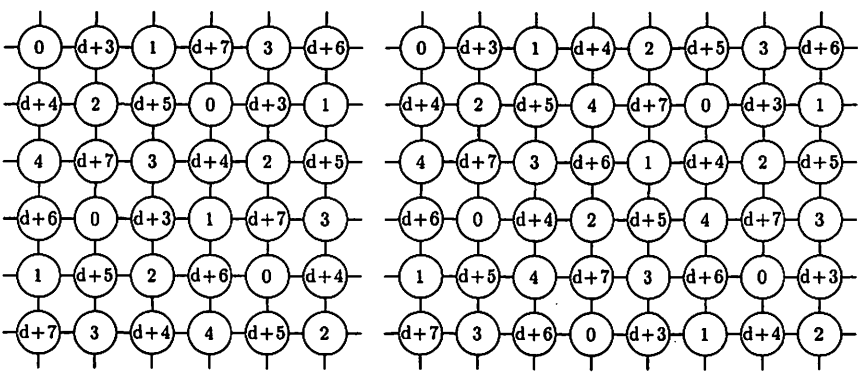
(c) $C_4 \Box C_3$ (d) $C_4 \Box C_4$

Figure 2. A $(2d+4)$-$L(d,1)$-labeling of $C_m \Box C_n$ for $d \geq 3$ if $m, n \neq 5$.

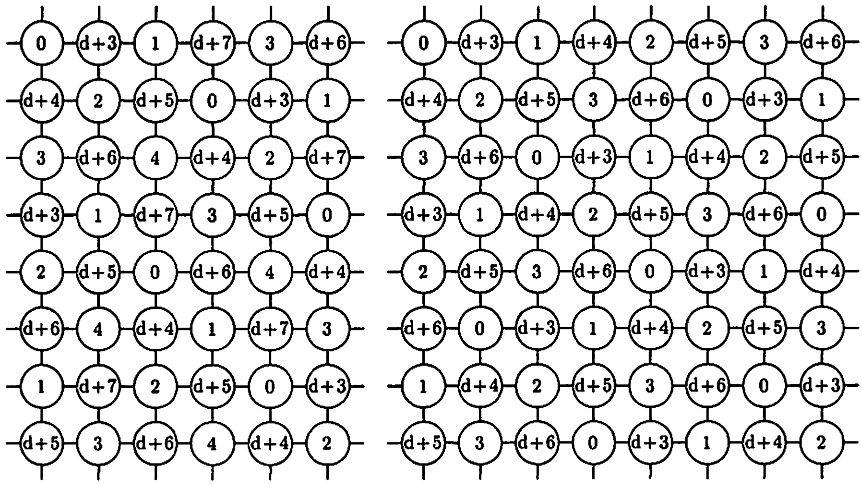


(a) $C_5 \square C_3$ (b) $C_5 \square C_4$ (c) $C_5 \square C_5$

Figure 3. A $(2d+4)$-$L(d,1)$-labeling of $C_5 \square C_n$ for $d \geq 3$.

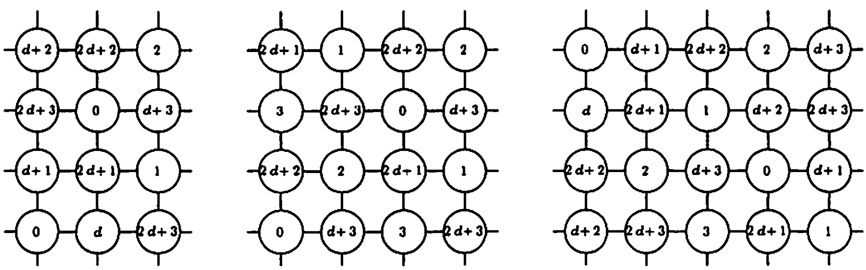


(a) $C_6 \square C_6$ (b) $C_6 \square C_8$



(c) $C_8\Box C_6$ (d) $C_8\Box C_8$

Figure 4. A $(d{+}7)$-$L(d,1)$-labeling of $C_{2m}\Box C_{2n}$ for $d \geq 3$ and $m, n \neq 2, 5$.



(a) $C_4\Box C_3$ (b) $C_4\Box C_4$ (c) $C_4\Box C_5$

Figure 5. A $(2d+3)$-$L(d,1)$-labeling of $C_m\Box C_{4n}$ for $d \geq 3$.

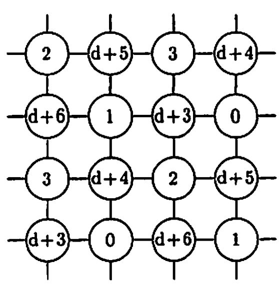


Figure 6. A $(d+6)$-$L(d,1)$-labeling of $C_{4m}\Box C_{4n}$.

Lemma 4 *[16] If f is a k-$L(d,1)$-labeling of a graph G, then the function $g : V(G) \to \{0, 1, \cdots, k\}$ defined by $g(v) = k - f(v)$ for each vertex v in G is a k-$L(d,1)$-labeling.*

Lemma 5 *Suppose $d \geq 3$ and f is a $(2d+2)$-$L(d,1)$-labeling of $C_m \Box C_n$. If $f(v_{i,j}) = d + k$, for some $k = 0, 1, 2$, then*
(a) either $\{f(v_{i+1,j}), f(v_{i-1,j})\} = \{k, 2d+k\}$ or $\{f(v_{i,j+1}), f(v_{i,j-1})\} = \{k, 2d+k\}$,
(b) $4 \notin \{m, n\}$.

Proof. (a) If $f(v_{i,j}) = d$, then $\{f(v_{i,j-1}), f(v_{i-1,j}), f(v_{i,j+1}), f(v_{i+1,j})\} = \{0, 2d, 2d+1, 2d+2\}$. Assume $f(v_{i,j-1}) = 0$ and $f(v_{i+1,j}) = 2d$. Then no labels can be assigned to $f(v_{i+1,j-1})$. So, either $\{f(v_{i+1,j}), f(v_{i-1,j})\} = \{0, 2d\}$ or $\{f(v_{i,j+1}), f(v_{i,j-1})\} = \{0, 2d\}$. It is similar to the cases for $f(v_{i,j}) = d+1$ and $f(v_{i,j}) = d+2$.

(b) Assume $f(v_{i,j}) = d+1$. By (a), let's assume $f(v_{i,j+1}) = 1$, $f(v_{i,j-1}) = 2d+1$, $f(v_{i-1,j}) = 0$, and $f(v_{i+1,j}) = 2d+2$. Then $f(v_{i+1,j+1}) = d+2$. By (a), we have $f(v_{i+1,j+2}) = 2$, $f(v_{i+2,j+1}) = 0$. This implies $m \neq 4$. If $n = 4$, then no labels can be assigned to $f(v_{i,j+2})$. Therefore, we have $n \neq 4$. It is similar to the cases for $f(v_{i,j}) = d$ and $f(v_{i,j}) = d+2$. ■

Lemma 6 *(a) Suppose f is a $(d+k+2)$-$L(d,1)$-labeling of $C_m \Box C_n$ and $d > k$. Then $f(v) \in \{0, 1, \cdots, k-1, d+3, \cdots, d+k+2\}$ for each $v \in V(C_m \Box C_n)$.*
(b) Suppose f is a $(2d+k)$-labeling of $C_3 \Box C_n$ and $d > k$. Then $f(v) \in \{0, \cdots, k, d, \cdots, d+k, 2d, \cdots, 2d+k\}$ for each $v \in V(C_3 \Box C_n)$.

Proof. (a) Assume $f(v) = i$ for some $v \in V(C_m \Box C_n)$ and $i \leq d$. Then $f(N(v)) \subseteq \{d+i, \cdots, d+k+2\}$ if $i < d$ and $f(N(v)) \subseteq \{0, 2d, \cdots, d+k+2\}$ if $i = d$. Since $|f(N(v))| = |N(v)| = 4$, we have $i < k$. Thus, $f(v) \in \{0, 1, \cdots, k-1\}$ if $f(v) \leq d$. Since $d > k$, we have $d+k+2 \leq 2d+1$. By Lemma 4, we have $f(v) \in \{0, 1, \cdots, k-1, d+3, \cdots, d+k+2\}$.

(b) Assume $f(v_{i,j}) = t$ for some $t \in \{0, 1, \cdots, d-1\}$. Then $v_{i-1,j}$ or $v_{i+1,j}$ must be labeled by a number which is at least $2d+t$. It implies $t \leq k$. Therefore, by Lemma 4, we have $f(v) \in \{0, 1, \cdots, k, d, \cdots, d+k, 2d, \cdots, 2d+k\}$ for each $v \in V(C_3 \Box C_n)$. ■

3 The $L(d,1)$-labeling number of $C_3 \square C_n$ for $d \geq 3$

In this section, we determine $\lambda_d(C_3 \square C_n)$ for $d \geq 3$.

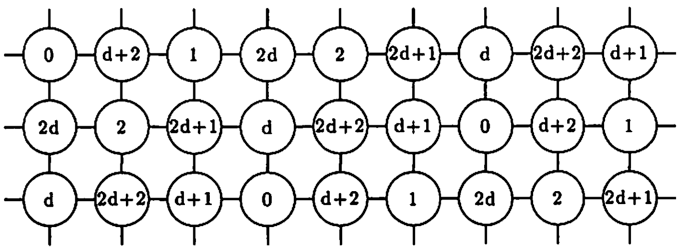


Figure 7. A $(2d+2)$-$L(d,1)$-labeling of $C_{3m} \square C_{9n}$ for $d \geq 3$.

Theorem 7 *If $d \geq 3$ and one of m and n is odd, then $\lambda_d(C_{3m} \square C_{9n}) = 2d+2$.*

Proof. By Lemma 2(b) and Figure 7, we have $\lambda_d(C_{3m} \square C_{9n}) = 2d+2$. ■

Theorem 8 *If $d \geq 3$, then*

$$\lambda_d(C_3 \square C_n) = \begin{cases} 2d+2, & n \equiv 0 (\text{mod } 9); \\ 2d+4, & n = 3 \text{ or } (n = 7 \text{ and } d = 3); \\ 2d+3, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 7, we have $\lambda_d(C_3 \square C_{9n}) = 2d+2$.

Suppose f is a $(2d+3)$-$L(d,1)$-labeling of $C_3 \square C_3$. By Lemma 6(b), we have $f(v) \in \{0, 1, 2, 3, d, d+1, d+2, d+3, 2d, 2d+1, 2d+2, 2d+3\}$ for each $v \in V(C_3 \square C_3)$.

Claim 1. $f(v) \neq d, d+3$ for each $v \in V(C_3 \square C_3)$.

Suppose $f(v_{i,j}) = d$. Then $f(N(v_{i,j})) \subseteq \{0, 2d, 2d+1, 2d+2, 2d+3\}$. Without loss of generality, we assume $f(v_{i-1,j}) = 0$. Since $|f(v_{i,j-1}) - f(v_{i,j+1})| \geq d \geq 3$, $v_{i,j-1}$ or $v_{i,j+1}$ must be labeled by $2d$, say $v_{i,j-1}$. This implies $f(v_{i-1,j-1}) = 2d+3$, Then no labels can be assigned to $v_{i,j+1}$. It is a contradiction. Thus, $f(v_{i,j}) \neq d$. By Lemma 4, we have $f(v) \neq d, d+3$ for each $v \in V(C_3 \square C_3)$.

Claim 2. $f(v) \neq 3, 2d$ for each $v \in V(C_3 \square C_3)$.

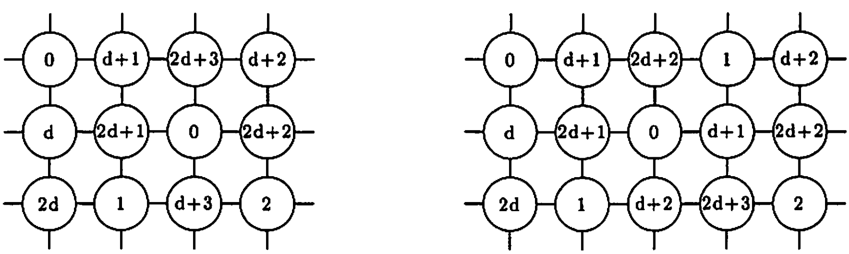
Assume $d \geq 4$ and $f(v_{i,j}) = 3$. By Lemma 6(b) and Claim 1, we have $f(N(v_{i,j})) \subseteq \{2d, 2d+1, 2d+2, 2d+3\}$. Then we can not assign $v_{i,j-1}$ and $v_{i,j+1}$ since $|f(v_{i,j-1}) - f(v_{i,j+1})| \geq d > 3$. It is a contradiction. Thus,

$f(v) \neq 3$ for each $v \in V(C_3 \square C_3)$. By Lemma 4 and Claim 1, we have $f(v) \neq d, d+3$ for each $v \in V(C_3 \square C_3)$.

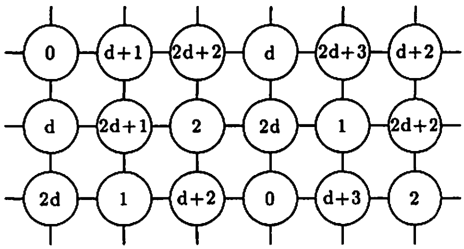
So, $f(v) \in \{0, 1, 2, d+1, d+2, 2d+1, 2d+2, 2d+3\}$ for each $v \in V(C_3 \square C_3)$. It is a contradiction to $|f^{-1}(V(C_3 \square C_3))| = 9$. Thus, $\lambda_d(C_3 \square C_3) \geq 2d+4$ for $d \geq 3$. By Lemma 3(a), $\lambda_d(C_3 \square C_3) = 2d+4$ for $d \geq 3$. According to the aid of computer program implementation, we have $\lambda_3(C_3 \square C_7) = 10$.

Suppose f is a $(2d+2)$-$L(d,1)$-labeling of $C_3 \square C_n$. By Lemma 6(b), $f(v) \in \{0, 1, 2, d, d+1, d+2, 2d, 2d+1, 2d+2\}$ for each $v \in V(C_3 \square C_n)$. Assume $f(v) = 2$ for some $v \in V(C_3 \square C_n)$. We may let $f(v_{1,1}) = 2$. Since $|f(v_{0,1}) - f(v_{2,1})| \geq d$, we have $\{f(v_{0,1}), f(v_{2,1})\} = \{d+2, 2d+2\}$. Without loss of generality, we assume $f(v_{2,1}) = d+2$, $f(v_{0,1}) = 2d+2$, $f(v_{1,0}) = 2d$, and $f(v_{1,2}) = 2d+1$. Then $f(v_{0,0}) = d$, $f(v_{2,0}) = 0$, $f(v_{0,2}) = d+1$, and $f(v_{2,2}) = 1$. This implies $f(v_{0,3}) = 0$ and $f(v_{1,3}) = d$. By Lemma 5(a), we have $f(v_{2,3}) = 2d$. By repeating the pattern mentioned above, we have $f(v_{i,j}) = 2$ if and only if $j \equiv 3i - 2 \pmod 9$. This implies $n \equiv 0 \pmod 9$.

Suppose $n \not\equiv 0 \pmod 9$. Then, $f(v) \neq 2$ for each $v \in V(C_3 \square C_n)$. Assume $f(v) = d+2$ for some $v \in V(C_3 \square C_n)$. Then $4 = |f(N(v))| \leq |\{0, 1, 2d+2\}| = 3$. It is a contradiction. Thus, $f(v) \neq d+2$ for each $v \in V(C_3 \square C_n)$. By Lemma 4, we have $f(v) \neq d$, $2d$ for each $v \in V(C_3 \square C_n)$. So, $f(v) \in \{0, 1, d+1, 2d+1, 2d+2\}$ for each $v \in V(C_3 \square C_n)$. This implies $5 \geq |f(C_3 \square C_n)| \geq |f(C_3 \square P_2)| = 6$, a contradiction. So, $\lambda_d(C_3 \square C_n) \geq 2d+3$ for $n \not\equiv 0 \pmod 9$ and $d \geq 3$. By combining (a), (b), and (c) of Figure 8, we have a $(2d+3)$-$L(d,1)$-labeling of $C_3 \square C_n$ for $d \geq 3$ and $n \neq 3, 7$. Figure 9 gives a $(2d+3)$-$L(d,1)$-labeling of $C_3 \square C_7$ for $d \geq 4$. Consequently, the proof is complete. ■



(a) $C_3 \square C_4$ (b) $C_3 \square C_5$



(c) $C_3\Box C_6$

Figure 8. A $(2d+3)$-$L(d,1)$-labeling of $C_3\Box C_n$ for $d \geq 3$ for $n \neq 3, 7$.

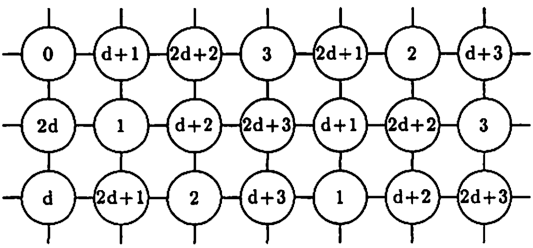


Figure 9. A $(2d+3)$-$L(d,1)$-labeling of $C_3\Box C_7$ for $d \geq 4$.

4 The $L(d,1)$-labeling number of $C_4\Box C_n$ for $d \geq 3$

In this section, we determine $\lambda_d(C_4\Box C_n)$ for $d \geq 3$.

Theorem 9 *Suppose $d \geq 3$ and $n \geq 1$. Then*
(a) $\lambda_d(C_4\Box C_{2n+1}) = 2d+3$.
(b) $\lambda_d(C_4\Box C_{4n}) = d+6$.

Proof. Let f be a $(2d+2)$-$L(d,1)$-labeling of $C_4\Box C_m$. By Lemma 5(b), we have $f(v_{i,j}) \in A_1 = \{0,1,2,\cdots,d-1\}$ or $f(v_{i,j}) \in A_2 = \{d+3,\cdots,2d+2\}$ for each $v_{i,j} \in V(C_4\Box C_{2n+1})$. So $f(v_{i,j}) \in A_1$ if and only if $f(v_{i',j'}) \in A_2$ when $i+j \not\equiv i'+j' \pmod 2$. This implies m is even. Hence $\lambda_d(C_4\Box C_{2n+1}) \geq 2d+3$. By Lemma 3(c), we have $\lambda_d(C_4\Box C_{2n+1}) = 2d+3$. By Lemma 2(c) and Lemma 3(d), we have $\lambda_d(C_4\Box C_{4n}) = d+6$. ■

For the case of $C_4\Box C_{4n+2}$, the $L(d,1)$-labeling number may be $d+6$, $d+7$, or $d+8$ when $d \geq 3$. In most cases, the number is $d+7$.

Lemma 10 *Suppose $m \leq n$ and $4 \nmid \gcd(m,n)$. Then $\lambda_d(C_m\Box C_n) > d+6$ if $d \geq 5$ or $d = m = 4$.*

Proof. Suppose f is a $(d+6)$-$L(d,1)$-labeling of $C_m \square C_n$. By Lemma 5(b) and Lemma 6(a), we have $f(v) \in \{0,1,2,3,d+3,d+4,d+5,d+6\}$ for $v \in V(C_m \square C_n)$ with $d \geq 5$ or $d = m = 4$. Without loss of generality, we assume $f(v_{0,0}) \in \{0,1,2,3\}$. Then $f(v_{i,j}) \in \{0,1,2,3\}$ if and only if $i+j$ is even. Let $A = \{f(v_{i,i}) | 0 \leq i \leq n-1\}$. It is trivial that $|A| \geq 2$. Suppose $|A| = 2$. We may assume $A = \{0,1\}$. Then $f(v_{i,j}) \in \{0,1\}$ if and only if $i - j \equiv 0 (\text{mod } 4)$ and $f(v_{i,j}) \in \{2,3\}$ if and only if $i - j \equiv 2(\text{mod } 4)$. This implies $4|m$ and $4|n$. For the case of $|A| \geq 2$, we may assume $f(v_{i,i}) = i$ for $i = 0,1,2$. Then,

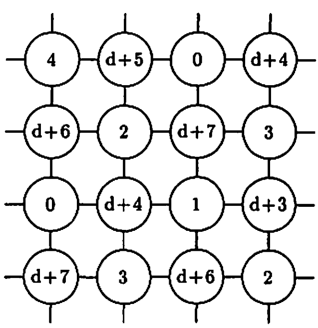
$$f(v_{i,j}) = \begin{cases} 0, & \text{if } i+j \equiv 0 (\text{mod}\, m) \text{ and } i \text{ is even.} \\ 0, & \text{if } i+j = 4 (\text{mod}\, m) \text{ and } i \text{ is odd.} \\ 1, & \text{if } i+j = 2 (\text{mod}\, m) \text{ and } i \text{ is odd.} \\ 2, & \text{if } i+j = 0 (\text{mod}\, m) \text{ and } i \text{ is odd.} \\ 2, & \text{if } i+j \equiv 4 (\text{mod}\, m) \text{ and } i \text{ is even.} \\ 3, & \text{if } i+j \equiv 2 (\text{mod}\, m) \text{ and } i \text{ is even.} \end{cases}$$

This implies $f(v_{i,j}) \in \{0,2\}$ if and only if $i+j \equiv 0(\text{mod } 4)$ and $f(v_{i,j}) \in \{1,3\}$ if and only if $i+j \equiv 2(\text{mod } 4)$. Thus, we have $4|m$ and $4|n$.

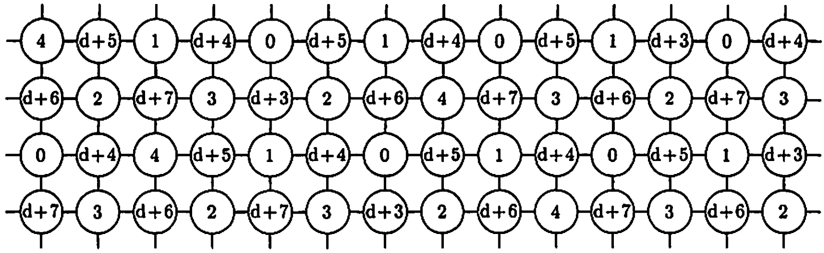
For any case, we have $4|m$ and $4|n$. It is a contradiction to $4 \nmid \gcd(m,n)$. So $\lambda_d(C_m \square C_n) > d+6$. ■

Theorem 11 *$\lambda_d(C_{4m} \square C_{4n+2}) = d+7$ for $d \geq 5$ and $n \geq 3$.*

Proof. By combining (a) and (b) of Figure 10, we have a $(d+7)$-$L(d,1)$-labeling of $C_{4m} \square C_{4n+2}$ for $d \geq 4$ and $n \geq 3$. By Lemma 10, we have $\lambda_d(C_{4m} \square C_{4n+10}) = d+7$ for $d \geq 5$ and $n \geq 3$. ■



(a) $C_4 \square C_4$



(b) $C_4 \Box C_{14}$

Figure 10. A $(d+7)$-$L(d,1)$-labeling of $C_{4m}\Box C_{4n+2}$ for $d \geq 4$ and $n \geq 3$.

Theorem 12 $\lambda_d(C_4\Box C_6) = d+8$ *for* $d \geq 5$.

Proof. Suppose f is a $(d+7)$-$L(d,1)$-labeling of $C_4\Box C_6$. For each $v \in V(C_4\Box C_6)$, by Lemma 5(b) and Lemma 6(a), $f(v)$ belong to $A_1 = \{0,1,2,3,4\}$ or $A_2 = \{d+3, d+4, d+5, d+6, d+7\}$. Then for each $uv \in E(C_4\Box C_6)$, $f(u) \in A_1$ if and only if $f(v) \in A_2$. It is not hard to check that $|f^{-1}(i)| \leq 2$ for each $i \in A_1 \cup A_2$. Then $24 = |V(C_4\Box C_6)| \leq 20$. It is a contradiction. Thus, $\lambda_d(C_4\Box C_6) \geq d+8$. Figure 11 gives a $(d+8)$-$L(d,1)$-labeling of $C_4\Box C_6$. Thus, $\lambda_d(C_4\Box C_6) = d+8$. ■

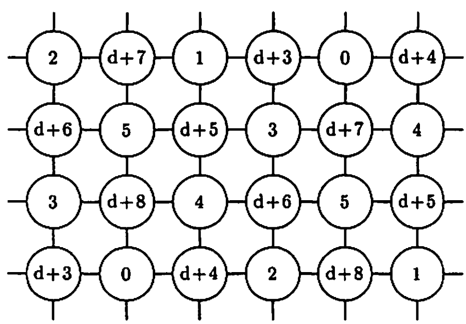


Figure 11. A $(d+8)$-$L(d,1)$-labeling of $C_4\Box C_6$.

Theorem 13 $\lambda_d(C_{2n}\Box C_{10}) = d+8$ *for* $d \geq 5$ *and* $n \leq 4$.

Proof. Suppose f is a $(d+7)$-$L(d,1)$-labeling of $C_{2n}\Box C_{10}$. By Lemma 5(b) and Lemma 6(a), we have $f(v) \in A_1 = \{0,1,2,3,4\}$ or $f(v) \in A_2 = \{d+3, d+4, d+5, d+6, d+7\}$ for each $v \in V(C_{2n}\Box C_{10})$. Without loss of generality, we assume $f(v_{i,j}) \in A_k$ for $i+j \equiv k (\text{mod } 2)$. Let $R_i = \{v_{i,0}, \cdots, v_{i,n-1}\}$ and $r_i^t = |f^{-1}(t) \cap R_i|$. The following rules are easy to check:

$\mathbf{T}_1.r_i^t \leq 2$.

$\mathbf{T}_2$. $r_i^t + r_{i+1}^t + r_{i+2}^t \leq 4$.

$\mathbf{T}_3$. If $r_i^t = 2$, then $r_{i-1}^t + r_i^t + r_{i+1}^t \leq 3$.

Claim 1. For each $t \in A_1 \cup A_2$, $|f^{-1}(t)| = 2n$ if $n \leq 4$.

Proof of Claim 1. Suppose $|f^{-1}(t)| \geq 2n+1$. Then $r_i^t + r_{i+1}^t \geq 3$ for some i. By Rules $\mathbf{T}_1$ and $\mathbf{T}_3$, we assume $r_1^t = 2$, $r_0^t = 0$, $r_2^t = 1$, and $f(v_{1,0}) = f(v_{1,4}) = f(v_{2,7}) = t$. This implies $r_3^t \leq 4 - (r_1^t + r_2^t) = 1$.

Case 1. $n = 2$. Then $|f^{-1}(t)| = r_0^t + r_1^t + r_2^t + r_3^t \leq 4$. It is a contradiction.

Case 2. $n = 3$. By Rules $\mathbf{T}_2$, we have $r_3^t + r_4^t + r_5^t = 4$. Then $r_3^t = r_4^t = 1$ and $r_5^t = 2$. This implies $f(v_{3,2}) = t$. We have $f(v_{5,0}), f(v_{5,2}), f(v_{5,4}) \neq t$. Then $r_5^t \leq 1$. It is a contradiction.

Case 3. $n = 4$. We have $r_3^t + r_4^t + r_5^t + r_6^t + r_7^t \geq 9 - 3 = 6$. If $r_4^t = 2$(resp. $r_5^t = 2$), then $r_3^t + r_4^t + r_5^t \leq 3$(resp. $r_4^t + r_5^t + r_6^t \leq 3$). Thus, we have either $r_6^t = 2$ or $r_7^t = 2$. Then we have $(r_1, r_2, \cdots, r_8) = (0,2,1,1,2,0,1,2)$, $(0,2,1,1,2,0,2,1)$, $(0,2,1,0,2,1,1,2)$, $(0,2,1,1,1,2,0,2)$, or $(0,2,1,1,1,1,1,2)$. For any one case, we can check that it is impossible.

By Case 1, Case 2, and Case 3, we have $|f^{-1}(t)| = 2n$ for $n \leq 4$ since they are $20n$ vertices and 10 labels. We complete the proof of Claim 1.

Noted that

(1). $d+3 \notin f(N(v))$ if $f(v) = 4$.

(2). $N(u) \cap N(v) = \varnothing$ if $f(u) = f(v)$.

(3). $f(v_{i,j}) \in A_k$ if and only if $i + j \equiv k (\text{mod } 2)$.

(4). The order of $C_{2n} \square C_{10}$ is $20n$.

So we have the following claim.

Claim 2. Suppose $f(u) \in A_2$. Then $f(u) = d+3$ if and only if $4 \notin f(N(u))$.

Without loss of generality, we assume $f(v_{2,3}) = 4$.

Suppose $n = 2$. Then we have $f(v_{1,6}) = 4$ or $f(v_{0,5}) = 4$ since $v_{1,5}$ and $v_{0,4}$ can not be both labeled by $d+3$. And we have $f(v_{3,6}) = 4$ or $f(v_{0,5}) = 4$ since $v_{3,5}$ and $v_{0,4}$ can not be both labeled by $d+3$. Thus, $f(v_{0,5}) = 4$. Similarly, we have $f(v_{2,7}) = f(v_{0,9}) = f(v_{2,1}) = 4$. It is a contradiction to $f(v_{2,3}) = 4$. So, $\lambda_d(C_4 \square C_{10}) = d + 8$.

When $n = 3$. Since at most one of $v_{4,2}, v_{5,3}, v_{4,4}$ can be labeled by $d+3$ and $4 \notin \{f(v_{0,3}), f(v_{3,2}), f(v_{4,3}), f(v_{3,4})\}$. We have $v_{5,2}$ or $v_{5,4}$ must be labeled by 4 or $f(v_{4,1}) = f(v_{4,5}) = 4$. If $f(v_{4,1}) = f(v_{4,5}) = 4$, then $f(v_{0,2}) = f(v_{0,4}) = d+3$ by Claim 2. It is a contradiction to $f(v_{0,2}) \neq f(v_{0,4})$. So, $f(v_{5,2}) = 4$ or $f(v_{5,4}) = 4$. Without loss of generality, we assume $f(v_{5,4}) = 4$. Then $v_{3,6}$ or $v_{1,6}$ must be labeled by 4 since $v_{3,5}$ and $v_{1,5}$ can not be both labeled by $d+3$. If $f(v_{3,6}) = 4$, then $f(v_{1,5}) = d+3$ and $f(v_{0,6}) \neq d+3$. This implies $f(v_{0,7}) = 4$. Then $f(v_{2,9}) = 4$ or $f(v_{4,9}) = 4$ since $v_{2,8}$ and $v_{4,8}$ cannot be both labeled by $d+3$. If $f(v_{2,9}) = 4$, then $f(v_{4,8}) = d+3$, $f(v_{5,9}) \neq d+3$, and $f(v_{5,0}) = 4$. This implies $f(v_{1,1}) = f(v_{3,1}) = d+3$. It is a contradiction. Thus, $f(v_{4,9}) = 4$. This implies $f(v_{3,1}) = f(v_{4,2}) = d+3$. It is a contradiction to $f(v_{3,1}) \neq f(v_{4,2})$.

Thus, $f(v_{3,6}) \neq 4$ and $f(v_{1,6}) = 4$. But by similar argument, we have a contradiction. This implies $\lambda_d(C_6 \Box C_{10}) > d+7$. So, $\lambda_d(C_6 \Box C_{10}) = d+8$.

Assume $n = 4$. Then one of $v_{3,6}, v_{4,5}, v_{5,4}$ must be labeled by 4 since $v_{3,5}$ and $v_{4,4}$ can not be both labeled by $d+3$ and $4 \notin \{f(v_{4,3}), f(v_{3,4}), f(v_{2,5})\}$.

Case 1. Let $f(v_{3,6}) = 4$. Then, either $v_{5,4}$ or $v_{6,5}$ must be labeled by 4 since $v_{5,5}$ and $v_{4,4}$ can not be both labeled by $d+3$.

Case 1.1. Let $f(v_{5,4}) = 4$. Then we have $f(v_{0,5}) = 4$ since $v_{0,4}$ and $v_{1,5}$ can not be both labeled by $d+3$. Then $f(v_{6,7}) = 4$ since at most one of $v_{5,7}, v_{6,6}, v_{7,7}$ can be labeled by $d+3$. Similarly, we have $f(v_{1,8}) = f(v_{4,9}) = 4$. This implies $f(v_{3,1}) = f(v_{2,0}) = d+3$ by Claim 2. It is a contradiction to $f(v_{3,1}) \neq f(v_{2,0})$.

Case 1.2. Let $f(v_{6,5}) = 4$. Then $f(v_{0,7}) = 4$ since at most one of $v_{7,7}, v_{0,6}, v_{1,7}$ can be labeled by $d+3$. Similarly, we have $f(v_{5,8}) = f(v_{2,9}) = f(v_{7,0}) = 4$. This implies $f(v_{1,1}) = f(v_{0,2}) = d+3$ by Claim 2. It is a contradiction to $f(v_{1,1}) \neq f(v_{0,2})$.

Case 2. Let $f(v_{5,4}) = 4$. Then $f(v_{0,5}) = 4$ since at most one of $v_{7,5}, v_{0,4}, v_{1,5}$ can be labeled by $d+3$. Similarly, we have $f(v_{3,6}) = 4$. It is similar to the Case 1.

Case 3. Let $f(v_{4,5}) = 4$. Then $f(v_{6,7}) = 4$. Otherwise, it is similar to the Case 1 and Case 2. Similarly, we have $f(v_{0,9}) = 4$ and $f(v_{2,1}) = 4$. It is a contradiction to $f(v_{2,3}) = 4$.

By Case 1, Case 2, and Case 3, we have $|f^{-1}(4)| = 0$. It is a contradiction to Claim 1. Thus, $\lambda_d(C_{2n} \Box C_{10}) \geq d+8$ for $n \leq 4$. By combining Figure 6 and Figure 11, we have $\lambda_d(C_{2n} \Box C_{10}) \leq d+8$ for $n = 2, 4$. The function $f : V(C_6 \Box C_{10}) \to \{0, 1, \cdots, d+8\}$ defined by $f(v_{i,j}) = ((i+j) \bmod 2)(d+4) + ((i+3j) \bmod 5)$ is a $(d+8)$-$L(d,1)$-labeling of $C_8 \Box C_{10}$. Hence, $\lambda_d(C_8 \Box C_{10}) = d+8$. ■

Theorem 14 *Suppose* $n \geq 1$. *Then*
(a) $\lambda_3(C_4 \Box C_{4n+2}) = 9$
(b) $\lambda_4(C_4 \Box C_{4n+2}) = 11$.

Proof. By Lemma 3(c), we have $\lambda_3(C_4 \Box C_{4n+2}) \leq 9$ and $\lambda_4(C_4 \Box C_{4n+2}) \leq 11$. By Lemma 2(c), we have $\lambda_3(C_4 \Box C_{4n+2}) = 9$. And by Lemma 10, we have $\lambda_4(C_4 \Box C_{4n+2}) = 11$. ■

Summarize the results in this section, we have for $d \geq 3$

$$\lambda_d(C_4 \Box C_n) = \begin{cases} 2d+3, & n \text{ is odd;} \\ d+6, & d=3 \text{ or } n \equiv 0 (\text{mod } 4); \\ d+8, & d \geq 5 \text{ and } (n=6 \text{ or } 10); \\ d+7, & \text{otherwise.} \end{cases}$$

5 The $L(3,1)$-labeling number of $C_m \square C_n$

Theorem 15 $\lambda_3(C_{3m} \square C_{9n}) = 8$

Proof. By Lemma 2(d) and Figure 7, we have $\lambda_3(C_{3m} \square C_{9n}) = 8$. ■

Lemma 16 $\lambda_3(C_m \square C_n) \geq 9$ *if* $3 \nmid \gcd(m,n)$ *or* $27 \nmid mn$.

Proof. Suppose f is an 8-$L(3,1)$-labeling of $C_m \square C_n$. Assume $f(v_{i,j}) = 4$. By Lemma 5(a), we assume $f(v_{i,j-1}) = 1$, $f(v_{i,j+1}) = 7$, $f(v_{i-1,j}) = 0$, and $f(v_{i+1,j}) = 8$. Then $f(v_{i-1,j+1}) = 3$, $f(v_{i+1,j-1}) = 5$. By Lemma 5(a), we have $f(v_{i-1,j+2}) = 6$ and $f(v_{i+1,j-2}) = 2$. This implies $f(v_{i-2,j+1}) = 8$, $f(v_{i+2,j-1}) = 0$, $f(v_{i-2,j}) = 5$, and $f(v_{i+2,j}) = 3$. By Lemma 5(a), we have $f(v_{i-2,j-1}) = 2$ and $f(v_{i+2,j+1}) = 6$. Then $f(v_{i-1,j-1}), f(v_{i-2,j-2}), f(v_{i,j-2}) \in \{6,7,8\}$. This implies $f(v_{i-1,j-2}) = 3$. By Lemma 5(a), we have $f(v_{i-1,j-1}) = 6$ and $f(v_{i-1,j-3}) = 0$. Then $f(v_{i,j-3}) = 4$ since $f(v_{i,j-2}) \in \{7,8\}$ and $f(v_{i+1,j-3}) \in \{6,7,8\}$. By Lemma 5(a), we have $f(v_{i,j-2}) = 7$. Then $f(v_{i-2,j-2}) = 8$, $f(v_{i-2,j-3}) = 5$, and $f(v_{i-3,j-1}) = 7$. By Lemma 5(a), we have $f(v_{i-2,j-4}) = 2$ and $f(v_{i-3,j-3}) = 1$. This implies $f(v_{i-3,j-2}) = 4$. Repeat the pattern, we have $f(v_{p,q}) = 4$ if and only if $(p,q) = (i-3+9x, j-2+3y), (i+9x, j-3+3y)$, or $(i+3+9x, j-1+3y)$ for some x, y. This implies $3|(m,n)$ and ($9|m$ or $9|n$). It is a contradiction. Thus, $f(v) \neq 4$ for each $v \in V(C_m \square C_n)$ if $3 \nmid \gcd(m,n)$ or $27 \nmid mn$. Similarly, $f(v) \neq 3, 5$. Then $f(v) \in \{0,1,2,6,7,8\}$ for each $v \in V(C_m \square C_n)$. It is impossible to label $C_m \square C_n$. So, $\lambda_3(C_m \square C_n) \geq 9$. ■

Theorem 17 *If* $3 \nmid \gcd(m,n)$ *or* $27 \nmid mn$, *then*
(a)$\lambda_3(C_m \square C_{4n}) = 9$
(b)$\lambda_3(C_m \square C_{3n}) = 9$ *if* $m \neq 7$ *and* $(m,n) \neq (3,1)$.

Proof. By Lemma 3(c) and Lemma 16, we have $\lambda_3(C_m \square C_{4n}) = 9$. If $m \neq 7$ and $(m,n) \neq (3,1)$, we have $\lambda_3(C_m \square C_{3n}) \leq 9$ by Theorem 8. From Lemma 16, (b) holds. ■

By Table 1 and our observation, we conjecture that

$$\lambda_3(C_m \square C_n) = \begin{cases} 8, & 3|\gcd(m,n) \text{ and } 27|mn; \\ 10, & \{m,n\} = \{3\}, \{5\}, \{3,7\}, \{5,7\}; \\ 9, & \text{otherwise.} \end{cases}$$

6 The $L(d,1)$-labeling number of $C_m \square C_n$ for $d \geq 5$ when m and n are even

In this section, we determine the $\lambda_d(C_m \square C_n)$ for $d \geq 5$ when m and n are even.

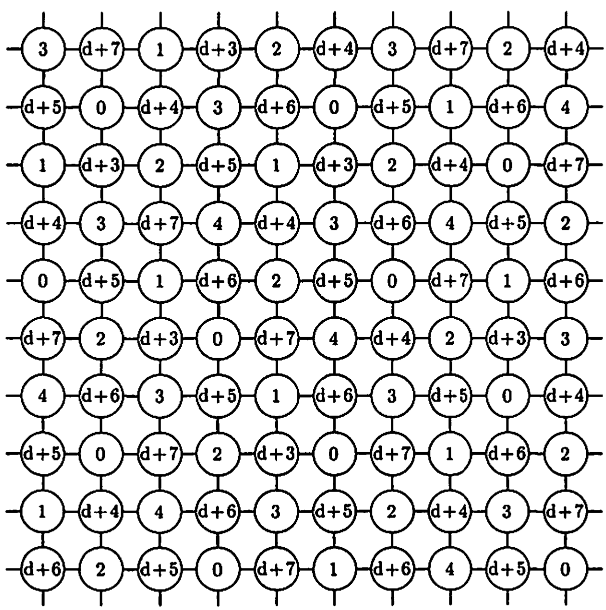
Lemma 18 $\lambda_d(C_{4m}\Box C_{4n}) = d + 6$ *for* $d \geq 4$.

Proof. By Lemma 2(a) and Lemma 3(d), we have $\lambda_d(C_{4m}\Box C_{4n}) = d+6$ for $d \geq 4$. ■

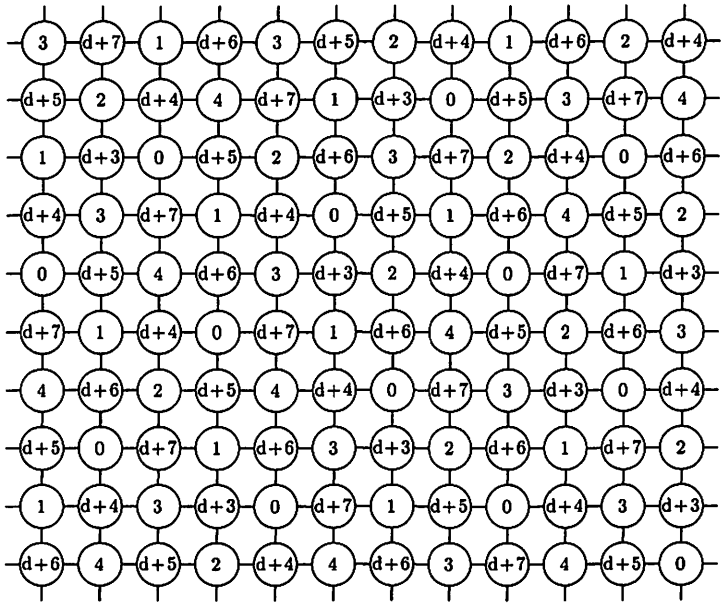
Theorem 19 *If m and n are even and $d \geq 5$, then*

$$\lambda_d(C_m\Box C_n) = \begin{cases} d+6, & 4|m \text{ and } 4|n; \\ d+8, & \{m,n\} = \{4,6\},\{4,10\},\{6,10\},\{8,10\}; \\ d+7, & \text{otherwise.} \end{cases}$$

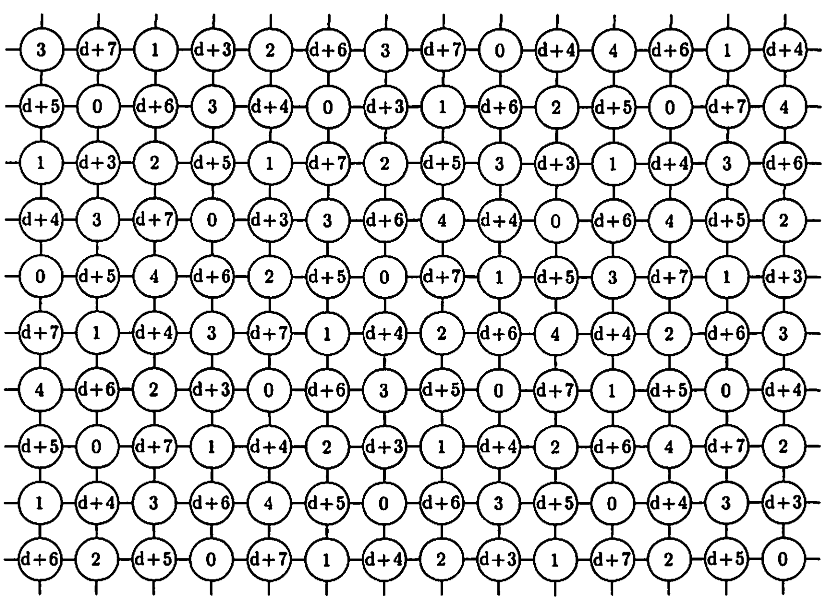
Proof. By Lemma 18, we have $\lambda_d(C_m\Box C_n) = d+6$ for $4|m$ and $4|n$. By Theorem 13, we have $\lambda_d(C_m\Box C_{10}) = d+8$ for $m \in \{4,6,8\}$. By Theorem 12, we have $\lambda_d(C_4\Box C_6) = d+8$. By Lemma 3(b), we have $\lambda_d(C_{2h}\Box C_{2k}) \leq d+7$ for $h,k \neq 2,5$. By Figure 13, Figure 14, and combining (a), (b), and (c) of Figure 12, we have a $(d+7)$-$L(d,1)$-labeling of $C_{10}\Box C_{2k}$ for $k \geq 5$. These results together with Lemma 10 and Theorem 11 complete the proof. ■



(a) $C_{10}\Box C_{10}$



(b) $C_{10}\square C_{12}$



(c) $C_{10}\square C_{14}$

Figure 12. A $(d+7)$-$L(d,1)$-labeling of $C_{10}\square C_n$ for $n \geq 5$ and $n \neq 8, 9$.

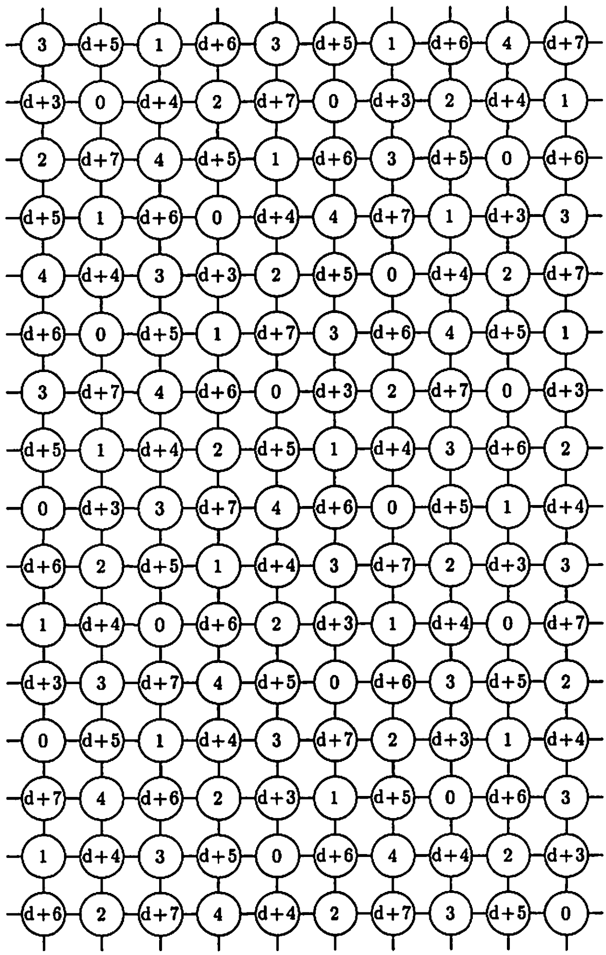


Figure 13. A $(d+7)$-$L(d,1)$-labeling of $C_{10}\square C_{16}$.

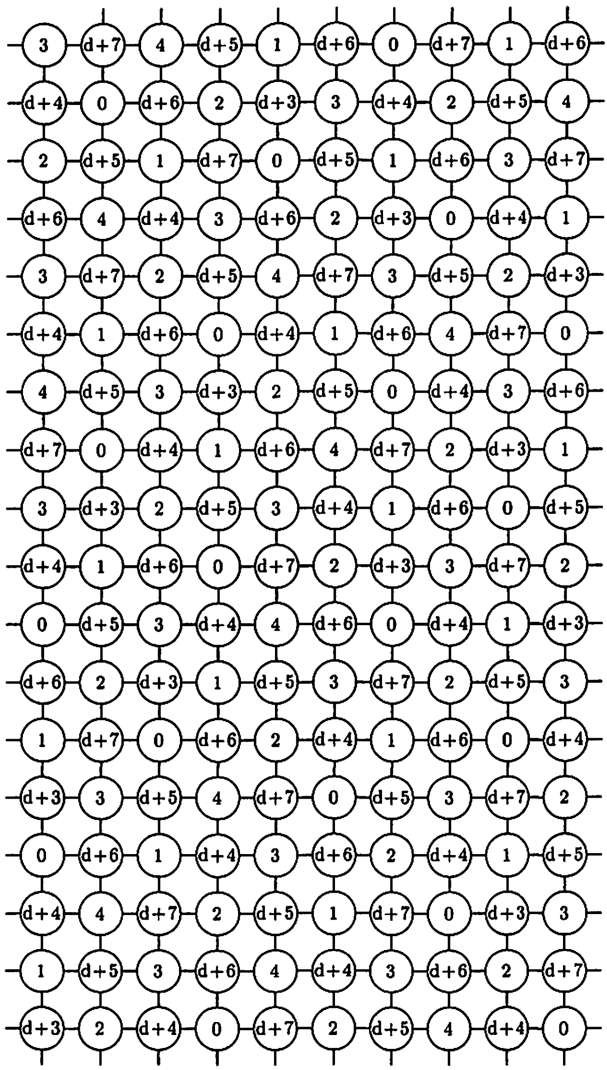


Figure 14. A $(d+7)$-$L(d,1)$-labeling of $C_{10}\square C_{18}$.

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