Extremal Tricyclic Graphs With Respect To Schultz Index

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Abstract

For a graph $G = (V, E)$, the Schultz index of G is defined as $S(G) = \sum_{\{u,v\}\subseteq V(G)} (d_G(u) + d_G(v))d_G(u, v)$ where $d_G(u)$ (or $d(u)$) is the degree of the vertex u in G, and $d_G(u, v)$ is the distance between u and v in G. In this paper, we investigate the Schultz index of tricyclic graphs. The n-tricyclic graphs with the minimum Schultz index are determined.

1 Introduction

We use Bondy and Murty [1] for terminologies and notions not defined here. Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$, and $|V(G)| = n$, $|E(G)| = m$ are the number of vertices and edges of G, *respectively*. For any $u, v \in V$, $d_G(u)$ (or simply by $d(u)$) and $d_G(u, v)$ denote the degree of u and the distance (i.e.,the number of edges on the shortest path) between u and v, *respectively*, $N_G(v) = \{u|uv \in E(G)\}$ denotes the neighbors of v, and $d_G(v) = |N_G(v)|$. P_n, C_n and $K_{1,n-1}$(or S_n) be the path, cycle and the star on n vertices.

Schultz [2] in 1989 introduced a graph-theoretical descriptor for characterizing alkanes by an integer, namely the *Schultz index*, defined as

$$S(G) = \sum_{\{u,v\}\subseteq V(G)} (d_G(u) + d_G(v))d_G(u, v) \tag{1}$$

In [3], the authors derived relations between Wiener index $W(G)$ and $S(G)$ for trees, i.e., $S(G) = 4W(G) - (n - 1)(2n - 1)$. In [4], the analogous results on (unbranched) hexagonal chain composed of n fused hexagons were derived as well, $S(G) = \frac{25}{4}W(G) - \frac{3}{4}(2n + 1)(20n + 7)$. More results in this direction can be found in references [5-10].

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The cyclomatic number of a connected graph G is defined as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called a $k-$ cyclic graph. For $c(G) = 3$, we named G as a tricyclic graph. Let $\mathscr{T}_n$ be the set of all tricyclic graphs with n vertices. We know, by Li *et al.*[11-15], that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in G. The authors in [16] characterized the laplace radius of tricyclic graphs. Let $\mathscr{T}_n = \mathscr{T}_n^3 \cup \mathscr{T}_n^4 \cup \mathscr{T}_n^6 \cup \mathscr{T}_n^7$, where $\mathscr{T}_n^i$ denotes the set of tricyclic graph on n vertices with exact i cycles for $i = 3, 4, 6, 7$. Note that the induced subgraphs of vertices on the cycles of $G \in \mathscr{T}_n^i$ $(i = 3, 4, 6, 7)$ are depicted in Figure 1.

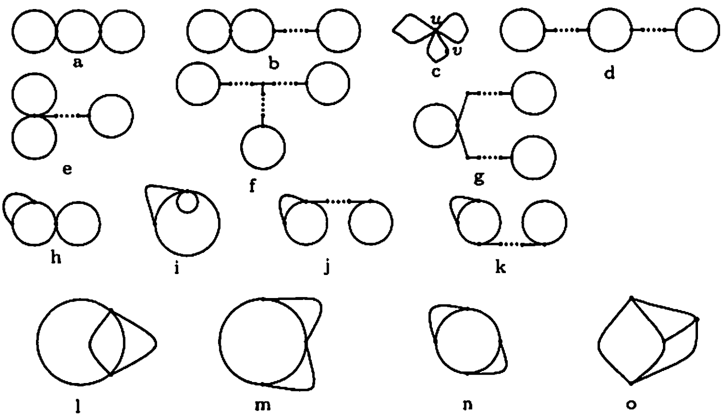


Figure 1. The arrangement of cycles in $\mathscr{T}_n^i (i = 3, 4, 6, 7)$

For any graph $G \in \mathscr{T}_n$, G can be obtained from some graphs showed in Figure 1 by attaching trees to some vertices.

In this paper, we shall investigate the Schultz index of tricyclic graphs by introducing some grafting transformations, and characterize the corresponding extremal graphs.

2 Preliminaries

Let $E' \subseteq E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E'. $V' \subseteq V(G)$, $G - V'$ denotes the subgraph of G obtained by deleting the vertices of V' and the edges incident with them. Let (G_1, v_1) and (G_2, v_2) be two graphs rooted at v_1 and v_2 respectively, then $G = (G_1, v_1)v(G_2, v_2)$ denote the graph obtained by identifying v_1 with v_2 as one common vertex v.

Lemma 1. Let C_p be the cycle of order p, and v is a vertex on C_p. Then

$$\sum_{x\in V(C_p)} d_{C_p}(x,v) = \begin{cases} \frac{p^2}{4}, & \text{if } p \text{ is even,} \\ \frac{p^2-1}{4}, & \text{if } p \text{ is odd} \end{cases}$$

For convenience, we provide some grafting transformations, which will decrease the Schultz index of graphs as follows:

Transformations A. Let uv be an edge G, $d_G(v) \geq 2$, $N_G(v) = \{u, w_1, w_2, \cdots, w_t\}$, and $w_1, w_2, \cdots, w_t$ are leaves adjacent to v. Then $G' = G - \{vw_1, vw_2, \cdots, vw_t\} + \{uw_1, uw_2, \cdots, uw_t\}$.

Lemma 2. Let G' be obtained from G by transformation A. Then we have $S(G') < S(G)$.

Proof. Let $G_0 = G - \{v, w_1, w_2, \cdots, w_s\}$, and $G_0' = G_0 - u$. By the definition of the Schultz index, we have

$$\begin{aligned} & S(G) \\ = \; & \sum_{x,y\in G_0'} [d_{G_0'}(x) + d_{G_0'}(y)] d_{G_0'}(x,y) + (s+2) \sum_{x\in G_0'} d_{G_0'}(x) d_{G_0'}(x,u) \\ & + [d_{G_0}(u) + 2s + 1] \sum_{x\in G_0'} d_{G_0'}(x,u) + (2s+1) \sum_{x\in G_0'} d_{G_0'}(x) \\ & + (3s+1)|V(G_0')| + (2s+1) d_{G_0}(u) + 3s^2 + 3s + 1 \end{aligned}$$

$$\begin{aligned} & S(G') \\ = \; & \sum_{x,y\in G_0'} (d_{G_0'}(x) + d_{G_0'}(y)) d_{G_0'}(x,y) + (s+2) \sum_{x\in G_0'} d_{G_0'}(x) d_{G_0'}(x,u) \\ & + (d_{G_0}(u) + 2s + 1) \sum_{x\in G_0'} d_{G_0'}(x,u) + (s+1) \sum_{x\in G_0'} d_{G_0'}(x) \\ & + (s+1)|G_0'| + (s+1) d_{G_0}(u) + 3s^2 + 4s + 1 \end{aligned}$$

Therefore, $S(G) - S(G') = s \sum_{x\in G_0'} d_{G_0'}(x) + 2s|V(G_0')| + s d_{G_0}(u) - s > 0.$

Remark 1. Repeating Transformation A, any tree can changed into a star, any cyclic graph can be changed into a cyclic such that all the edges not on the cycles are pendant edges.

Transformations B. Let u and v be two vertices in G. $u_1, u_2, \cdots, u_s$ are the leaves adjacent to u, $v_1, v_2, \cdots, v_t$ are the leaves adjacent to v. $G_0 = G - \{u_1, u_2, \cdots, u_s; v_1, v_2, \cdots, v_t\}$, $G' = G - \{vv_1, vv_2, \cdots, vv_t\} + \{uv_1, uv_2, \cdots, uv_t\}$, $G'' = G - \{uu_1, uu_2, \cdots, uu_s\} + \{vu_1, vu_2, \cdots, vu_s\}$.

Lemma 3. Let G', G'' are graphs obtained from G by transformation B. Then $S(G') < S(G)$, or $S(G'') < S(G)$.

Proof. Let $G_0^* = G_0 - \{u, v\}$. By the definition of Schultz index, we have

$$
\begin{aligned}
&S(G)\\
=&\sum_{x,y\in G_0^*}[d_{G_0^*}(x)+d_{G_0^*}(y)]d_{G_0^*}(x,y)+(s+1)\sum_{x\in G_0^*}d_{G_0^*}(x)d_{G_0^*}(x,u)\\
&+(t+1)\sum_{x\in G_0^*}d_{G_0^*}(x)d_{G_0^*}(x,v)+[d_{G_0}(u)+d_{G_0}(v)+s+t]d_{G_0}(u,v)\\
&+(s+t)[d_{G_0}(u)+d_{G_0}(v)]+(d_{G_0}(u)+2s)\sum_{x\in G_0^*}d_{G_0^*}(x,u)+\\
&(d_{G_0}(v)+2t)\sum_{x\in G_0^*}d_{G_0^*}(x,v)+(s+t)\sum_{x\in G_0^*}d_{G_0^*}(x)+(s+t)|V(G_0^*)|\\
&+3(s+t)^2+d_{G_0}(u,v)[4st+s+t+sd_{G_0}(v)+td_{G_0}(u)]
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&S(G')\\
=&\sum_{x,y\in G_0^*}[d_{G_0^*}(x)+d_{G_0^*}(y)]d_{G_0^*}(x,y)+\sum_{x\in G_0^*}d_{G_0^*}(x)d_{G_0^*}(x,u)\\
&+(s+t+1)\sum_{x\in G_0^*}d_{G_0^*}(x)d_{G_0^*}(x,v)+[d_{G_0}(u)+d_{G_0}(v)+s+t]\\
&d_{G_0}(u,v)+(s+t)[d_{G_0}(u)+d_{G_0}(v)]+d_{G_0}(u)\sum_{x\in G_0^*}d_{G_0^*}(x,u)\\
&+[d_{G_0}(v)+2s+2t]\sum_{x\in G_0^*}d_{G_0^*}(x,v)+(s+t)\sum_{x\in G_0^*}d_{G_0^*}(x)\\
&+(s+t)|V(G_0^*)|+3(s+t)^2+d_{G_0}(u,v)[s+t+sd_{G_0}(u)+td_{G_0}(u)]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta_1 &= S(G)-S(G')\\
&= t\{\sum_{x\in G_0^*}(2+d_{G_0^*}(x))[d_{G_0^*}(x,v)-d_{G_0^*}(x,u)]\\
&\quad +d_{G_0}(u,v)[4s+d_{G_0}(u)-d_{G_0}(v)]\}\\
\Delta_2 &= S(G)-S(G'')\\
&= s\{\sum_{x\in G_0^*}(2+d_{G_0^*}(x))[d_{G_0^*}(x,u)-d_{G_0^*}(x,v)]\\
&\quad +d_{G_0}(u,v)[4t+d_{G_0}(v)-d_{G_0}(u)]\}
\end{aligned}
$$

If $\Delta_1 < 0$, thus

$$\sum_{x\in G_0^*}(2+d_{G_0^*}(x))[d_{G_0^*}(x,u)-d_{G_0^*}(x,v)] > d_{G_0}(u,v)[4s+d_{G_0}(u)-d_{G_0}(v)]$$

Then, $\Delta_2 > 4s(s+t)d_{G_0}(u,v) > 0$

Otherwise, $\Delta_2 < 0$, we shall have $\Delta_1 > 0$.

Remark 2. Repeating Transformation B, any cyclic graph can be changed into a cyclic graph such that all the pendant edges are attached to the same vertex.

Following from the proof of Lemma 3, we have directly

Lemma 4. Let G' and G'' be the graphs depicted in transformation B, and $G_0^* = G_0 - \{u,v\}$. Then $S(G') > S(G'')$ if $d_{G_0}(u) > d_{G_0}(v)$ and $\sum_{x\in G_0^*} d_{G_0^*}(x)d_{G_0^*}(x,u) < \sum_{x\in G_0^*} d_{G_0^*}(x)d_{G_0^*}(x,v)$. Otherwise $S(G') < S(G'')$.

Lemma 5. Suppose that G is a graph of order $n \geq 7$ obtained from a connected graph $G_0 \ncong P_1$ and a cycle $C_P = v_0v_1\cdots v_{p-1}v_0(p \geq 4$ for p

is even. Otherwise $p \geq 5$) by identifying v_0 with a vertex v of the graph G_0. Let $G' = G - v_{p-1}v_{p-2} + vv_{p-2}$. We name above operation as grafting transformation C. Then $S(G') < S(G)$.

Proof. Let $G_0' = G_0 - v$, $C_p' = C_p - \{v, v_{p-1}\}$, $C_{p-1}' = C_{p-1} - v$. By the definition of Schultz index, we have

Case (i). p is even.

$$\begin{aligned} & S(G) \\ = & \sum_{x,y\in G_0'} (d_{G_0'}(x) + d_{G_0'}(y))d_{G_0'}(x,y) + \sum_{x\in G_0'} (d_{G_0'}(x) + d_G(v))d_{G_0'}(x,v) \\ & +4 \sum_{x,y\in C_p'} d_{C_p'}(x,y) + (2 + d_G(v)) \sum_{x\in C_p'} d_{C_p'}(x,v) + 2 + d_G(v) \\ & +4 \sum_{x\in C_p'} d_{C_p'}(x,v) + \sum_{x\in G_0'} (2 + d_{G_0'}(x)) \sum_{y\in C_p'} [d_{G_0'}(x,v) + \\ & \sum_{x\in G_0'} (2 + d_{G_0'}(x))](d_{G_0'}(x,v) + 1) \\ = & \sum_{x,y\in G_0'} [d_{G_0'}(x) + d_{G_0'}(y)]d_{G_0'}(x,y) + p \sum_{x\in G_0'} d_{G_0'}(x)d_{G_0'}(x,v) \\ & +[d_G(v) + 2p - 2] \sum_{x\in G_0'} d_{G_0'}(x,v) + \frac{p^2}{4} \sum_{x\in G_0'} d_{G_0'}(x) + \frac{p^2}{4} d_G(v) \\ & +\frac{p^3}{2} - \frac{p^2}{2} + \frac{p^2}{2}(|V(H)| - 1) \end{aligned}$$

Similarly, we have

$$\begin{aligned} & S(G') \\ = & \sum_{x,y\in G_0'} [d_{G_0'}(x) + d_{G_0'}(y)]d_{G_0'}(x,y) + p \sum_{x\in G_0'} d_{G_0'}(x)d_{G_0'}(x,v) + \\ & [d_G(v) + 2p - 2] \sum_{x\in G_0'} d_{G_0'}(x,v) + (\frac{p^2}{4} - \frac{p}{2} + 1) \sum_{x\in G_0'} d_{G_0'}(x) + \\ & \frac{p^2 - 2p + 4}{4} d_G(v) + \frac{p^3}{2} - p^2 + 3p - 4 + \frac{p^2 - 2p + 2}{2}(|V(H)| - 1) \end{aligned}$$

Thus,

$$\begin{aligned} & S(G') - S(G) \\ = & (1 - \frac{p}{2}) \sum_{x\in G_0'} d_{G_0'}(x) + (1 - \frac{p}{2})d_G(v) + (1 - p)(|V(H)| - 1) \\ & -\frac{1}{2}(p-3)^2 + \frac{1}{2} < 0, (\text{since } p \geq 4) \end{aligned}$$

Case (ii). p is odd.

Similar to case (i), we have

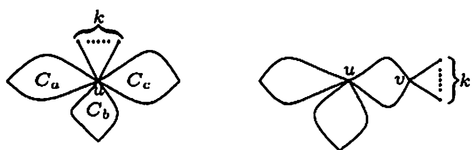
$$\begin{aligned} & S(G') - S(G) \\ = & (\frac{3}{2} - \frac{p}{2}) \sum_{x\in G_0'} d_{G_0'}(x) + (\frac{3}{2} - \frac{p}{2})d_G(v) + (2 - p)|V(H) - 1| \\ & -\frac{1}{2}(p-4)^2 + \frac{7}{2} < 0, (\text{since } p \geq 5) \end{aligned}$$

3 The minimum Schultz index of $\mathscr{T}_n^i$

In this section we shall determine the graphs achieve the minimum Schultz index in $\mathscr{T}_n^i(i = 3, 4, 6, 7)$, respectively.

3.1 The minimum Schultz index in $\mathscr{T}_n^3$

Let H be a graph formed by attaching three cycles C_a, C_b, C_c to a common vertex u; see Figure 1.(c). Then let $G^k_{a,b,c}$ is the graph on n vertices obtained from H by attaching k pendent edges to the vertex u. We also set $\mathcal{G} = \{G \in \mathscr{T}_n$: G is a graph obtained from H by attaching k pendent vertices to the vertex v of H except $u\}$, where $a+b+c+k = n+2$, and let $\widetilde{G}^k_{a,b,c}$ is one of the resulted graph. See Figure 2.



$G^k_{a,b,c}$ $\widetilde{G}^k_{a,b,c}$

Figure 2. The graphs $G^k_{a,b,c}$ and $\widetilde{G}^k_{a,b,c}$

Immediately, by Lemma 4, the following result is obvious.

Lemma 6. Let $G^k_{a,b,c}$, $\widetilde{G}^k_{a,b,c}$ are graphs depicted above, then

$$S(G^k_{a,b,c}) < S(\widetilde{G}^k_{a,b,c}).$$

Lemma 7. Let G is a n vertices tricyclic graph with exactly three cycles C_a, C_b and C_c, then $S(G) \geq S(G^k_{a,b,c})$ with the equalities if and only if $G \cong G^k_{a,b,c}$.

Proof. Let G is a n vertices tricyclic graph with exactly three cycles C_a, C_b and C_c, then the arrangement of the three cycles contained in G are depicted in Figure 1. a,b,c,d,e,f,g, respectively. Repeating transformation A and B on G, and by Lemma 2 and Lemma 3, we have $S(G) \geq S(G^k_{a,b,c})$ or $S(G) \geq S(\widetilde{G}^k_{a,b,c})$.

Hence, by Lemma 6, we have $S(G) \geq S(G^k_{a,b,c})$.

Lemma 8. For any given positive integers a, b, c and k, then

(i) $S(G^k_{a,b,c}) > S(G^{k+1}_{a-1,b,c})$, if $a \geq 4$, $b, c \geq 3$;

(ii) $S(G^k_{a,b,c}) > S(G^{k+1}_{a,b-1,c})$, if $a, c \geq 3$, $b \geq 4$;

(iii) $S(G^k_{a,b,c}) > S(G^{k+1}_{a,b,c-1})$, if $c \geq 4$, $a, b \geq 3$.

Proof. By the symmetry of three cycles C_a, C_b and C_c contained in G, here we only show that (i) holds. We omit the proofs for (ii) and (iii).

Let $G_0 = C_b u C_c u k K_1$, $G_0' = C_b u C_c u (k+1) K_1$, then $G^k_{a,b,c} = C_a u G_0$ and $G^{k+1}_{a-1,b,c} = C_{a-1} u G_0'$. Applying transformation C on $G^k_{a,b,c}$ and we get $G^{k+1}_{a-1,b,c}$, by Lemma 5, the results hold.

Combing Lemma 6, Lemma 7 with Lemma 8, we have

Theorem 3.1. Let $G \in \mathscr{T}_n^3$, then $S(G) \geq 3n^2 + 5n - 26$. The equality hold if and only if $G \cong G^{n-7}_{3,3,3}$.

Proof. Follows Lemmas 6, 7 and 8, for any graph $G \in \mathscr{T}_n^3$, $S(G) \geq S(G^{n-7}_{3,3,3})$. It is ease to calculate out that $S(G^{n-7}_{3,3,3}) = 3n^2 + 5n - 26$.

3.2 The minimum Schultz index in $\mathscr{T}_n^4$

Let P_{a+1}, P_{b+1}, P_{c+1} be three vertex disjoint paths with $a, b, c \geq 1$, and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, *resp.* The resulting graph, denote as Θ−graph $\Theta(a,b,c)$. Connecting the cycle C_d and $\Theta(a,b,c)$ by a path P_k, where $k \geq 1$, naming the resulting graph as $\widetilde{\Theta}$−graph. From [11-16], we know that the are exactly four types of $\widetilde{\Theta}$−graph, see Figure 1. h,i,j,k. $\mathscr{T}_n^4$ is the set of graphs each of which is a $\widetilde{\Theta}$−graph, has some trees attached, if possible. Let $\mathcal{H}_0 = \Theta(a,b,c) v C_d$, and $H^k_{a,b,c,d}$ is a n vertex graph formed from $\mathcal{H}_0$ by attaching $k (k = n + 5 - a - b - c - d)$ pendent vertices to v, see Figure 3.

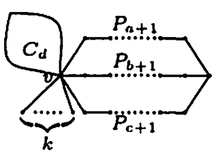


Figure 3. The graph $H^k_{a,b,c,d}$

Similar to the discussion way of section 3.1, we have

Lemma 9. Let $G \in \mathscr{T}_n^4$ such that G contains the $\Theta(a,b,c)$ and the cycle C_d with $E(\Theta(a,b,c)) \cap E(C_d) = \emptyset$. Then $S(G) \geq S(H^k_{a,b,c,d})$. The equality holds if and only if $G \cong H^k_{a,b,c,d}$.

Similarly, we have

Lemma 10. For any given positive integers a, b, c, d and k, then

(i) $S(H^k_{a,b,c,d}) > S(H^{k+1}_{a-1,b,c,d})$, for either $a \geq 4$, $b, c \geq 2$ and $bc \geq 6$, $d \geq 3$ or $a = 3$, $b, c, d \geq 3$;

(ii) $S(H^k_{a,b,c,d}) > S(H^{k+1}_{a,b-1,c,d})$, for either $b \geq 4$, $a, c \geq 2$ and $ac \geq 6$, $d \geq 3$ or $b = 3$, $a, c, d \geq 3$;

(iii) $S(H^k_{a,b,c,d}) > S(H^{k+1}_{a,b,c-1,d})$, for either $c \geq 4$, $a, b \geq 2$ and $ab \geq 6$, $d \geq 3$ or $c = 3$, $a, b, d \geq 3$;

(iv) $S(H^k_{a,b,c,d}) > S(H^{k+1}_{a,b,c,d-1})$, for $d \geq 4$, $a, b, c \geq 2$ and $abc \geq 18$.

And

Theorem 3.2. Let $G \in \mathscr{T}_n^4$, then $S(G) \geq 3n^2 + 5n - 28$, the equality holds if and only if $G \cong H_{2,3,3,3}^{n-6}$ (or $H_{3,2,3,3}^{n-6}$, $H_{3,3,2,3}^{n-6}$).

Proof. Note that $H_{2,3,3,3}^{n-6} \cong H_{3,2,3,3}^{n-6} \cong H_{3,3,2,3}^{n-6}$. By Lemma 10, for any graph $G \in \mathscr{T}_n^4$, $S(G) \geq S(H_{2,3,3,3}^{n-6})$, and $S(H_{2,3,3,3}^{n-6}) = 3n^2 + 5n - 28$.

3.3 The minimum Schultz index in $\mathscr{T}_n^6$

Let $I_{a,b,c,d}^k$ is a tricyclic graph with exact 6 cycles on n vertices obtained from Figure 1(l) by attaching k pendent vertices to v showed in Figure 4(i);

$J_{a,b,c}^k$ is a tricyclic graph with exact 6 cycles on m vertices obtained from Figure 1(m) by attaching k pendent vertices to v showed in Figure 4(ii);

$K_{a,b,c}^k$ is a tricyclic graph with exact 6 cycles on n vertices obtained from Figure 1(n) by attaching k pendent vertices to v showed in Figure 4(iii).

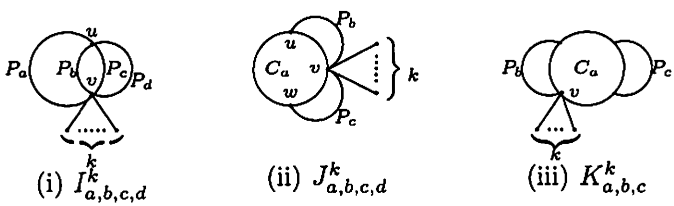


Figure 4. The graphs $I_{a,b,c,d}^k$, $J_{a,b,c}^k$, $K_{a,b,c}^k$

Lemma 11. Let $G \in \mathscr{T}_n^6$. Then

(i) $S(G) \geq S(I_{a,b,c,d}^k)$, if the six cycles in G are arranged the same way with the graphs depicted in Figure 1(l);

(ii) $S(G) \geq S(J_{a,b,c}^k)$, if the six cycles in G are arranged the same way with the graphs depicted in Figure 1(m);

(iii) $S(G) \geq S(K_{a,b,c}^k)$, if the six cycles in G are arranged the same way with the graphs depicted in Figure 1(n).

Similarly, we have

Theorem 3.3 Let $G \in \mathscr{T}_n^6$,

(i) If the arrangement of the six cycles is the same as Fig 1(l), then $S(G) \geq S(I_{3,3,3,2}^{n-5})$. The equality holds if and only if $G \cong I_{3,3,3,2}^{n-5}$;

(ii) If the arrangement of the six cycles is the same as Fig 1(m), then $S(G) \geq S(J_{3,3,3}^{n-5})$. The equality holds if and only if $G \cong J_{3,3,3}^{n-5}$;

(iii) If the arrangement of the six cycles is the same as Fig 1(n), then $S(G) \geq S(K_{4,3,3}^{n-6})$. The equality holds if and only if $G \cong K_{4,3,3}^{n-6}$.

Moreover, It is easily to compute out that

$S(I_{3,3,3,2}^{n-5}) = 3n^2 + 5n - 32$, $S(J_{3,3,3}^{n-5}) = 3n^2 + 5n - 30$, $S(K_{4,3,3}^{n-6}) = 3n^2 + 16n - 84$.

Combining above results, we arrive at:

Theorem 3.4. Let $G \in \mathscr{T}_n^6$, then $S(G) \geq 3n^2 + 5n - 32$. The equality holds if and only if $G \cong I_{3,3,3,2}^{n-5}$.

3.4 The minimum Schultz index in $\mathscr{T}_n^7$

Let $R^k(a, b, c, d, e, f)$ is a tricyclic graph with exact seven cycles on n vertices obtained from Figure 1(o) by attaching k pendent vertices to v showed in Figure 5, where $a + b + c + d + e + f + k = n + 8$.

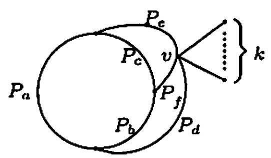


Figure 5. The graph $R^k_{a,b,c,d,e,f}$

Applying the similar methods above, we can obtain the following results, and we omit the proof here.

Theorem 3.5. Let $G \in \mathscr{T}_n^7$, then $S(G) \geq S(R^{n-4}_{2,2,2,2,2,2})$. The equality holds if and only if $G \cong R^{n-4}_{2,2,2,2,2,2}$.

Note that $S(R^{n-4}_{2,2,2,2,2,2}) = 3n^2 + 5n - 32$.

4 The minimum Schultz index of $\mathscr{T}_n$

Combining all the results above, we arrive at our main result:

Theorem 4.1 Let $G \in \mathscr{T}_n$, then $S(G) \geq 2n^2 + 13n - 30$. The equality holds if and only if $G \cong I^{n-5}_{3,3,3,2}$ or $G \cong R^{n-4}_{2,2,2,2,2,2}$.

Proof. Combining Theorem 3.1, 3.2, 3.4 with 3.5, for any graph $G \in \mathscr{T}_n$, we have

$S(G) \geq \min\{S(G^{n-7}_{3,3,3}), S(H^{n-6}_{2,3,3,3}), S(I^{n-5}_{3,3,3,2}), S(R^{n-4}_{2,2,2,2,2,2})\} = 3n^2 + 5n - 32$.

Therefore, $I^{n-5}_{3,3,3,2}$ and $R^{n-4}_{2,2,2,2,2,2}$ have the minimum Schultz index among all n-tricyclic graphs

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