Bounds for the b-chromatic number of the Mycielskian of some families of graphs

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Abstract

The b-chromatic number $b(G)$ of a graph G is defined as the maximum number k of colors in a proper coloring of the vertices of G in such a way that each color class contains at least one vertex adjacent to a vertex of every other color class. Let $\mu(G)$ denote the Mycielskian of G. In this paper, it is shown that if G is a graph with b-chromatic number b and for which the number of vertices of degree at least b is at most $2b-2$, then $b(\mu(G))$ lies in the interval $[b+1, 2b-1]$. As a consequence, it follows that $b(G)+1 \leq b(\mu(G)) \leq 2b(G)-1$ for G in any of the following families: split graphs, $K_{k,k}-\{$a 1-factor$\}$, the hypercubes Q_p, where $p \geq 3$, trees and a special class of bipartite graphs. We show further that for any positive integer b and every integer $k \in [b+1, 2b-1]$, there exists a graph G belonging to the family mentioned above, with $b(G) = b$ and $b(\mu(G)) = k$.

Key Words: b-chromatic number, Mycielskian, split graphs and hypercubes.
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1 Introduction

All graphs considered in this paper are non-trivial, simple and undirected. Let G be a graph with vertex set V and edge set E. The order of G will be denoted by n. A proper k-coloring of a graph G is a partition $P = \{V_1, V_2, \ldots, V_k\}$ of V into independent sets (also known as color classes). A proper k-coloring of a graph G is a b-coloring of G using k colors if each color class contains a color dominating vertex (c.d.v.), that is, a vertex adjacent to at least one vertex of every other color class. The b-chromatic number of a graph G, denoted by $b(G)$, is the maximum k such that G has a b-coloring using k colors.

The b-chromatic number was introduced by R.W. Irving and D.F. Manlove [8] by considering proper colorings that are minimal with respect to a partial order defined on the set of all partitions of $V(G)$. They have shown that the determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. There has been a lot of papers on b-coloring in recent times, for instance [1], [2], [3], [4], [5], [6], [7], [9], [10], [12], [13] and the references given in these papers.

The b-chromatic number is a graph parameter which is similar to the achromatic number. As it is well-known, the achromatic number of a graph G is the maximum number of colors used to give a proper coloring to the vertex set of G so that between any two distinct color classes, there is at least one edge. On the other hand, the b-chromatic number is the maximum number of color classes in a proper vertex coloring of G so that each color class contains a c.d.v.. Interestingly, the minimum number of color classes required in these two colorings is the same as $\chi(G)$, the chromatic number of G.

Suppose that the vertices of a graph G are ordered as $v_1, v_2, \ldots, v_n$ with $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. Then the m-degree, $m(G)$, of G is defined by [8]

$$m(G) = \max\{i : d(v_i) \geq i - 1,\ 1 \leq i \leq n\}.$$

It is easy to see that the following observations are true for $b(G)$ and $m(G)$.

Remark 1.1 *(i)* $\chi(G) \leq b(G) \leq \Delta(G) + 1$, *where* $\Delta(G)$ *is the maximum degree of* G.

(ii) $b(G) \leq m(G)$.

(iii) The number of vertices of degree at least $m(G)$ *is at most* $m(G)$.

A vertex $x \in V(G)$ is said to be dense if $d_G(x) \geq m(G) - 1$. A tree $T = (V, E)$ is *pivoted* (see [8]) if T has exactly $m(G)$ dense vertices and T contains a distinguished vertex v such that:
(i) v is not dense.
(ii) Each dense vertex is adjacent either to v or to a dense vertex adjacent to v.
(iii) Any dense vertex adjacent to v and to another dense vertex has degree $m(G) - 1$.

Such a vertex v is called the *pivot* of T [8]. Clearly, a pivot is unique, if it exists. In [8], Irving and Manlove have determined the b-chromatic number of trees.

Theorem 1.2 ([8])
If T is a pivoted tree, then $b(T) = m(T) - 1$. If not, $b(T) = m(T)$.

In [11], Kouider and Mahéo have determined the b-chromatic number of the hypercubes Q_p.

Theorem 1.3 ([11])
$b(Q_1) = b(Q_2) = 2$ and $b(Q_p) = p + 1$, $p \geq 3$, where Q_p is the hypercube of dimension p.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [14] developed an interesting graph transformation as follows. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is called the root of $\mu(G)$.

In this paper, we obtain bounds for the b-chromatic number of the Mycielskians of certain families of graphs.

2 Bounds for b($\mu(G)$)

In this section, we determine bounds for the b-chromatic number of certain families of the Mycielskian.

Theorem 2.1
Let G be a graph with $b(G) = b$, and let G have at most $2b - 2$ vertices of degree at least b. Then $b + 1 \leq b(\mu(G)) \leq 2b - 1$.

Proof. Let $\{V_1, V_2, \ldots, V_b\}$ be a b-coloring of G using b colors with x_i being a c.d.v. of V_i, $1 \leq i \leq b$. We shall extend this coloring to a $b-$coloring for $\mu(G)$ using $b + 1$ colors as follows. Let $U_i = V_i \cup V_i'$, $1 \leq i \leq b$, and $U_{b+1} = \{u\}$ (Here V_i' stands for the set of twins of the vertices of V_i). Clearly, x_i', the twin of x_i is a c.d.v. of U_i for each i, and u is trivially a c.d.v. of the color class U_{b+1}. Thus $b(\mu(G)) \geq b + 1$.

Let $K = K(G)$ denote the set of vertices of G of degree at least b, and $L(G) = V \backslash K$. By our assumption, $|K| \leq 2b - 2$. Let $P = \{V_1, V_2, \ldots, V_p\}$ be some b-coloring of $\mu(G)$ using p colors. By the definition of $\mu(G)$, $N_{\mu(G)}(x') = N_G(x) \cup \{u\}$. If x' is a c.d.v. with respect to P, then x has a neighbor in all V_i's except perhaps the color class containing x' and the class containing u. Thus $x \in V_j \cup V_k$, where $x' \in V_j$ and $u \in V_k$ for some $j, k \in \{1, 2, \ldots, p\}$. This is true for all $x' \in V'$ which are c.d.v.'s with respect to P. Hence we have the following conclusion: If x and x' are c.d.v.'s of distinct color classes, then x and u must belong to the same color class. Since by our assumption $|K| \leq 2b-2$, there can be at most $2b-2+1 = 2b-1$ c.d.v.'s of $\mu(G)$ in $K \cup K' \cup \{u\}$ belonging to distinct color classes in $\mu(G)$. Now for $b(\mu(G))$ to be strictly greater than $2b-1$, there should be at least $2b$ vertices in $\mu(G)$ each of degree at least $2b-1$. This is not possible since for any $x \in L(G) \cup L'(G)$, $d_{\mu(G)}(x) \leq (b-1)+(b-1) = 2b-2$. Therefore, $b(\mu(G)) \leq 2b-1$. ∎

Remark 2.2
(i) Note that $b(G)+1 \leq b(\mu(G))$ holds good for all graphs.
(ii) From the proof of Theorem 2.1, it follows that $b(\mu(G)) \leq n+1$ where n is the order of G.

Corollary 2.3
For any tree T with $b(T) = b$, $b+1 \leq b(\mu(T)) \leq 2b-1$.

Proof. We prove the result by establishing that $|K(T)| \leq 2b-2$. First assume that T is pivoted. Now by Theorem 1.2, $b(T) = m(T) - 1$. From the definition of a pivoted tree, $|K(T)| = m(T) = b+1 \leq 2b-2$ for any $b \geq 3$. For $b = 2$, we can see that T is not pivoted. Thus $|K(T)| \leq 2b-2$ for all pivoted trees T. For a tree T that is not pivoted, by Theorem 1.2, $b(T) = m(T)$ and by (iii) of Remark 1.1, $|K(T)| \leq m(T)$. Therefore $|K(T)| \leq 2b-2$. ∎

Corollary 2.4
For a graph G with $b(G) = \Delta(G)+1$, $b(G)+1 \leq b(\mu(G)) \leq 2b(G)-1$.

Proof. Since there are no vertices of degree $b(G)$, $|K| = 0$. ∎

As a consequence, we see that $b(G)+1 \leq b(\mu(G)) \leq 2b(G)-1$ for $G = K_n, K_{n,n}-\{\text{a 1-factor}\}$ and the hypercubes Q_p, where $p \geq 3$. In particular, since $b(K_n)+1 = n+1 \leq b(\mu(K_n)) \leq 1+\Delta(\mu(K_n)) = 1+n$, $b(\mu(K_n)) = n+1$.

Recall that a graph G is said to be a split graph if its vertex set $V(G)$ can be partitioned into two subsets such that the subgraph induced by one set is a clique and the other is totally disconnected.

Corollary 2.5
For any non-trivial split graph S, $b(S)+1 \leq b(\mu(S)) \leq 2b(S)-1$.

Proof. As S is a split graph, $V(S)$ can be partitioned into V_1, V_2 where $< V_1 >$ is a clique of maximum order, say k, and $< V_2 >$ is totally disconnected. For each $x \in V_2$, $d_S(x) \leq k-1$. As S is a split graph,

$k = \omega(S) = \chi(S) \leq b(S)$, where $\omega(S)$ denotes the clique number of S. As the number of vertices with degree at least k is at most k, $b(S) = k$. Hence $|K(S)| \leq k = b(S) \leq 2b(S) - 2$. ∎

Corollary 2.6
Let $G = (V, E)$ be a bipartite graph with edge disjoint cycles and $b(G) = b \geq 3$. If there exists a vertex $x \in V$ such that $d(x, y) \leq 2$ for all $y \in K(G) = K$, then $b(G) + 1 \leq b(\mu(G)) \leq 2b(G) - 1$.

Proof. The proof is based on the result that for these bipartite graphs, $|K| \leq b + 1$ for $b \leq 6$ and $|K| \leq b + 2$ for $b \geq 7$. We omit the details of the proof (which is somewhat lengthy) of this result as this pertains to a rather restricted family of bipartite graphs. ∎

3 All values in $[b + 1, 2b - 1]$ are attainable

In this section, we show that for any positive integer b and every integer $k \in [b + 1, 2b - 1]$, there exists a graph G with $b(G) = b$ and $b(\mu(G)) = k$. We establish this by showing that for any positive integer $q \geq 2$ and for any integer p, where $0 \leq p \leq q - 2$ there exists a tree T with $b(T) = q$ and $b(\mu(T)) = p + q + 1$.

Let $T_{p,q}$ be the caterpillar (Recall that a caterpillar is a tree in which the removal of the pendant vertices results in a path) constructed as in *Figure 1.* In $T_{p,q}$, $v_1, v_2, \dots, v_q$ are vertices of degree $2q - 2$, and let $N(v_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,2q-2}\}$ for $i = 1, \dots, q$. Let $u_1, u_2, \dots, u_p$ be the vertices of degree $q - 1$ with $N(u_j) = \{u_{j,1}, u_{j,2}, \dots, u_{j,q-1}\}$ for $j = 1, \dots, p$. Also $d(v_i, v_{i+1}) = d(u_j, u_{j+1}) = d(v_q, u_1) = 4$ for $i = 1, \dots, q - 1$ and $j = 1, \dots, p-1$. Further w_i is the middle vertex of the path of length 4 between v_i and v_{i+1}, for $i = 1, 2, \dots, q - 1$, w_{q+j} is the middle vertex of the path of length 4 between u_j and u_{j+1}, for $j = 1, 2, \dots, p - 1$ and w_q is the middle vertex of the path between v_q and u_1. In Theorem 3.1, we show that for $q \geq 4$ and $0 \leq p \leq q - 2$, $T_{p,q}$ satisfies the required conditions.

Theorem 3.1
For $T_{p,q}$ with $q \geq 4$ and $0 \leq p \leq q-2$, $b(T_{p,q}) = q$ and $b(\mu(T_{p,q})) = p+q+1$.

Proof. Observing $b(T_{p,q}) \leq q$ is not difficult as $T_{p,q}$ does not have $q + 1$ vertices of degree at least q. Moreover, it is easy to show that $b(T_{p,q}) \geq q$. A b-coloring using q colors can easily be given to $T_{p,q}$: Simply color the vertices $v_1, v_2, \dots, v_q$ by the colors $1, 2, \dots, q$ respectively and extend it to a b-coloring of $T_{p,q}$. Such an extension is easily obtainable. Consequently, $b(T_{p,q}) = q$. We shall next show that $\mu(T_{p,q})$ has a b-coloring using $p+q+1$

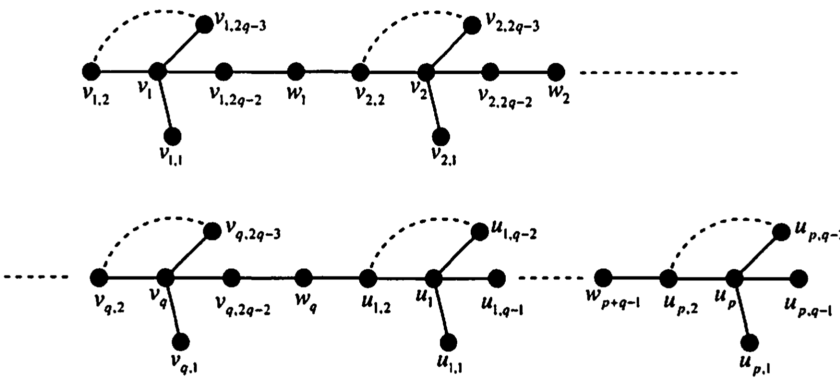

Figure 1: $T_{p,q}$

colors. For each $i = 1, \ldots, q$ and $j = 1, \ldots, p$, color v_i with i and u_j with $q + j$ and u with $p + q + 1$. Since $d_{T_{p,q}}(v_i) = 2q - 2$ for all $i = 1, \ldots, q$ and $N_{T_{p,q}}[x] \cap N_{T_{p,q}}[y] = \emptyset$ for all $x, y \in \{v_1, \ldots, v_q, u_1, \ldots, u_p\}$, we can make v_i a c.d.v. of color class i by giving distinct colors to the neighbors of v_i in $T_{p,q}$. Now color each vertex in $(N_{T_{p,q}}[v_i])'$ by the color of its twin except for the twins of those vertices for which the color $p + q + 1$ is given. For these twins, give any color except i and $p + q + 1$. This would make u a c.d.v. of the color class $p + q + 1$. For each j, $1 \leq j \leq p$, $d_{\mu(T_{p,q})}(u_j) = 2q - 2$ and thus it is possible to make u_j a c.d.v. of the color class $q + j$ by giving distinct colors to the neighbors of u_j in $\mu(T_{p,q})$. While doing so, we have to be careful not to give color $p + q + 1$ to any of the twins of $N_{T_{p,q}}(u_j)$. Note that, for $x \in \{v_1, \ldots, v_q, u_1, \ldots, u_p\}$, if some vertices in $N_{T_{p,q}}[x]$ or $(N_{T_{p,q}}[x])'$ are not colored, giving a proper color to it is not difficult. At this stage, the vertices that have been left out without coloring are only the vertices of degree 2 in $T_{p,q}$. Such vertices are of degree 4 in $\mu(T_{p,q})$ and their twins are of degree 3 in $\mu(T_{p,q})$. Clearly since $p + q + 1 \geq 5$, these vertices can be properly colored. Thus $p + q + 1 \leq b(\mu(T_{p,q}))$. If $b(\mu(T_{p,q})) > p + q + 1$, then there should be at least $p + q + 2$ vertices of degree at least $p + q + 1 \geq 5$. But this is not possible as all vertices other than those in $\{u\} \cup \{v_1, v_2, \ldots, v_q\} \cup \{u_1, u_2, \ldots, u_p\}$ and their twins (except for u) are of degree at most 4 in $\mu(T_{p,q})$. Thus $b(\mu(T_{p,q})) = p + q + 1$. ∎

We now consider the cases not covered by Theorem 3.1. For $b(G) = 3$, $2b(G) - 1 = 5$. Thus $b(\mu(G)) = 4$ or 5. We can easily see that $b(P_5) = 3$ and $b(\mu(P_5)) = 4$ and for P_8, $b(P_8) = 3$ and $b(\mu(P_8)) = 5$ (P_n stands for the path on n vertices). For $b(G) = 2$, $b(G) + 1 = 2b(G) - 1 = 3 = b(\mu(G))$. Thus we see that there exists a tree T with $b(T) = b$ such that for each k,

$b+1 \leq k \leq 2b-1$, $b(\mu(T)) = k$.

Open Problems

(1) Does there exist a graph G for which $b(\mu(G)) \geq 2b(G)$?

(2) Is it true that for chordal graphs G, $|K| \leq 2b-2$? (where K is as defined in Theorem 2.1)

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