K_5^--factor in a Graph

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Abstract

Let G be a graph and let $\delta(G)$ denote the minimum degree of G. Let F be a given connected graph. Suppose that $|V(G)|$ is a multiple of $|V(F)|$. A spanning subgraph of G is called an F-factor if its components are all isomorphic to F.

In 2002, Kawarabayashi [5] conjectured that if G is a graph of order $\ell k(\ell \geq 3)$ with $\delta(G) \geq \frac{\ell^2-3\ell+1}{\ell-2}k$, then G has a K_ℓ^--factor, where K_ℓ^- is the graph obtained from K_ℓ by deleting just one edge. In this paper, we prove that this conjecture is true when $\ell = 5$.

Keywords: K_5^--factor, minimum degree

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1 Introduction

We use [1] for notations and terminology not defined here, and consider finite undirected simple graphs only. Let H be a subgraph of G. Denote $N_G(v) = \{u \in V(G) | uv \in E(G)\}$, $N_H(v) = N_G(v) \cap V(H)$, $d_G(v) = |N_G(v)|$, and $d_H(v) = |N_H(v)|$. The minimum degree of G is denoted by $\delta(G)$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $G[S]$. For a subgraph H of G and the vertices x and y of G with $x \in V(H)$ and $y \notin V(H)$, $G - H = G[V(G) - V(H)]$, $H - x = G[V(H) - \{x\}]$, $H + y = G[V(H) \cup \{y\}]$, and $H - x + y = G[(V(H) - \{x\}) \cup \{y\}]$. The complement of the subgraph H of G is denoted by $\overline{H}$. Let H and K be two vertex-disjoint graphs. $H \cup K$ denotes the union of H and K. The joint $H \vee K$ is the graph with $V(H \vee K) = V(H) \cup V(K)$ and $E(H \vee K) = E(H) \cup E(K) \cup \{uv | u \in V(H), v \in V(K)\}$.

Let F be a given connected graph. Suppose that $|V(G)|$ is a multiple of $|V(F)|$. A spanning subgraph of G is called an F-factor if its components are all isomorphic to F. There are many results concerning minimum degree conditions for a graph to have an F-factor. Hajnal and Szemerédi [4] proved that for $F = K_\ell$, $\delta(G) \geq \frac{\ell-1}{\ell} \cdot |V(G)|$ suffices. Let K_ℓ^- be the graph obtained from K_ℓ by deleting just one edge. It is natural to ask what condition in terms of the minimum degree can guarantee the existence of a K_ℓ^--factor.

For the case $\ell = 3$, since $K_3^- = P_3$, Enomoto, Kaneko, and Tuza [3] proved that a connected graph G with order $3k$ has a P_3-factor if $\delta(G) \geq k$. For the case $\ell = 4$, Kawarabayashi proved the following in 2002.

Theorem 1.1 (Kawarabayashi [5]) *Let G be a graph of order $4k$ with $\delta(G) \geq \frac{5}{2}k$. Then G has a K_4^--factor.*

Consider the graph $G = K_{k-1} \vee G'$, where G' is a $K_{\ell-1}$-free graph. It is obvious that G contains at most $k-1$ vertex-disjoint $K_{\ell-1}$, and a K_ℓ^--factor needs at least k vertex-disjoint $K_{\ell-1}$. So G does not have a K_ℓ^--factor. Since G' is a $K_{\ell-1}$-free graph, by Turan's Theorem, the minimum degree of G is at most

$$\frac{(\ell-1)k+1}{\ell-2} \cdot (\ell-3) + k - 1 = \frac{\ell^2 - 3\ell + 1}{\ell - 2}k - \frac{1}{\ell-2}.$$

Based on this fact, Kawarabayashi [5] proposed the following conjecture.

Conjecture 1.2 (Kawarabayashi [5]) *Let G be a graph of order ℓk with $\delta(G) \geq \frac{\ell^2-3\ell+1}{\ell-2}k$, where $\ell \geq 3$. Then G has a K_ℓ^--factor.*

Conjecture 1.2 is true for the case that $\ell = 3, 4$ (in the case $\ell = 3$, the assumption "connected" is necessary). Also, this conjecture was verified in [2] if the order of the graph is sufficiently large. The remaining question is what if the order of the graph is not so large. In this paper we give a complete answer for the case $\ell = 5$.

Theorem 1.3 *Let G be a graph of order $5k$ with $\delta(G) \geq \frac{11}{3}k$. Then G has a K_5^--factor.*

The condition on $\delta(G)$ is best possible in a sense. Consider the graph $G = \overline{K_{k-1}} \vee \overline{K_{\frac{4k+1}{3}}} \vee \overline{K_{\frac{4k+1}{3}}} \vee \overline{K_{\frac{4k+1}{3}}}$. Then $n = 5k$, $\delta(G) = \frac{11}{3}k - \frac{1}{3}$. Since $\overline{K_{\frac{4k+1}{3}}} \vee \overline{K_{\frac{4k+1}{3}}} \vee \overline{K_{\frac{4k+1}{3}}}$ is K_4-free, G does not have a K_5^--factor.

Let S be the graph obtained from K_5 by removing two edges incident with a common vertex, and let L be the graph obtained from K_5 by removing three edges incident with a common vertex. Since both S and L are subgraphs of K_5^- and they have K_4 as a subgraph, the condition on $\delta(G)$ is also best possible because of the same example as in Theorem 1.3. Hence we can get the following corollaries.

Corollary 1.4 *Let G be a graph of order $5k$ with $\delta(G) \geq \frac{11}{3}k$. Then G has an S-factor.*

Corollary 1.5 *Let G be a graph of order $5k$ with $\delta(G) \geq \frac{11}{3}k$. Then G has an L-factor.*

2 Preparation for the Proof of Theorem 1.3

Let G be an edge-maximal counterexample. Since a complete graph of order $5k$ has a K_5^--factor, G is not a complete graph. Let u and v be nonadjacent vertices of G and let G' be the graph obtained from G by adding the edge uv. Then G' is not a counterexample by the maximality of G and G' has a K_5^--factor, that is, G' contains k vertex-disjoint subgraphs $D_1, D_2, \cdots, D_k$, where D_i is isomorphic to K_5^- or K_5. Since G is a counterexample, the edge uv lies in one of $D_1, D_2, \cdots, D_k$. Without loss of generality, we may assume $uv \in E(D_k)$, that is, G has $k-1$ vertex-disjoint subgraphs $D_1, D_2, \cdots, D_{k-1}$ such that $\sum_{i=1}^{k-1} |D_i| = 5k-5$. Let H be the subgraph of G induced by $\bigcup_{i=1}^{k-1} V(D_i)$ and $M = G-H$. Since $uv \in E(D_k)$, M is obtained from K_5^- by removing just one edge. So there are two possibilities for M, namely S and W_4 (see Figure 1).

Figure 1

Now we choose $D_1, D_2, \cdots, D_{k-1}$ so that

(1) M is either S or W_4.

(2) Subject to the condition (1), if there are two possibilities for M, namely S and W_4, then we choose S.

(3) Subject to the conditions (1) and (2), $\sum_{i=1}^{k-1} |E(D_i)|$ is as large as possible, that is, the D_i's $(1 \le i \le k-1)$ take K_5 instead of K_5^- as many times as possible.

In Section 3, we shall prove the case where M is isomorphic to S. In Section 4, we shall settle the case where M is isomorphic to W_4 by reducing the situation to the case where M is isomorphic to S.

3 The case where M is isomorphic to S

Let a, b, c, d, e be the vertices of M such that $d_M(a) = 2$, $d_M(b) = d_M(e) = 4$ and $d_M(c) = d_M(d) = 3$. Let $V(D_i) = \{a_i, b_i, c_i, d_i, e_i\}$ with $d_{D_i}(b_i) = d_{D_i}(c_i) = d_{D_i}(e_i) = 4$, and $d_{D_i}(a_i) \ge 3$ and $d_{D_i}(d_i) \ge 3$ (see Figure 2). For a subgraph N of G, let $\theta_N = d_N(a) + d_N(b) + d_N(c) + d_N(d) + d_N(e)$. Next we will evaluate θ_{D_i} for each $D_i (i = 1, 2, \cdots, k-1)$.

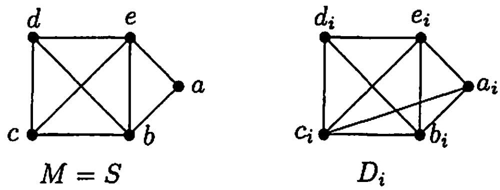

Figure 2

Lemma 3.1 *If $d_{D_i}(c) = 5$ or $d_{D_i}(d) = 5$, then, for any $y \in V(D_i)$, $d_M(y) \leq 4$. Therefore, $\theta_{D_i} \leq 20$.*

Proof. Without loss of generality, we assume that $d_{D_i}(c) = 5$. If $d_M(y) = 5$ for some $y \in V(D_i)$, then $M - c + y$ and $D_i - y + c$ contain K_5^-, a contradiction. Therefore, for any $y \in V(D_i)$, $d_M(y) \leq 4$. ∎

Lemma 3.2 *If $d_{D_i}(a) \leq 2$, then $\theta_{D_i} \leq 20$.*

Proof. By Lemma 3.1, we assume that $d_{D_i}(x) \leq 4$ for $x \in \{c, d\}$. Since $d_{D_i}(x) \leq 5$ for $x \in \{b, e\}$, $\theta_{D_i} = d_{D_i}(a)+d_{D_i}(b)+d_{D_i}(c)+d_{D_i}(d)+d_{D_i}(e) \leq 2+5+4+4+5 = 20$.∎

Lemma 3.3 *If $d_{D_i}(a) = 5$, then $\theta_{D_i} \leq 15$.*

Proof. Since $d_{D_i}(a) = 5$, for any $x \in V(D_i)$, $|N_G(x) \cap \{b, c, d, e\}| \leq 2$ (Otherwise, $D_i - x + a$ and $M - a + x$ contain K_5^-, a contradiction). Therefore, $\theta_{D_i} \leq 5 + 2 \times 5 = 15$. ∎

Lemma 3.4 *If $d_{D_i}(a) = 3$, then $\theta_{D_i} \leq 20$.*

Proof. Assume that $\theta_{D_i} \geq 21$. By Lemma 3.1, $d_{D_i}(c) \leq 4$ and $d_{D_i}(d) \leq 4$. Thus $d_{D_i}(b) = d_{D_i}(e) = 5$ and $d_{D_i}(c) = d_{D_i}(d) = 4$. As $|N_{D_i}(e) \cap N_{D_i}(d)| \geq 4$, we may assume $a_i, e_i \in N_{D_i}(e) \cap N_{D_i}(d)$. If $aa_i, ae_i \in E(G)$, then $G[\{a_i, e_i, a, e, d\}]$ and $G[\{b_i, c_i, d_i, b, c\}]$ contain K_5^-, a contradiction. So $|N_{D_i}(a) \cap \{b_i, c_i, d_i\}| \geq 2$. Thus $G[\{a, b, b_i, c_i, d_i\}]$ and $G[\{c, d, e, a_i, e_i\}]$ contain K_5^-, a contradiction. ∎

Lemma 3.5 *If $d_{D_i}(a) = 3$ and $\theta_{D_i} \geq 20$, then $\theta_{D_i} = 20$, $D_i = K_5^-$, and the subgraph induced by $V(D_i) \cup V(M)$ is isomorphic to H_1 shown in Figure 3.*

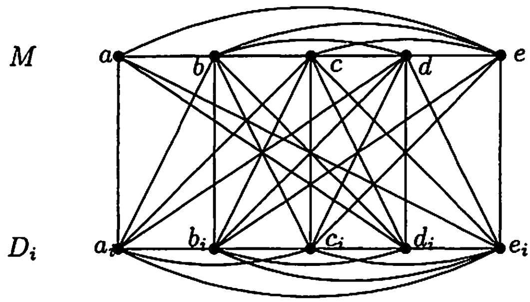

Figure 3. H_1

Proof. By Lemma 3.4, $\theta_{D_i} = 20$.

Claim 1. $D_i = K_5^-$.

Assume $D_i = K_5$. Without loss of generality, we may assume $N_{D_i}(a) = \{a_i, b_i, c_i\}$. Then $d_M(e_i) \leq 2$ (Otherwise, $D_i - e_i + a$ and $M - a + e_i$ contain K_5^-, a contradiction). Similarly, $d_M(d_i) \leq 2$. As $d_M(x) \leq 5$ for $x \in \{a_i, b_i, c_i\}$, we have $\theta_{D_i} = d_M(a_i)+d_M(b_i)+d_M(c_i)+d_M(d_i)+d_M(e_i) \leq 3 \cdot 5 + 2 \cdot 2 = 19$, a contradiction. So Claim 1 holds.

By Claim 1, $a_i d_i \notin E(G)$.

Claim 2. $N_{D_i}(a) \neq \{b_i, c_i, e_i\}$.

Assume $N_{D_i}(a) = \{b_i, c_i, e_i\}$. Then $d_M(a_i) \leq 2$ (Otherwise, $D_i - a_i + a$ and $M - a + a_i$ contain K_5^-, a contradiction). Similarly, $d_M(d_i) \leq 2$. As $d_M(x) \leq 5$ for $x \in \{b_i, c_i, e_i\}$, we have $\theta_{D_i} = d_M(a_i) + d_M(b_i) + d_M(c_i) + d_M(d_i) + d_M(e_i) \leq 3 \cdot 5 + 2 \cdot 2 = 19$, a contradiction, and hence Claim 2 follows.

Claim 3. $\{a_i, d_i\} \subseteq N_{D_i}(a)$.

Assume not. By Claim 2, we may assume that $N_{D_i}(a) = \{a_i, b_i, c_i\}$. If $d_M(a_i) = 5$, then $D_i - a_i + a$ is isomorphic to S, and $M - a + a_i$ is K_5. This contradicts the extremality condition (3). So $d_M(a_i) \leq 4$. Similarly, $d_M(e_i) \leq 3$. If $d_M(d_i) \geq 3$, then $D_i - d_i + a$ and $M - a + d_i$ contain K_5^-, a contradiction. So $d_M(d_i) \leq 2$. As $d_M(x) \leq 5$ for $x \in \{b_i, c_i\}$, we have $\theta_{D_i} = d_M(a_i) + d_M(b_i) + d_M(c_i) + d_M(d_i) + d_M(e_i) \leq 4+3+2+5+5 = 19$, a contradiction, and hence the result follows.

By Claim 3, we may assume that $N_{D_i}(a) = \{a_i, d_i, e_i\}$. If $d_M(a_i) = 5$, then

$D_i - a_i + a$ is S, and $M - a + a_i$ is K_5. It would contradict the extremality condition (3). So $d_M(a_i) \leq 4$. Similarly, $d_M(d_i) \leq 4$.

Claim 4. $d_{D_i}(b) \leq 4$ and $d_{D_i}(e) \leq 4$.

Assume that $d_{D_i}(b) = 5$. Then $G[\{a_i, d_i, e_i, a, b\}]$ is K_5^-. Next we consider the subgraph induced by $\{b_i, c_i, c, d, e\}$ to get a contradiction. As $d_M(a_i) \leq 4$ and $d_M(d_i) \leq 4$, $d_M(b_i)+d_M(c_i) \geq 20-(4+4+5) = 7$. As $ab_i, ac_i \notin E(G)$, $|N_M(b_i)\cap\{c,d,e\}|+|N_M(c_i)\cap\{c,d,e\}| \geq 5$. Thus $G[\{b_i, c_i, c, d, e\}]$ contains K_5^-, a contradiction. So $d_{D_i}(b) \leq 4$. Similarly, $d_{D_i}(e) \leq 4$. Therefore, Claim 4 holds.

As $d_{D_i}(a) = 3$, $d_{D_i}(b) + d_{D_i}(c) + d_{D_i}(d) + d_{D_i}(e) = 17$. By Claim 4, $d_{D_i}(c) + d_{D_i}(d) \geq 9$. Thus we have either $d_{D_i}(c) = 5$ or $d_{D_i}(d) = 5$. Without loss of generality, we assume that $d_{D_i}(c) = 5$. By Lemma 3.1, $20 \leq d_M(a_i)+d_M(b_i)+d_M(c_i)+d_M(d_i)+d_M(e_i) \leq 20$, and so as $ab_i, ac_i \notin E(G)$, we have both $d_M(x) = 4$ for any $x \in V(D_i)$ and $N_M(b_i) = N_M(c_i) = \{b, c, d, e\}$. Since $d_M(a_i) = d_M(d_i) = d_M(e_i) = 4$, $|N_M(a_i) \cap N_M(e_i)| \geq 3$ and $|N_M(d_i) \cap N_M(e_i)| \geq 3$.

Claim 5. (i) $b, e \notin N_M(a_i) \cap N_M(e_i)$. Therefore, $N_M(a_i) \cap N_M(e_i) = \{a, c, d\}$.
(ii) $b, e \notin N_M(d_i) \cap N_M(e_i)$. Therefore, $N_M(d_i) \cap N_M(e_i) = \{a, c, d\}$.

By contradiction, we assume that $b \in N_M(a_i)\cap N_M(e_i)$. Then $G[\{a, b, a_i, e_i, b_i\}]$ is K_5^-. As $d_M(d_i) = 4$, $|N_M(d_i)\cap\{c, d, e\}| \geq 2$. Thus $G[\{c_i, d_i, c, d, e\}]$ contains K_5^-, a contradiction. So, $b \notin N_M(a_i) \cap N_M(e_i)$. Similarly, $e \notin N_M(a_i) \cap N_M(e_i)$, and $b, e \notin N_M(d_i) \cap N_M(e_i)$. Therefore, Claim 5 holds.

By Claim 5, $\{a, c, d\} \subseteq N_M(e_i)$. As $d_M(e_i) = 4$, we have either $be_i \in E(G)$ or $ee_i \in E(G)$. By symmetry of M, we assume that $ee_i \in E(G)$. Then $be_i \notin E(G)$. If $a_ie \in E(G)$, then $G[\{a, e, a_i, b_i, e_i\}]$ and $G[\{b, c, d, c_i, d_i\}]$ contain K_5^-, a contradiction. So, $a_ie \notin E(G)$. Similarly, $d_ie \notin E(G)$. Therefore, $N_M(a_i) = N_M(d_i) = \{a, b, c, d\}$, and the subgraph induced by $V(D_i) \cup V(M)$ is isomorphic to H_1 shown in Figure 3. ∎

Lemma 3.6 *Suppose that $d_{D_i}(a) = 4$. Then the following statements hold.*
(i) If $D_i = K_5$, then $\theta_{D_i} \leq 14$.
(ii) If $D_i = K_5^-$ and $\theta_{D_i} \geq 17$, then $a_i, d_i \in N_G(a)$.
(iii) If $D_i = K_5^-$, then $\theta_{D_i} \leq 18$.

Proof. (i) Assume $D_i = K_5$. Then for each $x \in V(D_i)$, $|N_G(x) \cap$

$\{b, c, d, e\}| \leq 2$ (Otherwise, $D_i - x + a$ and $M - a + x$ contain K_5^-, a contradiction). So $\theta_{D_i} \leq 4 + 2 \times 5 = 14$.

(ii) By contradiction, we assume that $a_i a \notin E(G)$. Then $N_{D_i}(a) = \{b_i, c_i, d_i, e_i\}$. If $d_M(a_i) \geq 2$, then $M - a + a_i$ contains S, and $D_i - a_i + a$ is K_5. It contradicts the extremality condition (3). So $d_M(a_i) \leq 1$. If $d_M(e_i) = 5$, then $D_i - e_i + a$ is S and $M - a + e_i$ is K_5. It contradicts the extremality condition (3) again. So $d_M(e_i) \leq 4$. Similarly, $d_M(b_i) \leq 4$ and $d_M(c_i) \leq 4$. If $d_M(d_i) \geq 4$, then $M - a + d_i$ and $D_i - d_i + a$ contain K_5^-, a contradiction, and hence $d_M(d_i) \leq 3$. Therefore, $\theta_{D_i} \leq 4 \times 3 + 1 + 3 = 16$, a contradiction. So, $a_i a \in E(G)$. Similarly, $d_i a \in E(G)$.

(iii) By (ii), we may assume that $N_{D_i}(a) = \{a_i, b_i, c_i, d_i\}$. Then $D_i - e_i + a$, $D_i - a_i + a$, and $D_i - d_i + a$ are isomorphic to K_5^-. Thus $|N_G(e_i) \cap \{b, c, d, e\}| \leq 2$, $|N_G(a_i) \cap \{b, c, d, e\}| \leq 2$ and $|N_G(d_i) \cap \{b, c, d, e\}| \leq 2$. Therefore, $d_M(a_i) \leq 3, d_M(d_i) \leq 3$, and $d_M(e_i) \leq 2$. Clearly, $d_M(c_i) \leq 5$ and $d_M(b_i) \leq 5$. So, $\theta_{D_i} \leq 5 \times 2 + 3 + 3 + 2 = 18$. ∎

Lemma 3.7 *If $d_{D_i}(a) = 4$, then $\theta_{D_i} \leq 17$.*

Proof. By contradiction, we assume that $\theta_{D_i} \geq 18$. By Lemma 3.6, $D_i = K_5^-$, $a_i, d_i \in N_G(a)$, and $\theta_{D_i} = 18$. Without loss of generality, we assume that $N_{D_i}(a) = \{a_i, b_i, c_i, d_i\}$. Then $D_i - e_i + a$, $D_i - a_i + a$, and $D_i - d_i + a$ are isomorphic to K_5^-. Thus, $|N_G(e_i) \cap \{b, c, d, e\}| \leq 2$, $|N_G(a_i) \cap \{b, c, d, e\}| \leq 2$ and $|N_G(d_i) \cap \{b, c, d, e\}| \leq 2$. Therefore, $d_M(a_i) \leq 3, d_M(d_i) \leq 3$, and $d_M(e_i) \leq 2$. Clearly, $d_M(c_i) \leq 5$ and $d_M(b_i) \leq 5$. So $18 \leq \theta_{D_i} \leq 5 \times 2 + 3 + 3 + 2 = 18$. This implies that $\theta_{D_i} = 18$, $d_M(b_i) = d_M(c_i) = 5$, $d_M(a_i) = d_M(d_i) = 3$, and $d_M(e_i) = 2$. Hence, $M - a + d_i$, $M - a + a_i$, and $M - a + e_i$ are isomorphic to S.

Let $T = \{b, c, d, e, d_i, e_i\}$, and let F be the subgraph induced by $V(M) \cup V(D_i)$. Since $d_F(d_i) = 6$ and $d_F(e_i) = 6$, and since $\sum_{x \in \{b,c,d,e\}} d_F(x) = (18 - 4) + \sum_{x \in \{b,c,d,e\}} d_M(x) = 14 + 14 = 28$, we have $\sum_{x \in T} d_F(x) = 28 + 6 + 6 = 40$. To get a contradiction, we consider $\sum_{x \in T} d_G(x)$. Denote $\tau_j = \sum_{x \in T} d_{D_j}(x)$, where $j \in \{1, 2, \cdots, k-1\}$ and $j \neq i$. If $\tau_j \leq 22$ for all j, then

$$22k \leq \sum_{x \in T} d_G(x) = 40 + \sum_{j \neq i} \tau_j \leq 40 + 22(k-2) = 22k - 4,$$

a contradiction. So there is some j such that $\tau_j \geq 23$.

Note that $D_i - e_i + a$, $D_i - a_i + a$, and $D_i - d_i + a$ are isomorphic to K_5^-, and note that $M - a + d_i$, $M - a + a_i$, and $M - a + e_i$ are isomorphic to S. To get a contradiction, we need to reset M and D_i. For example, we will reset M and D_i to be either $M := M - a + d_i$ and $D_i := D_i - d_i + a$, or $M := M - a + e_i$ and $D_i := D_i - e_i + a$.

If $d_{D_j}(d_i) = 5$, by resetting $M := M - a + d_i$ and $D_i := D_i - d_i + a$, we have $\tau_j \leq 15 + d_{D_j}(e_i) \leq 15 + 5 = 20$ by Lemma 3.3, a contradiction. So $d_{D_j}(d_i) \leq 4$. Similarly, $d_{D_j}(e_i) \leq 4$. If $d_{D_j}(d_i) = 4$, by resetting $M := M - a + d_i$ and $D_i := D_i - d_i + a$ again, we have $\tau_j \leq 18 + d_{D_j}(e_i) \leq 18 + 4 = 22$ by Lemma 3.6, a contradiction. So $d_{D_j}(d_i) \leq 3$. Similarly, $d_{D_j}(e_i) \leq 3$. Furthermore, if either $d_{D_j}(d_i) < 3$ or $d_{D_j}(e_i) < 3$, without loss of generality, we assume that $d_{D_j}(d_i) \leq 3$ and $d_{D_j}(e_i) < 3$. We reset $M := M - a + d_i$ and $D_i := D_i - d_i + a$. By Lemmas 3.2 and 3.4, $\sum_{x \in V(M)} d_{D_j}(x) \leq 20$. Thus $\tau_j \leq 20 + d_{D_j}(e_i) \leq 22$, a contradiction. So $d_{D_j}(d_i) = 3$ and $d_{D_j}(e_i) = 3$.

Since $\tau_j \geq 23$, we have $\sum_{x \in \{b,c,d,e,d_i\}} d_{D_j}(x) \geq 20$ and $\sum_{x \in \{b,c,d,e,e_i\}} d_{D_j}(x) \geq 20$. By Lemma 3.5 and by resetting M and D_i to be either $M := M - a + d_i$ and $D_i := D_i - d_i + a$, or $M := M - a + e_i$ and $D_i := D_i - e_i + a$, the subgraphs induced by $\{b, c, d, e, d_i\} \cup V(D_j)$ and $\{b, c, d, e, e_i\} \cup V(D_j)$ are isomorphic to H_1 (see Figure 1), and $D_j = K_5^-$ (i.e., $a_j d_j \notin E(G)$). Therefore, $a_j d_i, a_j e_i, d_j d_i, d_j e_i \in E(G)$. As $d_{D_j}(e_i) = 3$, we may assume that $e_i e_j \in E(G)$. By the structure of H_1, $b, c, d, e \in N_G(b_j)$ and $b, c, d, e \in N_G(c_j)$. By using the structure of H_1 again, $d_i e_j \in E(G)$. Since $ae, b_i e, c_i e \in E(G)$, the subgraphs induced by $\{a, e, a_i, b_i, c_i\}$, $\{d_i, e_i, a_j, d_j, e_j\}$, and $\{b, c, d, b_j, c_j\}$ are isomorphic to either K_5^- or K_5, a contradiction. ∎

Now we are ready to prove Theorem 1.3 when M is isomorphic to S. By Lemmas 3.2, 3.3, 3.4, and 3.7, we can evaluate $d_G(a)$ and θ_G. For $0 \leq j \leq 5$, let q_j denote the number of indices i such that $d_{D_i}(a) = j$. Since there are $k - 1$ such D_i's and since $d_G(a) \geq \delta(G) \geq \frac{11}{3}k$, we have

$$\sum_{j=0}^{5} q_j = k - 1 \tag{1}$$

$$d_G(a) = q_1 + 2q_2 + 3q_3 + 4q_4 + 5q_5 + 2 \geq \frac{11}{3}k \tag{2}$$

Since $\theta_M = 16$, we can get the following.

$$\frac{11}{3}k \times 5 \leq \theta_G \leq 20q_0 + 20q_1 + 20q_2 + 20q_3 + 17q_4 + 15q_5 + 16 \tag{3}$$

From $5 \times (2) + 2 \times (3)$, we can get the following

$$40q_0 + 45q_1 + 50q_2 + 55q_3 + 54q_4 + 55q_5 + 42 \geq 55k \qquad (4)$$

From (1), we can get the following

$$\begin{aligned} & 40q_0 + 45q_1 + 50q_2 + 55q_3 + 54q_4 + 55q_5 + 42 \\ \leq \; & 55\sum_{j=0}^{5} q_j + 42 = 55(k-1) + 42 = 55k - 13. \end{aligned}$$

But, this contradicts to (4). This completes the proof of Theorem 1.3 when M is isomorphic to S.

4 The case where M is isomorphic to W_4

We need additional notations. Let $V(D_i) = \{a_i, b_i, c_i, d_i, e_i\}$ with $d_{D_i}(b_i) = d_{D_i}(c_i) = d_{D_i}(e_i) = 4$ and $d_{D_i}(a_i) \geq 3$ and $d_{D_i}(d_i) \geq 3$ (If $D_i = K_5^-$, then $d_{D_i}(a_i) = d_{D_i}(d_i) = 3$. If $D_i = K_5$, then $d_{D_i}(a_i) = d_{D_i}(d_i) = 4$). Suppose $M = W_4$. Let $V(M) = \{v_1, v_2, v_3, v_4, v_5\}$ with $d_M(v_1) = 4$ and $d_M(v_2) = d_M(v_3) = d_M(v_4) = d_M(v_5) = 3$ (see Figure 4). For a subgraph N of G, let $\theta_N = d_N(v_1) + d_N(v_2) + d_N(v_3) + d_N(v_4) + d_N(v_5)$. Next we will evaluate θ_{D_i} for each $D_i (i = 1, 2, \cdots, k-1)$.

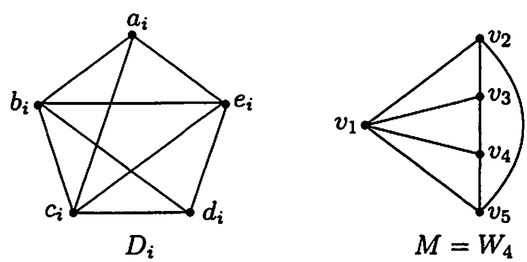

Figure 4

Lemma 4.1 *Let $x \in \{v_2, v_3, v_4, v_5\}$ with $d_{D_i}(x) = 5$. Then the following holds.*
(i) For any $y \in V(D_i)$, $d_{M-x}(y) \leq 3$.
(ii) Let x', x'' be the vertices adjacent to x in the 4-cycle $v_2v_3v_4v_5v_2$. If, for some $y \in V(D_i)$, $d_{M-x}(y) = 3$, then $yx', yx'' \in E(G)$. Therefore, $M - x + y$ is W_4.

Proof. (i) If $d_{M-x}(y) = 4$, then $D_i - y + x$ and $M - x + y$ contain K_5^-, a contradiction.

(ii) Without loss of generality, we may assume that $d_{D_i}(v_2) = 5$ and $d_{M-v_2}(y) = 3$ for some $y \in V(D_i)$. If $yv_3 \notin E(G)$, then $yv_1, yv_4, yv_5 \in E(G)$, and so $M - v_2 + y$ is isomorphic to S. Since $d_{D_i}(v_2) = 5$, $D_i - y + v_2$ contains K_5^-, which is contrary to the extremality condition (2). ∎

Lemma 4.2 *If* $d_{D_i}(x) = 5$, $x \in \{v_2, v_3, v_4, v_5\}$, *then* $\sum_{y \in V(M)-\{x\}} d_{D_i}(y) \leq 13$. *Therefore,* $\theta_{D_i} \leq 18$.

Proof. Without loss of generality, we assume that $d_{D_i}(v_2) = 5$, and $d_{D_i}(v_1) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) \geq 14$. Suppose that $D_i = K_5^-$. If $d_{M-v_2}(d_i) \geq 3$, then, by Lemma 4.1, $d_{M-v_2}(d_i) = 3$ and $M - v_2 + d_i$ is W_4. But $D_i - d_i + v_2$ is K_5. It contradicts to the extremality condition (3). So $d_{M-v_2}(d_i) \leq 2$. Similarly, $d_{M-v_2}(a_i) \leq 2$. By Lemma 4.1, $d_{M-v_2}(b_i) \leq 3$, $d_{M-v_2}(c_i) \leq 3$, and $d_{M-v_2}(e_i) \leq 3$. So $d_{D_i}(v_1) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) \leq 2 + 2 + 3 + 3 + 3 = 13$, a contradiction. Therefore, $D_i = K_5$.

Since $d_{M-v_2}(a_i) + d_{M-v_2}(b_i) + d_{M-v_2}(c_i) + d_{M-v_2}(d_i) + d_{M-v_2}(e_i) = d_{D_i}(v_1) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) \geq 14$, by Lemma 4.1(i), there are at least four vertices x in $V(D_i)$ such that $d_{M-v_2}(x) = 3$. Since $D_i = K_5$, we may assume that $d_{M-v_2}(a_i) = d_{M-v_2}(b_i) = d_{M-v_2}(c_i) = d_{M-v_2}(d_i) = 3$ and $d_{M-v_2}(e_i) \geq 2$. By Lemma 4.1, $zv_3, zv_5 \in E(G)$, where $z \in \{a_i, b_i, c_i, d_i\}$.

If $v_1a_i \in E(G)$, then $M - v_5 + a_i$ is S and $D_i - a_i + v_5$ contains K_5^-, which is contrary to the extremality condition (2). So $v_1a_i \notin E(G)$. Therefore, $v_4a_i \in E(G)$. Similarly, $v_1b_i, v_1c_i, v_1d_i \notin E(G)$ and $v_4b_i, v_4c_i, v_4d_i \in E(G)$.

Notice that $d_{M-v_2}(e_i) \geq 2$. If $v_1e_i \in E(G)$, then $G[\{v_1, v_2, e_i, a_i, b_i\}]$ is S and $G[\{v_3, v_4, v_5, c_i, d_i\}]$ is K_5^-, which is contrary to the extremality condition (2). So $v_1e_i \notin E(G)$. If $v_5e_i \in E(G)$, then $G[\{v_1, v_2, v_5, e_i, d_i\}]$ and $G[\{v_3, v_4, a_i, b_i, c_i\}]$ are isomorphic to S and K_5, respectively. It contradicts to the extremality condition (2) again. So $v_5e_i \notin E(G)$. It forces that $v_3e_i, v_4e_i \in E(G)$. Hence $G[\{v_1, v_3, v_4, e_i, d_i\}]$ is S and $G[\{v_2, v_5, a_i, b_i, c_i\}]$ is K_5, a contradiction. ∎

Lemma 4.3 *If* $d_{D_i}(v_1) = 5$, *then* $\theta_{D_i} \leq 19$.

Proof. By contradiction. Suppose that $\theta_{D_i} \geq 20$. Then $d_{D_i}(v_2) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) \geq 20 - 5 = 15$. By Lemma 4.2, for $x \in$

$\{v_2, v_3, v_3, v_5\}$, $d_{D_i}(x) \leq 4$. Thus there are three vertices in $\{v_2, v_3, v_4, v_5\}$, say v_2, v_3, v_4, such that $d_{D_i}(v_2) = d_{D_i}(v_3) = d_{D_i}(v_4) = 4$. In addition, $d_{D_i}(v_5) \geq 3$. Thus, $N_{D_i}(v_3) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5) \neq \emptyset$, $N_{D_i}(v_2) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5) \neq \emptyset$, and $N_{D_i}(v_2) \cap N_{D_i}(v_3) \cap N_{D_i}(v_5) \neq \emptyset$. Let $z \in N_{D_i}(v_3) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5)$.

Assume that $D_i = K_5$, or $D_i = K_5^-$ and $z \in \{a_i, d_i\}$. Then $D_i - z$ is K_4. Since $d_{D_i}(v_2) = 4$, $|N_G(v_2) \cap (V(D_i) - \{z\})| \geq 3$. Thus $D_i - z + v_2$ and $M - v_2 + z$ contain K_5^-, a contradiction. So $D_i = K_5^-$, and $N_{D_i}(v_3) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5) \subseteq \{b_i, c_i, e_i\}$. Similarly, $N_{D_i}(v_2) \cap N_{D_i}(v_3) \cap N_{D_i}(v_5) \subseteq \{b_i, c_i, e_i\}$, and $N_{D_i}(v_2) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5) \subseteq \{b_i, c_i, e_i\}$. Without loss of generality, we assume $e_i \in N_{D_i}(v_3) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5)$.

If $v_2a_i \notin E(G)$, then $N_{D_i}(v_2) = \{b_i, c_i, d_i, e_i\}$ since $d_{D_i}(v_2) = 4$. Thus $D_i - e_i + v_2$ is S and $M - v_2 + e_i$ is K_5^-, which is contrary to the extremality condition (2). So $a_iv_2 \in E(G)$. Similarly, $d_iv_2 \in E(G)$. Applying the same discussion on $N_{D_i}(v_2) \cap N_{D_i}(v_4) \cap N_{D_i}(v_5)$ and $N_{D_i}(v_2) \cap N_{D_i}(v_3) \cap N_{D_i}(v_5)$, we have $v_3a_i, v_3d_i, v_4a_i, v_4d_i \in E(G)$. Thus, $M - v_5 + a_i$ is K_5^-. Since $d_{D_i}(v_5) \geq 3$ and $e_iv_5 \in E(G)$, we have $|N_G(v_5) \cap \{b_i, c_i, d_i\}| \geq 1$. Hence $D_i - a_i + v_5$ is isomorphic to either S, K_5^-, or K_5, a contradiction. ∎

Lemma 4.4 *If $d_{D_i}(v_1) = 5$, and $\theta_{D_i} \geq 19$, then $\theta_{D_i} = 19$, $D_i = K_5^-$ and the subgraph induced by $V(D_i) \cup V(M)$ is isomorphic to H_2 shown in Figure 5.*

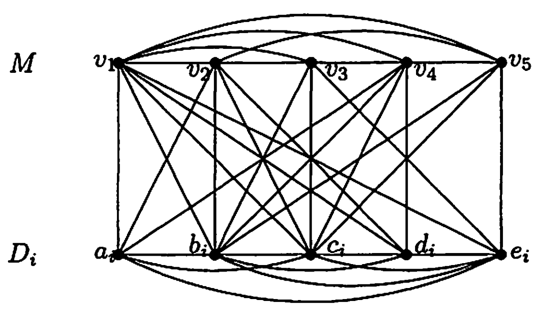

Figure 5. H_2

Proof. By Lemma 4.3, $\theta_{D_i} = 19$, and so $d_{D_i}(v_2) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) = 14$. By Lemma 4.2, for any $x \in \{v_2, v_3, v_4, v_5\}$, $d_{D_i}(x) \leq 4$. Thus, there is a vertex in $\{v_2, v_3, v_4, v_5\}$, say v_2, such that $d_{D_i}(v_2) = 4$. Thus, $d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) = 14 - 4 = 10$.

Claim 1. $D_i = K_5^-$.

Suppose that $D_i = K_5$ and $N_{D_i}(v_2) = \{a_i, b_i, c_i, d_i\}$. If there is a vertex in $\{a_i, b_i, c_i, d_i\}$, say a_i, such that $d_M(a_i) = 5$, then $D_i - a_i + v_2$ and $M - v_2 + a_i$ would be isomorphic to K_5^-, a contradiction. So, for $x \in \{a_i, b_i, c_i, d_i\}$, $d_M(x) \leq 4$. It forces that $d_M(e_i) \geq 3$, and hence $|N_G(e_i) \cap \{v_3, v_4, v_5\}| \geq 2$.

If $v_4e_i \in E(G)$, then either $v_3e_i \in E(G)$ or $v_5e_i \in E(G)$. Thus $D_i - e_i + v_2$ is K_5 and $M - v_2 + e_i$ contains S, which is contrary to the extremality condition (2). So $v_4e_i \notin E(G)$. Therefore, $N_M(e_i) = \{v_1, v_3, v_5\}$, and $d_M(x) = 4$ for $x \in \{a_i, b_i, c_i, d_i\}$.

Assume $v_4a_i \in E(G)$. Since $a_iv_1, a_iv_2 \in E(G)$, we have either $v_3a_i \in E(G)$ or $v_5a_i \in E(G)$. Then $D_i - a_i + v_2$ is K_5^- and $M - v_2 + a_i$ contains S, a contradiction. So $v_4a_i \notin E(G)$. Similarly, $v_4b_i, v_4c_i, v_4d_i \notin E(G)$. Therefore, $N_M(x) = \{v_1, v_2, v_3, v_5\}$ for $x \in \{a_i, b_i, c_i, d_i\}$. It implies that $G[\{v_1, v_4, v_5, a_i, b_i\}]$ is S and $G[\{v_2, v_3, c_i, d_i, e_i\}]$ is K_5^-, a contradiction. So Claim 1 holds.

By Claim 1, $a_id_i \notin E(G)$.

Claim 2. $G[N_{D_i}(v_2)] = K_4^-$.

Assume not. We may assume that $N_{D_i}(v_2) = \{a_i, b_i, c_i, e_i\}$. As $D_i - d_i + v_2$ is K_5, $d_{M-v_2}(d_i) \leq 2$ (Otherwise, $M - v_2 + d_i$ contains either S or W_4, which is contrary to the extremality condition (2) or (3)). Since $d_iv_1 \in E(G)$, $|N_G(d_i) \cap \{v_3, v_4, v_5\}| \leq 1$. Thus, $|N_G(a_i) \cap \{v_3, v_4, v_5\}| + |N_G(b_i) \cap \{v_3, v_4, v_5\}| + |N_G(c_i) \cap \{v_3, v_4, v_5\}| + |N_G(e_i) \cap \{v_3, v_4, v_5\}| \geq 19 - 5 - 4 - 1 = 9$, and so there is a vertex $x \in \{a_i, b_i, c_i, e_i\}$ such that $xv_3, xv_4, xv_5 \in E(G)$.

If $x = a_i$, then $M - v_2 + a_i$ and $D_i - a_i + v_2$ are K_5^-, a contradiction. Thus $x \neq a_i$, and so $x \in \{b_i, c_i, e_i\}$. Without loss of generality, we assume that $x = b_i$. Then $M - v_2 + b_i$ is K_5^-, and $D_i - b_i + v_2$ is S, a contradiction, and hence Claim 2 follows.

By Claim 2, we may assume $N_{D_i}(v_2) = \{a_i, b_i, c_i, d_i\}$.

Claim 3. $N_{D_i}(v_3) \cap N_{D_i}(v_4) \subseteq \{b_i, c_i\}$ and $N_{D_i}(v_4) \cap N_{D_i}(v_5) \subseteq \{b_i, c_i\}$. Therefore, $d_{D_i}(v_3) + d_{D_i}(v_4) \leq 7$ and $d_{D_i}(v_4) + d_{D_i}(v_5) \leq 7$.

Suppose that there is a vertex $x \in \{a_i, d_i, e_i\}$ such that $xv_3, xv_4 \in E(G)$. Then $D_i - x + v_2$ is K_5^-, and $M - v_2 + x$ contains S, which is contrary to the extremality condition (2). So $N_{D_i}(v_3) \cap N_{D_i}(v_4) \subseteq \{b_i, c_i\}$. Similarly, $N_{D_i}(v_4) \cap N_{D_i}(v_5) \subseteq \{b_i, c_i\}$. So Claim 3 holds.

As $d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) = 10$, by Claim 3, we have $d_{D_i}(v_5) \geq 3$ and $d_{D_i}(v_3) \geq 3$.

Claim 4. $d_{D_i}(v_4) = 4$. Therefore, $d_{D_i}(v_3) = d_{D_i}(v_5) = 3$.

By contradiction, we assume that $d_{D_i}(v_4) \leq 3$. If $d_{D_i}(v_4) \leq 2$, then $d_{D_i}(v_3) = d_{D_i}(v_5) = 4$ and $d_{D_i}(v_4) = 2$. We re-label $(v_1, v_5, v_2, v_3, v_4)$ to be $(y_1, y_2, y_3, y_4, y_5)$. Then $d_{D_i}(y_2) = 4$, and $d_{D_i}(y_3) + d_{D_i}(y_4) = 8$, which is contrary to Claim 3. So $d_{D_i}(v_4) = 3$. Therefore, $d_{D_i}(v_3) + d_{D_i}(v_5) = 7$. Without loss of generality, we assume that $d_{D_i}(v_3) = 4$ and $d_{D_i}(v_5) = 3$. Applying the same argument in Claim 2 on $N_{D_i}(v_3)$, we have $G[N_{D_i}(v_3)] = K_4^-$. Thus $a_i, d_i \in N_G(v_3)$.

Since $d_{D_i}(v_3) = 4$ and $d_{D_i}(v_4) = 3$, $|N_{D_i}(v_3) \cap N_{D_i}(v_4)| \geq 2$. By Claim 3, $N_{D_i}(v_3) \cap N_{D_i}(v_4) = \{b_i, c_i\}$. Thus $N_{D_i}(v_3) = \{a_i, b_i, c_i, d_i\}$ and $N_{D_i}(v_4) = \{b_i, c_i, e_i\}$. Therefore, $G[\{a_i, c_i, d_i, v_2, v_3\}]$ is K_5^-, and $G[\{b_i, e_i, v_1, v_4, v_5\}]$ contains S, a contradiction, and hence Claim 4 follows.

By Claims 3 and 4, $N_{D_i}(v_3) \cap N_{D_i}(v_4) = N_{D_i}(v_4) \cap N_{D_i}(v_5) = \{b_i, c_i\}$. Applying the same argument in Claim 2 on $N_{D_i}(v_4)$, we have $G[N_{D_i}(v_4)] = K_4^-$. Thus $N_{D_i}(v_4) = \{a_i, b_i, c_i, d_i\}$. By Claim 3, $N_{D_i}(v_3) = N_{D_i}(v_5) = \{b_i, c_i, e_i\}$. Therefore, the subgraph induced by $V(M) \cup V(D_i)$ is isomorphic to H_2. ∎

Lemma 4.5 *If $d_{D_i}(x) \leq 4$ for $x \in V(M)$, then $\theta_{D_i} \leq 19$. Moreover, if $\theta_{D_i} = 19$, then $D_i = K_5^-$, $d_{D_i}(v_1) = 3$, $d_{D_i}(v_j) = 4$ for $j = 2, 3, 4, 5$, and the subgraph induced by $V(D_i) \cup V(M)$ is isomorphic to H_3 shown in Figure 6.*

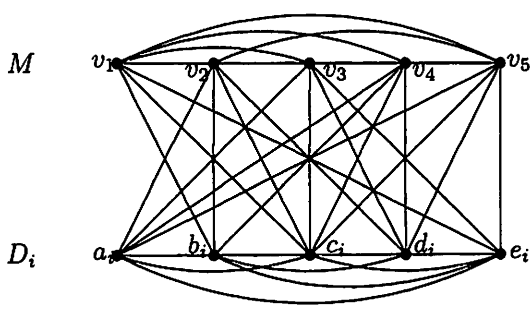

Figure 6. H_3

Proof. Suppose that $\theta_{D_i} \geq 19$. Since $d_{D_i}(x) \leq 4$ for any $x \in V(M)$, there are at least three vertices in $\{v_2, v_3, v_4, v_5\}$, say v_2, v_3, v_4, such that $d_{D_i}(v_2) = d_{D_i}(v_3) = d_{D_i}(v_4) = 4$. Thus $d_{D_i}(v_1) + d_{D_i}(v_5) \geq 7$.

Claim 1. $D_i = K_5^-$.

Suppose that $D_i = K_5$. Consider v_2. Since $d_{D_i}(v_2) = 4$, we may assume that $N_{D_i}(v_2) = \{a_i, b_i, c_i, d_i\}$. Clearly, for any $x \in V(D_i)$, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$. Moreover, if $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$, then $xv_3, xv_5 \in E(G)$ (Otherwise, $D_i - x + v_2$ contains K_5^-, and $M - v_2 + x$ is S, a contradiction). Therefore, $\theta_{D_i} = 19$, $d_M(a_i) = d_M(b_i) = d_M(c_i) = d_M(d_i) = 4$, $d_M(e_i) = 3$, and $xv_3, xv_5 \in E(G)$ for $x \in V(D_i)$. It implies that $d_{D_i}(v_3) = d_{D_i}(v_5) = 5$, a contradiction. So Claim 1 holds.

By Claim 1, $a_i d_i \notin E(G)$.

Claim 2. Let $x \in \{v_2, v_3, v_4, v_5\}$ with $d_{D_i}(x) = 4$. Then $G[N_{D_i}(x)] = K_4^-$.

By contradiction, we assume that $G[N_{D_i}(v_2)] = K_4$ and $N_{D_i}(v_2) = \{a_i, b_i, c_i, e_i\}$. If $d_{M-v_2}(d_i) \geq 3$, then $D_i - d_i + v_2$ is K_5 and $M - v_2 + d_i$ contains either S or W_4 as a subgraph. This contradicts to the extremality condition (2) or (3). So $d_{M-v_2}(d_i) \leq 2$. For $x \in \{a_i, b_i, c_i, e_i\}$, if $d_{M-v_2}(x) = 4$, then $D_i - x + v_2$ is isomorphic to either S or K_5^-, and $M - v_2 + x$ is K_5^-, a contradiction. So $d_{M-v_2}(x) \leq 3$. Thus, $\theta_{D_i} \leq 4 + 2 + 3 \cdot 4 = 18$, a contradiction, and hence Claim 2 holds.

By Claim 2, $\{a_i, d_i\} \subseteq N_{D_i}(v_2) \cap N_{D_i}(v_3) \cap N_{D_i}(v_4)$. Since $d_{D_i}(v_2) = 4$, we may assume that $N_{D_i}(v_2) = \{a_i, b_i, c_i, d_i\}$. Thus $D_i - a_i + v_2$, $D_i - d_i + v_2$, and $D_i - e_i + v_2$ are isomorphic to K_5^-. So, for $x \in \{a_i, e_i, d_i\}$, $d_{M-v_2}(x) \leq 3$. Furthermore, if $d_{M-v_2}(x) = 3$, then $xv_3, xv_5 \in E(G)$. As $a_i v_3, a_i v_4 \in E(G)$, we have $a_i v_1 \notin E(G)$. Similarly, $d_i v_1 \notin E(G)$. Since $d_{D_i}(v_1) \geq 3$, $N_{D_i}(v_1) = \{b_i, c_i, e_i\}$. Therefore, $d_{D_i}(v_5) = 4$, $\{a_i, d_i\} \subseteq N_{D_i}(v_5)$, and $\theta_{D_i} = 19$.

Claim 3. $e_i v_4 \notin E(G)$. Therefore, $N_{D_i}(v_4) = \{a_i, b_i, c_i, d_i\}$.

Assume that $e_i v_4 \in E(G)$. As $e_i v_1 \in E(G)$, we have $e_i v_3, e_i v_5 \notin E(G)$ (Otherwise, $D_i - e_i + v_2$ is K_5^-, and $M - v_2 + e_i$ contains S, a contradiction). Thus, $N_{D_i}(v_3) = N_{D_i}(v_5) = \{a_i, b_i, c_i, d_i\}$. As $d_{D_i}(v_4) = 4$, we have either $c_i v_4 \in E(G)$ or $b_i v_4 \in E(G)$. Without loss of generality, we assume $c_i v_4 \in E(G)$. Then $G[\{c_i, d_i, e_i, v_4, v_5\}]$ and $G[\{a_i, b_i, v_1, v_2, v_3\}]$ are isomorphic to K_5^-, a contradiction. So Claim 3 holds.

Claim 4. $e_i v_5, e_i v_3 \in E(G)$.

Assume that $e_iv_5 \notin E(G)$. Then $N_{D_i}(v_5) = \{a_i, b_i, c_i, d_i\}$. Since $d_{D_i}(v_3) = 4$ and $a_iv_3, d_iv_3 \in E(G)$, $|N_G(v_3) \cap \{b_i, c_i, e_i\}| = 2$. Thus $|N_G(v_3) \cap \{b_i, c_i\}| \geq 1$. Without loss of generality, we assume $b_iv_3 \in E(G)$. Since $N_{D_i}(v_2) = N_{D_i}(v_4) = \{a_i, b_i, c_i, d_i\}$, we have $G[\{a_i, b_i, v_1, v_2, v_3\}$ is K_5^- and $G[\{c_i, d_i, e_i, v_4, v_5\}]$ is S, a contradiction. So $e_iv_5 \in E(G)$. Similarly, $e_iv_3 \in E(G)$. Therefore, Claim 4 holds.

As $d_{D_i}(v_5) = 4$ and $e_iv_5 \in E(G)$, we have either $c_iv_5 \in E(G)$ or $b_iv_5 \in E(G)$. Since b_i and c_i are symmetric, we may assume that $c_iv_5 \in E(G)$. Then $N_{D_i}(v_5) = \{a_i, c_i, d_i, e_i\}$. If $b_iv_3 \in E(G)$, then $G[\{c_i, e_i, d_i, v_4, v_5\}]$ and $G[\{v_1, v_2, v_3, a_i, b_i\}]$ are K_5^-, a contradiction, and hence $b_iv_3 \notin E(G)$ and $N_{D_i}(v_3) = \{a_i, c_i, d_i, e_i\}$. Therefore, the subgraph induced by $V(M) \cup V(D_i)$ is isomorphic to H_3. ∎

Lemma 4.6 *If $d_{D_i}(v_1) \leq 4$, then $\theta_{D_i} \leq 18$.*

Proof. By contradiction, we assume that $\theta_{D_i} \geq 19$. By Lemma 4.2, for $x \in \{v_2, v_3, v_4, v_5\}$, $d_{D_i}(x) \leq 4$. By Lemma 4.5, $\theta_{D_i} = 19$, $D_i = K_5^-$, and the subgraph induced by $V(M) \cup V(D_i)$ is isomorphic to H_3. Let $T = \{v_1, v_2, \cdots, v_5, a_i, b_i, d_i, e_i\}$ (see Figure 6), and denote F the graph induced by $V(M) \cup V(D_i)$. Then $\sum_{x \in T} d_F(x) = 9 \cdot 7 = 63$. To get a contradiction, we consider $\tau_j = \sum_{x \in T} d_{D_j}(x)$, where $j \in \{1, 2, \cdots, k-1\}$ and $j \neq i$. If $\tau_j \leq 33$ for all j, then

$$33k = 9 \cdot \frac{11k}{3} \leq \sum_{x \in T} d_G(x) \leq 63 + 33(k-2) = 33k - 3,$$

a contradiction. So there is some j such that $\tau_j \geq 34$. Note that $G[\{a_i, b_i, e_i, v_1, v_3\}]$, $G[\{d_i, v_1, v_2, v_4, v_5\}]$, $G[\{a_i, e_i, v_2, v_3, v_5\}]$, and $G[\{a_i, v_2, v_3, v_4, v_5\}]$ are isomorphic to W_4, and $G[\{c_i, d_i, v_2, v_4, v_5\}]$, $G[\{a_i, b_i, c_i, e_i, v_3\}]$, $G[\{b_i, c_i, d_i, v_1, v_4\}]$, and $G[\{b_i, c_i, d_i, e_i, v_1\}]$ are isomorphic to K_5^-. To get a contradiction, we need to reset D_i and M. For example, we will reset D_i and M to be either $D_i := G[\{c_i, d_i, v_2, v_4, v_5\}]$ and $M := G[\{a_i, b_i, e_i, v_1, v_3\}]$, or $D_i := G[\{a_i, b_i, c_i, e_i, v_3\}]$ and $M := G[\{d_i, v_1, v_2, v_4, v_5\}]$, etc.

Claim 1. For each $x \in T$, $d_{D_j}(x) \leq 4$.

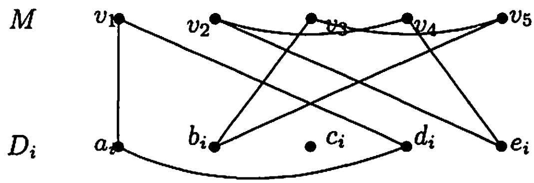

Figure 7. The complement of H_3

Note that the complement of H_3 is $K_3 \cup K_3 \cup K_3 \cup K_1$, and T is the collection of the vertices of these three triangles (see Figure 7). By symmetry of the complement of H_3, we just need to prove $d_{D_j}(v_1) \leq 4$.

By contradiction, we assume $d_{D_j}(v_1) = 5$. By resetting $M := G[\{e_i, v_1, a_i, b_i, v_3\}]$ and $D_i := G[\{v_2, v_4, v_5, c_i, d_i\}]$, we have $d_{D_j}(e_i) + d_{D_j}(v_1) + d_{D_j}(a_i) + d_{D_j}(b_i) + d_{D_j}(v_3) \leq 18$ by Lemma 4.2. Thus $d_{D_j}(e_i) + d_{D_j}(a_i) + d_{D_j}(b_i) + d_{D_j}(v_3) \leq 13$. Similarly, by resetting $M := G[\{v_5, v_1, d_i, v_2, v_4\}]$ and $D_i := G[\{v_3, a_i, b_i, c_i, e_i\}]$, we have $d_{D_j}(v_5) + d_{D_j}(d_i) + d_{D_j}(v_2) + d_{D_j}(v_4) \leq 13$. Thus, $\tau_j \leq 13 + 13 + 5 = 31$, a contradiction, and hence Claim 1 follows.

Claim 2. For each $x \in T$, $d_{D_j}(x) \leq 3$.

By symmetry of the complement of H_3, we just need to prove $d_{D_j}(v_1) \leq 3$. By contradiction, we assume $d_{D_j}(v_1) = 4$. Let's consider $M := G[\{e_i, v_1, a_i, b_i, v_3\}]$ and $D_i := G[\{v_2, v_4, v_5, c_i, d_i\}]$. By Claim 1 and Lemma 4.5, $d_{D_j}(e_i) + d_{D_j}(v_1) + d_{D_j}(a_i) + d_{D_j}(b_i) + d_{D_j}(v_3) \leq 19$. Similarly, by redefining $M := G[\{v_5, v_1, d_i, v_2, v_4\}]$ and $D_i := G[\{v_3, a_i, b_i, c_i, e_i\}]$, we have $d_{D_j}(v_5) + d_{D_j}(v_1) + d_{D_j}(d_i) + d_{D_j}(v_2) + d_{D_j}(v_4) \leq 19$. Thus, $38 = 34 + 4 \leq \tau_j + d_{D_j}(v_1) \leq 38$, and hence $d_{D_j}(e_i) + d_{D_j}(v_1) + d_{D_j}(a_i) + d_{D_j}(b_i) + d_{D_j}(v_3) = 19$ and $d_{D_j}(v_5) + d_{D_j}(v_1) + d_{D_j}(d_i) + d_{D_j}(v_2) + d_{D_j}(v_4) = 19$. By Lemma 4.5, $d_{D_j}(e_i) = 3$ and $d_{D_j}(v_1) = d_{D_j}(a_i) = d_{D_j}(b_i) = d_{D_j}(v_3) = 4$, and $d_{D_j}(v_5) = 3$ and $d_{D_j}(d_i) = d_{D_j}(v_2) = d_{D_j}(v_4) = 4$. By resetting $M := G[\{b_i, v_1, a_i, v_2, v_4\}]$ and $D_i := G[\{v_3, v_5, c_i, d_i, e_i\}]$, we have $20 = d_{D_j}(b_i) + d_{D_j}(v_1) + d_{D_j}(a_i) + d_{D_j}(v_2) + d_{D_j}(v_4) \leq 19$ by Lemma 4.5, a contradiction, and hence Claim 2 follows.

By Claim 2, $\tau_j \leq 3 \cdot 9 = 27$, a contradiction. ∎

Next we will prove $\theta_{D_i} \leq 18$ if $d_{D_i}(v_1) = 5$. Before doing that, we will prove Lemmas 4.7–4.11.

Lemma 4.7 $d_{D_i}(v_2) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) \leq 16$.

Proof. By contradiction, we assume that $d_{D_i}(v_2) + d_{D_i}(v_3) + d_{D_i}(v_4) + d_{D_i}(v_5) \geq 17$. Then there are two vertices t, s in $V(D_i)$ such that $\{v_2, v_3, v_4, v_5\} \subseteq N_G(t)$ and $\{v_2, v_3, v_4, v_5\} \subseteq N_G(s)$. In addition, there is a vertex in $\{v_2, v_3, v_4, v_5\}$, say v_2, such that $V(D_i) \subseteq N_G(v_2)$.

Suppose that $ts \in E(G)$ and $D_i - \{t, s\}$ is a triangle, without loss of generality, we assume $\{t, s\} = \{a_i, b_i\}$. Then $|N_G(v_3) \cap \{c_i, d_i, e_i\}| + |N_G(v_4) \cap \{c_i, d_i, e_i\}| + |N_G(v_5) \cap \{c_i, d_i, e_i\}| \geq 17 - 5 - 6 = 6$. Since $|N_G(v_4) \cap \{c_i, d_i, e_i\}| \leq 3$, $|N_G(v_3) \cap \{c_i, d_i, e_i\}| + |N_G(v_5) \cap \{c_i, d_i, e_i\}| \geq 3$. So we may assume that $|N_G(v_3) \cap \{c_i, d_i, e_i\}| \geq 2$. Hence $G[\{a_i, b_i, v_4, v_5, v_1\}]$ contains S and $G[\{c_i, d_i, e_i, v_2, v_3\}]$ contains K_5^-, a contradiction. So, we have either $ts \notin E(G)$ or $D_i - \{s, t\}$ is not a triangle. Therefore, $D_i = K_5^-$.

If $ts \notin E(G)$, then $\{t, s\} = \{a_i, d_i\}$. Then $D_i - d_i + v_2$ is K_5 and $M - v_2 + d_i$ contains W_4, a contradiction, and hence $D_i - \{s, t\}$ is not a triangle. Without loss of generality, we assume $\{t, s\} = \{b_i, c_i\}$.

If $|N_G(a_i) \cap \{v_3, v_4, v_5\}| = 3$, then $D_i - a_i + v_2$ is K_5 and $M - v_2 + a_i$ contains W_4, a contradiction, and hence $|N_G(a_i) \cap \{v_3, v_4, v_5\}| \leq 2$, and so $|N_G(a_i) \cap \{v_2, v_3, v_4, v_5\}| \leq 3$. Similarly, $|N_G(d_i) \cap \{v_3, v_4, v_5\}| \leq 2$ and $|N_G(d_i) \cap \{v_2, v_3, v_4, v_5\}| \leq 3$. Thus $|N_G(e_i) \cap \{v_2, v_3, v_4, v_5\}| \geq 17 - 8 - 6 = 3$, which implies that either $e_i v_3 \in E(G)$ or $e_i v_5 \in E(G)$. Without loss of generality, we assume $e_i v_3 \in E(G)$.

Suppose that $v_3 a_i \in E(G)$. Then $N_G(d_i) \cap \{v_4, v_5\} = \emptyset$ (Otherwise, $G[\{a_i, e_i, v_2, v_3, v_1\}]$ contains S and $G[\{b_i, c_i, d_i, v_4, v_5\}]$ contains K_5^-, a contradiction). Thus $9 \leq |N_G(a_i) \cap \{v_2, v_3, v_4, v_5\}| + |N_G(d_i) \cap \{v_2, v_3, v_4, v_5\}| + |N_G(e_i) \cap \{v_2, v_3, v_4, v_5\}| \leq 3+2+4 = 9$. and hence $|N_G(a_i) \cap \{v_2, v_3, v_4, v_5\}| = 3$, $|N_G(d_i) \cap \{v_2, v_3, v_4, v_5\}| = 2$, and $|N_G(e_i) \cap \{v_2, v_3, v_4, v_5\}| = 4$. Therefore, $|N_G(a_i) \cap \{v_4, v_5\}| \geq 1$ and $d_i v_2, d_i v_3 \in E(G)$. So $G[\{d_i, e_i, v_2, v_3, v_1\}]$ contains S and $G[\{a_i, b_i, c_i, v_4, v_5\}]$ contains K_5^-, a contradiction. So $v_3 a_i \notin E(G)$. Similarly, $d_i v_3 \notin E(G)$, and so $d_{D_i}(v_3) \leq 3$. Thus $d_{D_i}(v_4) + d_{D_i}(v_5) \geq 17-5-3 = 9$. Furthermore, $e_i v_5 \notin E(G)$ (otherwise, by using the same argument above, we have $a_i, d_i \notin N_G(v_5)$, and so $d_{D_i}(v_5) \leq 3$. Thus, $d_{D_i}(v_4) + d_{D_i}(v_5) \leq 5+3 = 8$, a contradiction). Therefore, $d_{D_i}(v_4) = 5$ and $d_{D_i}(v_5) = 4$. Hence $G[\{a_i, c_i, d_i, v_2, v_5\}]$ is K_5^-, and $G[\{b_i, e_i, v_3, v_4, v_1\}]$ contains S, a contradiction. ∎

Lemma 4.8 *Suppose that $d_{D_i}(v_2) = d_{D_i}(v_4) = 5$. Then $\theta_{D_i} \leq 17$.*

Proof. Assume $\theta_{D_i} \geq 18$. By Lemma 4.2, $\theta_{D_i} = 18$. Then $\sum_{x \in V(D_i)} |N_G(x)$

$\cap\{v_1, v_3, v_4, v_5\}| = 13$. We consider two cases.

Case 1. $D_i = K_5$.

By Lemma 4.1(i), for any $x \in V(D_i)$, $d_{M-v_2}(x) \leq 3$. Thus there are at three vertices in $V(D_i)$, say a_i, b_i, c_i, such that $d_{M-v_2}(a_i) = 3$, $d_{M-v_2}(b_i) = 3$, and $d_{M-v_2}(c_i) = 3$. By Lemma 4.1(ii), $a_i, b_i, c_i \in N_G(v_3)$ and $a_i, b_i, c_i \in N_G(v_5)$. By Lemma 4.7, $d_{D_i}(v_3) = d_{D_i}(v_5) = 3$, and so $d_{D_i}(v_1) = 2$ and $d_{M-v_2}(e_i) = d_{M-v_2}(d_i) = 2$. Thus, $N_{D_i}(v_3) = N_{D_i}(v_5) = \{a_i, b_i, c_i\}$ and $N_{D_i}(v_1) = \{d_i, e_i\}$. Hence, $G[\{v_3, v_5, a_i, b_i, c_i\}$ and $G[\{d_i, e_i, v_1, v_2, v_4\}]$ are K_5^-, a contradiction.

Case 2. $D_i = K_5^-$.

Then $a_i d_i \notin E(G)$. If $d_{M-v_2}(d_i) \geq 3$, then, by Lemma 4.1, $d_{M-v_2}(d_i) = 3$ and $M - v_2 + d_i$ is W_4. But $D_i - d_i + v_2$ is K_5, which is contrary to the extremality condition (3). So $d_{M_2}(d_i) \leq 2$. Similarly, $d_{M-v_2}(a_i) \leq 2$. By Lemma 4.1, $d_{M-v_2}(b_i) \leq 3$, $d_{M-v_2}(c_i) \leq 3$ and $d_{M-v_2}(e_i) \leq 3$. So $13 = \sum_{x \in V(D_i)} |N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 2 \cdot 2 + 3 \cdot 3 = 13$. Therefore, $d_{M-v_2}(b_i) = d_{M-v_2}(c_i) = d_{M-v_2}(e_i) = 3$ and $d_{M-v_2}(a_i) = d_{M-v_2}(d_i) = 2$. By Lemma 4.1(ii), $b_i, c_i, e_i \in N_G(v_3)$ and $b_i, c_i, e_i \in N_G(v_5)$. By Lemma 4.7, $d_{D_i}(v_3) = d_{D_i}(v_5) = 3$ and $d_{D_i}(v_1) = 2$. Thus $N_{D_i}(v_3) = N_{D_i}(v_5) = \{b_i, c_i, e_i\}$, and $N_{D_i}(v_1) = \{a_i, d_i\}$. Therefore, the complement of the graph induced by $V(M) \cup V(D_i)$ is the following graph.

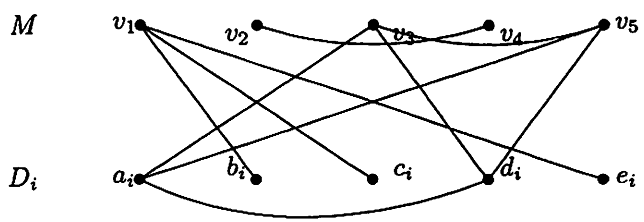

Figure 8

Let $T = \{v_1, v_2, v_4, v_3, v_5, a_i, d_i, e_i\}$, and denote F the graph induced by $V(M) \cup V(D_i)$. Then $\sum_{x \in T} d_F(x) + d_F(v_1) + d_F(e_i) = 68$. To get a contradiction, we consider $\tau_j = \sum_{x \in T} d_{D_j}(x) + d_{D_j}(v_1) + d_{D_j}(e_i)$ for $j \in \{1, 2, \cdots, k-1\}$ and $j \neq i$. If $\tau_j \leq 36$ for all j, then

$$10 \cdot \frac{11k}{3} \leq \tau_j \leq 36(k-2) + 68 = 36k - 4,$$

a contradiction. So there is some j such that $\tau_j \geq 37$. Since v_2 and v_4 are symmetric, we may assume $d_{D_j}(v_2) \geq d_{D_j}(v_4)$.

Claim 1. $d_{D_j}(v_2) = 5$.

Assume $d_{D_j}(v_2) \leq 4$. By resetting $M := G[\{v_2, v_1, e_i, v_3, a_i\}]$ and $D_i := G[\{b_i, c_i, v_4, v_5, d_i\}]$, we have $d_{D_j}(v_2)+d_{D_j}(v_1)+d_{D_j}(e_i)+d_{D_j}(v_3)+d_{D_j}(a_i) \leq 18$ by Lemma 4.6. Thus $d_{D_j}(v_1) + d_{D_j}(e_i) + d_{D_j}(v_3) + d_{D_j}(a_i) \leq 18 - d_{D_j}(v_2)$. Similarly, by resetting $M := G[\{v_2, v_1, e_i, v_5, d_i\}]$ and $D_i := G[\{b_i, c_i, v_4, v_3, a_i\}]$, we have $d_{D_j}(v_1) + d_{D_j}(e_i) + d_{D_j}(v_5) + d_{D_j}(d_i) \leq 18 - d_{D_j}(v_2)$. Therefore, $\tau_j \leq (18 - d_{D_j}(v_2)) + (18 - d_{D_j}(v_2)) + d_{D_j}(v_2) + d_{D_j}(v_4) = 36 - d_{D_j}(v_2) + d_{D_j}(v_4) \leq 36$, a contradiction.

Claim 2. $d_{D_j}(v_4) \geq 4$.

Assume that $d_{D_j}(v_4) \leq 3$. By resetting $M := G[\{v_2, v_1, e_i, v_3, a_i\}]$ and $D_i := G[\{b_i, c_i, v_4, v_5, d_i\}]$, we have $d_{D_j}(v_2)+d_{D_j}(v_1)+d_{D_j}(e_i)+d_{D_j}(v_3)+d_{D_j}(a_i) \leq 19$. Thus $d_{D_j}(v_1)+d_{D_j}(e_i)+d_{D_j}(v_3)+d_{D_j}(a_i) \leq 14$. Similarly, by considering $M := G[\{v_2, v_1, e_i, v_5, d_i\}]$ and $D_i := G[\{b_i, c_i, v_4, v_3, a_i\}]$, we have $d_{D_j}(v_1) + d_{D_j}(e_i) + d_{D_j}(v_5) + d_{D_j}(d_i) \leq 14$. Therefore, $\tau_j \leq 14 + 14 + d_{D_j}(v_2) + d_{D_j}(v_4) \leq 36$, a contradiction.

Claim 3. $d_{D_j}(v_4) \neq 5$.

Assume $d_{D_j}(v_4) = 5$. By Lemma 4.2, $d_{D_j}(v_1) + d_{D_j}(v_3) + d_{D_j}(v_5) \leq 18 - d_{D_j}(v_2) - d_{D_j}(v_4) = 8$. Similarly, by resetting $M := G[\{v_1, v_2, v_4, a_i, d_i\}]$ and $D_i := G[\{b_i, c_i, e_i, v_3, v_5\}]$, we have $d_{D_j}(v_1) + d_{D_j}(a_i) + d_{D_j}(d_i) \leq 8$. Thus, $\tau_j \leq 8 + 8 + 2d_{D_j}(e_i) + d_{D_j}(v_2) + d_{D_j}(v_4) \leq 16 + 20 = 36$, a contradiction, and hence Claim 3 follows.

By Claims 2 and 3, $d_{D_j}(v_4) = 4$. By resetting $M := G[\{v_2, v_1, e_i, v_3, a_i\}]$ and $D_i := G[\{b_i, c_i, v_4, v_5, d_i\}]$, we have $d_{D_j}(v_2) + d_{D_j}(v_1) + d_{D_j}(e_i) + d_{D_j}(v_3) + d_{D_j}(a_i) \leq 19$. Thus $d_{D_j}(v_1) + d_{D_j}(e_i) + d_{D_j}(v_3) + d_{D_j}(a_i) \leq 14$. Similarly, by considering $M := G[\{v_2, v_1, e_i, v_5, d_i\}]$ and $D_i := G[\{a_i, b_i, c_i, v_3, v_4\}]$, we have $d_{D_j}(v_1) + d_{D_j}(e_i) + d_{D_j}(v_5) + d_{D_j}(d_i) \leq 14$. Therefore, $37 \leq \tau_j \leq 14+14+d_{D_j}(v_2)+d_{D_j}(v_4) = 37$, and hence $d_{D_j}(v_2)+d_{D_j}(v_1)+d_{D_j}(e_i)+d_{D_j}(v_3)+d_{D_j}(a_i) = 19$ and $d_{D_j}(v_2)+d_{D_j}(v_1)+d_{D_j}(e_i)+d_{D_j}(v_5)+d_{D_j}(d_i) = 19$. By Lemma 4.4, either $d_{D_j}(v_1) = d_{D_j}(e_i) = 3$ and $d_{D_j}(v_3) = d_{D_j}(v_5) = d_{D_j}(a_i) = d_{D_j}(d_i) = 4$, or $d_{D_j}(v_1) = d_{D_j}(e_i) = 4$ and $d_{D_j}(v_3) = d_{D_j}(v_5) = d_{D_j}(a_i) = d_{D_j}(d_i) = 3$, and so $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5)$ is either 19 or 20. However, since $d_{D_j}(v_2) = 5$, $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 18$ by Lemma 4.2, a contradiction. ∎

Lemma 4.9 *Suppose $d_{D_i}(v_1) = 2$ and $d_{D_i}(x) = 4$ for $x \in \{v_2, v_3, v_4, v_5\}$. Then the complement of the graph induced by $V(D_i) \cup V(M)$ must be isomorphic to one of the graphs in Figure 9.*

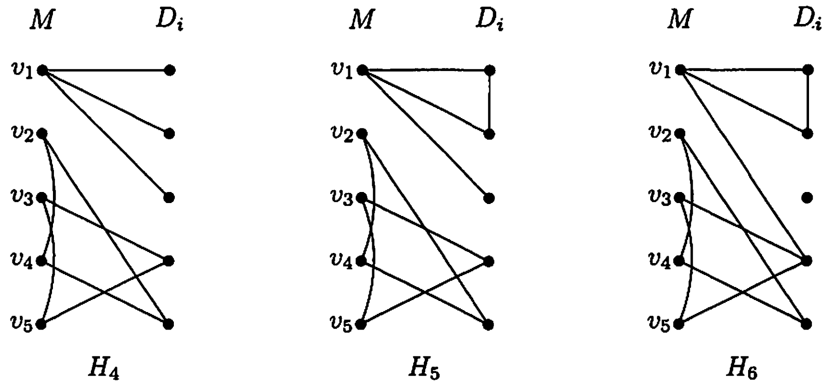

Figure 9

Proof. We consider two cases.

Case 1. $D_i = K_5$.

Since $d_{D_i}(v_2) = 4$, without loss of generality, we assume $N_{D_i}(v_2) = \{a_i, b_i, c_i, d_i\}$. Let $x \in V(D_i)$. Since $D_i - x + v_2$ contains K_5^-, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$. Moreover, if $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$, then $xv_3, xv_5 \in E(G)$. Since $\theta_{D_i} = 18$ and $d_{D_i}(v_2) = 4$, $\sum_{x \in V(D_i)} |N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 14$. Thus there are four vertices $x \in V(D_i)$ such that $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$.

Assume that $|N_G(e_i) \cap \{v_1, v_3, v_4, v_5\}| \leq 2$. Then, for $x \in \{a_i, b_i, c_i, d_i\}$, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$, and hence $xv_3, xv_5 \in E(G)$. So $N_{D_i}(v_3) = N_{D_i}(v_5) = \{a_i, b_i, c_i, d_i\}$. Since $d_{D_i}(v_1) = 2$ and $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$ for $x \in V(D_i)$, $N_{D_i}(v_4) \neq \{a_i, b_i, c_i, d_i\}$. Without loss of generality, we assume $N_{D_i}(v_4) = \{a_i, b_i, c_i, e_i\}$. Then $N_{D_i}(v_1) = \{d_i, e_i\}$. Thus $D_i - d_i + v_4$ is K_5 and $M - v_4 + d_i$ is K_5^-, a contradiction. So $|N_G(e_i) \cap \{v_1, v_3, v_4, v_5\}| = 3$.

Note that there are four vertices $x \in V(D_i)$ such that $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$. Without loss of generality, we assume $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$ for $x \in \{a_i, b_i, c_i, e_i\}$ and $|N_G(d_i) \cap \{v_1, v_3, v_4, v_5\}| = 2$. Then $N_{D_i}(v_3) =$

$N_{D_i}(v_5) = \{a_i, b_i, c_i, e_i\}$.

Let's consider $N_{D_i}(v_4)$. Suppose that $e_iv_4 \in E(G)$. Since $d_{D_i}(v_1) = 2$ and $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$ for $x \in V(D_i)$, $N_{D_i}(v_4) \neq \{a_i, b_i, c_i, e_i\}$. Therefore, $d_iv_4 \in E(G)$. Without loss of generality, we assume $N_{D_i}(v_4) = \{a_i, b_i, d_i, e_i\}$. Then $N_{D_i}(v_1) = \{c_i, d_i\}$. Thus $D_i - c_i + v_4$ and $M - v_4 + c_i$ contain K_5^-, a contradiction, and hence $e_iv_4 \notin E(G)$. Therefore, $N_{D_i}(v_4) = \{a_i, b_i, c_i, d_i\}$, and so $N_{D_i}(v_1) = \{d_i, e_i\}$, and the complement of the graph is isomorphic to H_4 in Figure 9.

Case 2. $D_i = K_5^-$.

Then $a_id_i \notin E(G)$. We consider two subcases.

Case 2.1. There is a vertex x in $\{v_2, v_3, v_4, v_5\}$ such that the subgraph induced by $N_{D_i}(x)$ is K_4.

Without loss of generality, we assume $N_{D_i}(v_2) = \{a_i, b_i, c_i, e_i\}$. Then $D_i - d_i + v_2$ is K_5, $D_i - a_i + v_2$ is K_5^-, and $D_i - x + v_2$ is S for $x \in \{b_i, c_i, e_i\}$. Since $D_i = K_5^-$, we have $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$ for $x \in \{a_i, b_i, c_i, e_i\}$, and $|N_G(d_i) \cap \{v_1, v_3, v_4, v_5\}| \leq 2$. Since $\theta_{D_i} = 18$, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$ for $x \in \{a_i, b_i, c_i, e_i\}$, and $|N_G(d_i) \cap \{v_1, v_3, v_4, v_5\}| = 2$, and hence $a_iv_3, a_iv_5 \in E(G)$ (Otherwise, $D_i - a_i + v_2$ is K_5^- and $M - v_2 + a_i$ is S, a contradiction).

We claim $d_iv_3 \in E(G)$. By contradiction, we assume $d_iv_3 \notin E(G)$. Then $N_{D_i}(v_3) = \{a_i, b_i, c_i, e_i\}$. Since $d_{D_i}(v_4) = d_{D_i}(v_5) = 4$, $|N_{D_i}(v_4) \cap N_{D_i}(v_5)| \geq 3$. Thus $|(N_{D_i}(v_4) \cap N_{D_i}(v_5)) \cap \{b_i, c_i, e_i\}| \geq 1$. Without loss of generality, we assume $e_i \in N_{D_i}(v_4) \cap N_{D_i}(v_5)$. If $d_i \in N_{D_i}(v_4) \cap N_{D_i}(v_5)$, then $G[\{d_i, e_i, v_4, v_5, v_1\}]$ contains S and $G[\{a_i, b_i, c_i, v_2, v_3\}]$ is K_5, a contradiction, and hence $d_i \notin N_{D_i}(v_4) \cap N_{D_i}(v_5)$. Since $|N_G(d_i) \cap \{v_1, v_3, v_4, v_5\}| = 2$ and $d_iv_3 \notin E(G)$, $d_iv_1 \in E(G)$ and $|N_G(d_i) \cap \{v_4, v_5\}| = 1$. Thus $G[\{d_i, e_i, v_4, v_5, v_1\}]$ contains W_4 and $G[\{a_i, b_i, c_i, v_2, v_3\}]$ is K_5, which is contrary to the extremality condition (3). So $d_iv_3 \in E(G)$. Similarly, $d_iv_5 \in E(G)$. Since $|N_G(d_i) \cap \{v_1, v_3, v_4, v_5\}| = 2$, $d_iv_1 \notin E(G)$ and $d_iv_4 \notin E(G)$, and hence $N_{D_i}(v_4) = \{a_i, b_i, c_i, e_i\}$.

Note that $a_i, d_i \in N_{D_i}(v_3) \cap N_{D_i}(v_5)$. If $N_{D_i}(v_3) = N_{D_i}(v_5)$, without loss of generality, we may assume $N_{D_i}(v_3) \cap N_{D_i}(v_5) = \{a_i, d_i, b_i, c_i\}$. Then $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$ for $x \in \{a_i, b_i, c_i\}$. Thus $a_iv_1, b_iv_1, c_iv_1 \notin E(G)$, and so $d_{D_i}(v_1) \leq 1$, a contradiction. So $N_{D_i}(v_3) \neq N_{D_i}(v_5)$. Since b_i, c_i, e_i are symmetric, we may assume $N_{D_i}(v_3) = \{a_i, d_i, b_i, c_i\}$ and $N_{D_i}(v_5) = \{a_i, d_i, b_i, e_i\}$. Then $N_{D_i}(v_1) = \{c_i, e_i\}$. Hence $G[\{d_i, v_4, b_i, c_i,$

$v_3\}]$ and $G[\{v_2, a_i, v_1, v_5, e_i\}]$ are K_5^-, a contradiction.

Case 2.2. For each x in $\{v_2, v_3, v_4, v_5\}$, the subgraph induced by $N_{D_i}(x)$ is K_4^-.

In this case, $a_i, d_i \in N_{D_i}(x)$ for $x \in \{v_2, v_3, v_4, v_5\}$. Without loss of generality, we assume $N_{D_i}(v_2) = \{a_i, d_i, b_i, c_i\}$. Thus, $D_i - x + v_2$ is K_5^- for $x \in \{a_i, d_i, e_i\}$, and $D_i - x + v_2$ is W_4 for $x \in \{b_i, c_i\}$. So, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$ for $x \in \{a_i, d_i, e_i\}$, and $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 4$ for $x \in \{b_i, c_i\}$.

We claim that $N_{D_i}(v_2) \neq N_{D_i}(v_3)$ and $N_{D_i}(v_2) \neq N_{D_i}(v_5)$. By contradiction, we assume $N_{D_i}(v_3) = \{a_i, b_i, c_i, d_i\}$. If $e_i \in N_{D_i}(v_4) \cap N_{D_i}(v_5)$, then $G[\{d_i, e_i, v_4, v_5, v_1\}]$ contains S and $G[\{a_i, b_i, c_i, v_2, v_3\}]$ is K_5, a contradiction, and hence $e_i \notin N_{D_i}(v_4) \cap N_{D_i}(v_5)$. Since $d_{D_i}(v_4) = 4$ and $d_{D_i}(v_5) = 4$, we have $|N_{D_i}(v_4) \cap N_{D_i}(v_5)| \geq 3$. Thus $|(N_{D_i}(v_4) \cap N_{D_i}(v_5)) \cap \{b_i, c_i\}| \geq 1$. Without loss of generality, we assume $b_i \in N_{D_i}(v_4) \cap N_{D_i}(v_5)$. Since $d_{D_i}(v_1) = 2$, we have either $N_G(v_1) \cap \{a_i, b_i\} \neq \emptyset$ or $N_G(v_1) \cap \{c_i, d_i\} \neq \emptyset$. If $N_G(v_1) \cap \{a_i, b_i\} \neq \emptyset$, then $G[\{v_1, v_4, v_5, a_i, b_i\}]$ contains K_5^- and $G[\{c_i, d_i, e_i, v_2, v_3\}]$ contains S; if $N_G(v_1) \cap \{c_i, d_i\} \neq \emptyset$, then $G[\{v_1, v_2, v_3, c_i, d_i\}]$ contains K_5^- and $G[\{a_i, b_i, e_i, v_4, v_5\}]$ contains S, a contradiction, and hence $N_{D_i}(v_2) \neq N_{D_i}(v_3)$. Similarly, $N_{D_i}(v_2) \neq N_{D_i}(v_5)$. Therefore, $e_i v_3, e_i v_5 \in E(G)$.

Since $d_{D_i}(v_3) = 4$ and b_i, c_i are symmetric, we may assume $N_{D_i}(v_3) = \{a_i, d_i, e_i, b_i\}$. If $e_i v_4 \in E(G)$, then $G[\{d_i, e_i, v_4, v_5, v_1\}]$ contains S and $G[\{a_i, b_i, c_i, v_2, v_3\}]$ is K_5^-, a contradiction, and hence $e_i v_4 \notin E(G)$ and $N_{D_i}(v_4) = \{a_i, b_i, c_i, d_i\} = N_{D_i}(v_2)$. Similarly, $N_{D_i}(v_3) = N_{D_i}(v_5) = \{a_i, b_i, d_i, e_i\}$. Since $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$, $N_{D_i}(v_1) \subseteq \{b_i, c_i, e_i\}$. If $N_{D_i}(v_1) = \{b_i, c_i\}$ or $N_{D_i}(v_1) = \{b_i, e_i\}$, then the complement of the graph induced by $V(M) \cup V(D_i)$ is isomorphic to H_6; if $N_{D_i}(v_1) = \{c_i, e_i\}$, then the complement of the graph induced by $V(M) \cup V(D_i)$ is isomorphic to H_5. ∎

Lemma 4.10 *If $d_{D_i}(x) = 4$ for $x \in \{v_2, v_3, v_4, v_5\}$, then $\theta_{D_i} \leq 17$.*

Proof. By contradiction, we assume $\theta_{D_i} \geq 18$. By Lemmas 4.3, 4.4 and 4.6, $\theta_{D_i} = 18$. Thus $d_{D_i}(v_1) = 2$. By Lemma 4.9, the complement of the graph induced by $V(M) \cup V(D_i)$ is isomorphic to either H_4, or H_5 or H_6 in Figure 9. Next we will exclude H_4. Use the same method we could exclude H_5 and H_6. For the convenience of discussion, the vertices of D_i from the top to the bottom in H_4, H_5, H_6 are labelled by b_1, b_2, b_3, b_4, and

b_5, respectively.

For the graph H_4, denote $T = \{v_1, b_1, v_2, v_3, v_4, v_5\}$, and denote F the graph induced by $V(M) \cup V(D_i)$ in G, that is, H_4 is the complement of F. Then $\sum_{x \in T} d_F(x) = 42$. To get a contradiction, we consider $\tau_j = \sum_{x \in T} d_{D_j}(x)$ for each $j \in \{1, 2, \cdots, k-1\}$ and $j \neq i$. If $\tau_j \leq 22$ for all j, then

$$22k = 6 \cdot \frac{11k}{3} \leq \sum_{j \neq i} \tau_j + 42 \leq 22(k-2) + 42 = 22k - 2,$$

a contradiction. So there is a D_j such that $\tau_j \geq 23$.

First we claim that $d_{D_j}(x) \leq 4$ for each $x \in \{v_2, v_3, v_4, v_5\}$. By contradiction, we assume $d_{D_j}(v_2) = 5$. By resetting $M := G[\{v_3, v_1, b_1, v_2, v_4\}]$ and $D_i := G[\{b_2, b_3, b_4, b_5, v_5\}]$, we have $d_{D_j}(v_3) + d_{D_j}(v_1) + d_{D_j}(b_1) + d_{D_j}(v_2) + D_{D_j}(v_4) \leq 18$ by Lemma 4.2. Since $\tau_j \geq 23$, $d_{D_j}(v_5) = 5$. Similarly, by resetting $M := G[\{v_5, v_1, b_1, v_2, v_4\}]$ and $D_i := G[\{b_2, b_3, b_4, b_5, v_3\}]$, we have $d_{D_j}(v_3) = 5$. By Lemma 4.8, $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 17$. Thus, $d_{D_j}(b_1) \geq 23 - 17 = 6$, a contradiction. So $d_{D_j}(v_2) \leq 4$. Similarly, $d_{D_j}(v_3) \leq 4$, $d_{D_j}(v_4) \leq 4$, and $d_{D_j}(v_5) \leq 4$.

Again, let's reset $M := G[\{v_3, v_1, b_1, v_2, v_4\}]$ and $D_i := G[\{b_2, b_3, b_4, b_5, v_5\}]$. By Lemma 4.6, $d_{D_j}(v_3) + d_{D_j}(v_1) + d_{D_j}(b_1) + d_{D_j}(v_2) + D_{D_j}(v_4) \leq 18$. Since $\tau_j \geq 23$, we have $d_{D_j}(v_5) = 5$, a contradiction. This contradiction implies that H_4 should be excluded.

For the graph H_5 and H_6, we still consider $T = \{b_1, v_1, v_2, v_3, v_4, v_5\}$. Arguing similarly as above, we can obtain a contradiction. ∎

Lemma 4.11 *If $d_{D_i}(v_1) = 5$, then $\theta_{D_i} \leq 18$.*

Proof. By contradiction, we assume that $\theta_{D_i} \geq 19$. By Lemma 4.4, $\theta_{D_i} = 19$, $D_i = K_5^-$ and the subgraph induced by $V(M) \cup V(D_i)$ is isomorphic to the graph H_2 in Figure 5. The following graph is the complement of H_2.

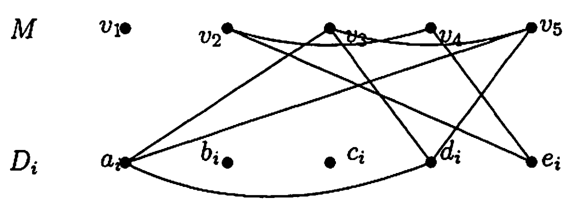

Figure 10. The complement of H_2

For $j \neq i$, denote $T = \{v_1, v_2, \cdots, v_5, a_i, d_i, e_i\}$ and $\tau_j = \sum_{x \in T} d_{D_j}(x) + d_{D_j}(v_2)$, and let F be the subgraph induced by $V(M) \cup V(D_i)$ in G. Then $\sum_{x \in T} d_F(x) + d_F(v_2) = 61$. To get a contradiction, we consider τ_j for each $j \in \{1, 2, \cdots, k-1\}$ and $j \neq i$. If $\tau_j \leq 33$ for all j, then

$$33k = 9 \cdot \frac{11k}{3} \leq \sum_{j \neq i} \tau_j + 61 = 33(k-2) + 61 = 33k - 5,$$

a contradiction. So there is a D_j such that $\tau_j \geq 34$.

Claim 1. $d_{D_j}(v_1) \neq 5$.

Assume $d_{D_j}(v_1) = 5$. Then $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 19$. By resetting $M := G[\{v_1, v_2, e_i, a_i, d_i\}]$ and $D_i := G[\{b_i, c_i, v_3, v_4, v_5\}]$, we have $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(e_i) + d_{D_j}(a_i) + d_{D_j}(d_i) \leq 19$. Thus $34 + 5 \leq d_{D_j}(v_1) + \tau_j \leq 19 + 19 = 38$, a contradiction, and hence Claim 1 follows.

Claim 2. $d_{D_j}(v_1) \leq 2$.

By contradiction, we assume $d_{D_j}(v_1) \geq 3$. By Lemma 4.6 and Claim 1, $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 18$. By resetting $M := G[\{v_1, v_2, e_i, a_i, d_i\}]$ and $D_i := G[\{b_i, c_i, v_3, v_4, v_5\}]$, we have $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(e_i) + d_{D_j}(a_i) + d_{D_j}(d_i) \leq 18$. Thus $37 \leq d_{D_j}(v_1) + \tau_j \leq 18 + 18 = 36$, a contradiction. So Claim 2 holds.

Claim 3. $d_{D_j}(v_1) = 2$.

By Claim 2, we assume that $d_{D_j}(v_1) \leq 1$. Since $\tau_j \geq 34$, $[d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5)] + [d_{D_j}(v_2) + d_{D_j}(e_i) + d_{D_j}(a_i) + d_{D_j}(d_i)] = \tau_j - d_{D_j}(v_1) \geq 33$. Thus we have either $d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \geq 17$ or $d_{D_j}(v_2) + d_{D_j}(e_i) + d_{D_j}(a_i) + d_{D_j}(d_i) \geq 17$, which is contrary to Lemma 4.7. So Claim 3 holds.

Claim 4. Let S be W_4 in F and $F - V(S)$ be K_5^-. If $v_1, v_2 \in V(S)$ and $d_S(v_1) = 4$, then $\sum_{x \in V(S) - \{v_1\}} d_{D_j}(x) = 16$.

Without loss of generality, we assume $S = G[\{v_1, v_2, d_i, e_i, v_3\}]$. By resetting $M := G[\{v_1, v_2, d_i, e_i, v_3\}]$ and $D_i := G[\{b_i, c_i, v_4, v_5, a_i\}]$, we have $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(d_i) + d_{D_j}(e_i) + d_{D_j}(v_3) \leq 18$. By considering $M := G[\{v_1, v_2, v_4, a_i, v_5\}]$ and $D_i := G[\{b_i, c_i, v_3, d_i, e_i\}]$, we have $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_4) + d_{D_j}(a_i) + d_{D_j}(v_5) \leq 18$. Then $36 \leq d_{D_j}(v_1) + \tau_j \leq$

$18+18 = 36$, and hence $d_{D_j}(v_1)+d_{D_j}(v_2)+d_{D_j}(d_i)+d_{D_j}(e_i)+d_{D_j}(v_3) = 18$. Therefore, $d_{D_j}(v_2) + d_{D_j}(d_i) + d_{D_j}(e_i) + d_{D_j}(v_3) = 16$.

Claim 5. $d_{D_j}(v_4) = d_{D_j}(e_i)$, and $d_{D_j}(v_3) = d_{D_j}(v_5) = d_{D_j}(a_i) = d_{D_j}(d_i)$.

Consider $G[\{v_1, v_2, v_3, v_4, v_5\}]$ and $G[\{v_1, v_2, e_i, v_3, v_5\}]$. Since D_i and $D_i - e_i + v_4$ are K_5^-, by Claim 4, $d_{D_j}(v_2)+d_{D_j}(v_3)+d_{D_j}(v_4)+d_{D_j}(v_5) = 16$ and $d_{D_j}(v_2)+d_{D_j}(e_i)+d_{D_j}(v_3)+d_{D_j}(v_5) = 16$. Then $d_{D_j}(v_4) = d_{D_j}(e_i)$. Similarly, by considering $G[\{v_1, v_2, e_i, v_3, a_i\}]$ and $G[\{v_1, v_2, e_i, v_5, a_i\}]$, $d_{D_j}(v_3) = d_{D_j}(v_5)$, by considering $G[\{v_1, v_2, e_i, v_5, a_i\}]$ and $G[\{v_1, v_2, e_i, v_3, v_5\}]$, $d_{D_j}(v_3) = d_{D_j}(a_i)$, and by using $G[\{v_1, v_2, e_i, v_3, v_5\}]$ and $G[\{v_1, v_2, e_i, v_5, d_i\}]$, $d_{D_j}(v_3) = d_{D_j}(d_i)$. So Claim 5 holds.

Claim 6. $d_{D_j}(v_2) \geq d_{D_j}(v_4)$ and $d_{D_j}(v_2) \geq d_{D_j}(e_i)$.

By resetting $M := G[\{v_1, v_3, v_4, d_i, e_i\}]$ and $D_i := G[\{b_i, c_i, v_2, v_5, a_i\}]$, we have $d_{D_j}(v_1) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(d_i) + d_{D_j}(e_i) \leq 18$. By Claim 4, $d_{D_j}(v_2) + d_{D_j}(v_4) + d_{D_j}(a_i) + d_{D_j}(v_5) = 16$. Thus $34 + d_{D_j}(v_4) - d_{D_j}(v_2) \leq \tau_j + d_{D_j}(v_4) - d_{D_j}(v_2) \leq 18 + 16 = 34$. So $d_{D_j}(v_2) \geq d_{D_j}(v_4)$. Similarly, since $d_{D_j}(v_1) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(d_i) + d_{D_j}(e_i) \leq 18$ and $d_{D_j}(v_2)+d_{D_j}(e_i)+d_{D_j}(a_i)+d_{D_j}(v_5) = 16$, we have $d_{D_j}(v_2) \geq d_{D_j}(e_i)$. So Claim 6 holds.

Claim 7. $d_{D_j}(v_3) \leq 4$.

By contradiction, assume that $d_{D_j}(v_3) = 5$. Then for $x \in \{v_3, v_5, a_i, d_i\}$, $d_{D_j}(x) = 5$. By Lemma 4.8, $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 17$. Thus $d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 17 - 2 = 15$, which is contrary to Claim 4. Therefore, Claim 7 holds.

By Claims 3, 5, 7 and $\tau_j \geq 34$, we have $d_{D_j}(v_2) + d_{D_j}(v_4) \geq 8$.

Claim 8. $d_{D_j}(v_2) \leq 4$.

By contradiction, we assume that $d_{D_j}(v_2) = 5$. If $d_{D_j}(v_3) \leq 3$, then $d_{D_j}(v_3) = d_{D_j}(v_5) = 3$ and $d_{D_j}(v_4) = 5$. By Lemma 4.8, $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 17$. Thus $d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) \leq 17 - 2 = 15$, which is contrary to Claim 4, and hence $d_{D_j}(v_3) = 4$. By Claim 5, $d_{D_j}(v_5) = d_{D_j}(a_i) = d_{D_j}(d_i) = 4$. By Claim 4, $d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) = 16$, and hence $d_{D_j}(v_4) = 3$. By Claim 5, $d_{D_j}(e_i) = 3$. Since $d_{D_j}(v_1) = 2$, there are two vertices $t, s \in V(D_j)$ such that $v_1 t, v_1 s \in E(G)$. Next we consider the subgraph induced by

$V(M) \cup V(D_j)$.

If $ts \in E(G)$ and $D_j - \{s,t\}$ is a triangle, without loss of generality, we assume that $\{s,t\} = \{a_j, b_j\}$. Note that $N_G(v_4) \cap \{c_j, d_j, e_j\} \neq \emptyset$ since $d_{D_j}(v_4) = 3$. If $\{c_j, d_j, e_j\} \subseteq N_G(v_5)$, then $G[\{c_i, d_j, e_j, v_4, v_5\}]$ contains S and $G[\{a_j, b_j, v_1, v_2, v_3\}]$ contains K_5^-, a contradiction. So $|N_G(v_5) \cap \{c_j, d_j, e_j\}| \leq 2$. Similarly, $|N_G(v_3) \cap \{c_j, d_j, e_j\}| \leq 2$. Since $d_{D_j}(v_3) = d_{D_j}(v_5) = 4$, we have $a_jv_3, b_jv_3, a_jv_5, b_jv_5 \in E(G)$. Thus $G[\{a_j, b_j, v_1, v_3, v_4\}]$ contains S and $G[\{c_j, d_j, e_j, v_2, v_5\}]$ contains K_5^-, a contradiction. So we have either $ts \notin E(G)$ or $D_j - \{s,t\}$ is not a triangle. Therefore, $D_j = K_5^-$.

If $ts \notin E(G)$, then $\{t,s\} = \{a_j, d_j\}$. Since $d_{D_j}(v_3) = 4$, we have either $a_jv_3 \in E(G)$ or $d_jv_3 \in E(G)$. Without loss of generality, we assume $a_jv_3 \in E(G)$. Thus $G[\{a_j, v_1, v_2, v_3, v_4\}]$ contains S and $G[\{b_j, c_j, d_j, e_j, v_5\}]$ contains K_5^-, a contradiction. So $D_j - \{s,t\}$ is not a triangle. Without loss of generality, we assume that $v_1b_j, v_1c_j \in E(G)$.

If $|N_G(d_j) \cap \{v_1, v_3, v_4, v_5\}| \geq 3$, then, by Lemma 4.2, $G[\{d_j, v_1, v_3, v_4, v_5\}]$ is W_4. However, $G[\{v_2, a_j, b_j, c_j, e_j\}]$ is K_5, a contradiction. So $|N_G(d_j) \cap \{v_1, v_3, v_4, v_5\}| \leq 2$. Similarly, $|N_G(a_j) \cap \{v_1, v_3, v_4, v_5\}| \leq 2$. By Lemma 4.1, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 3$ for $x \in \{b_j, c_j, e_j\}$. Thus

$$13 = d_{D_j}(v_1) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5)$$

$$= \sum_{x \in V(D_j)} |N_G(x) \cap \{v_1, v_3, v_4, v_5\}| \leq 2 \times 2 + 3 \times 3 = 13.$$

Hence, $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 2$ for $x \in \{a_j, d_j\}$, and $|N_G(x) \cap \{v_1, v_3, v_4, v_5\}| = 3$ for $x \in \{b_j, c_j, e_j\}$. By Lemma 4.1, $b_jv_3, b_jv_5, c_jv_3, c_jv_5, e_jv_3, e_jv_5 \in E(G)$. If $|N_G(v_4) \cap \{a_j, e_j, d_j\}| = 3$, then $G[\{a_j, e_j, d_j, v_4, v_5\}]$ contains S and $G[\{b_j, c_j, v_1, v_2, v_3\}]$ is K_5, a contradiction, and hence $|N_G(v_4) \cap \{a_j, e_j, d_j\}| \leq 2$. Since $d_{D_j}(v_4) = 3$, $N_G(v_4) \cap \{b_j, c_j\} \neq \emptyset$. Thus $G[\{b_j, c_j, v_1, v_3, v_4\}]$ contains K_5^- and $G[\{a_j, e_j, d_j, v_2, v_5\}]$ contains S, a contradiction. So Claim 8 holds.

By Claims 7 and 8, we have $d_{D_j}(x) = 4$ for $x \in \{v_2, v_3, v_4, v_5, a_i, d_i, e_i\}$. By Claim 3, $d_{D_j}(v_1) = 2$. Thus $d_{D_j}(v_1) + d_{D_j}(v_2) + d_{D_j}(v_3) + d_{D_j}(v_4) + d_{D_j}(v_5) = 18$, which is contrary to Lemma 4.10. ∎

Now we are ready to prove Theorem 1.3 when M is isomorphic to W_4. By Lemmas 4.6 and 4.11, $\theta_{D_i} \leq 18$ for each $i \in \{1, 2, \cdots, k-1\}$. Since

$\theta_M = 16$, we have

$$5 \cdot \frac{11}{3} k \leq \sum_{x \in V(M)} d_G(x) \leq 18(k-1) + 16 = 18k - 2,$$

a contradiction. This finishes the proof of Theorem 1.3 when M is isomorphic to W_4.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, Elsevier North-Holland, 1976.

[2] O. Cooley, D. Kühn, and D. Osthus, Perfect packings with complete graphs minus an edge, European J. Combin. 28 (2007), 2143-2155.

[3] H. Enomoto, A. Kaneko, and Z. Tuza, P_3-factors and covering cycles in graphs of minimum degree $n/3$, Colloq. Math. Soc. János Bolyai 52 (1987), 213-220.

[4] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős, Colloq. Math. Soc. János Bolyai 4 (1970), 601-623.

[5] K. Kawarabayashi, K_4^--factor in a graph, J. Graph Theory, 39 (2002), 111-128.