Maximum degree of 3-independent vertices and group connectivity

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Abstract

Let A be an abelian group with $|A| \geq 4$. Suppose that G is a 3-edge-connected simple graph on $n \geq 19$ vertices. We show in this paper that if $\max\{d(x), d(y), d(z)\} \geq n/6$ for every 3-independent vertices $\{x, y, z\}$ of G, then either G is A-connected or G can be T-reduced to the Petersen graph, which generalizes the result of Zhang and Li (Graphs and Combin., 30 (2014), 1055-1063).

Keywords nowhere-zero flow· k-independent vertices· Group connectivity

1 Introduction

The graphs considered in this paper are finite, loopless, and may have multiple edges. Terminology and notation not defined here can be referred to in [1]. Let X and Y be two disjoint sets of vertices of a graph $G = (V, E)$. We denote by $E[X, Y]$ the set of edges of G with one end in X and the other end in Y, and by $e(X, Y)$ the cardinality of $E[X, Y]$. If $Y = \{v\}$, we write $e(v, X)$ instead of $e(\{v\}, X)$. When $Y = V \setminus X$, we refer to $e(X, Y)$ as the boundary of X and denote it by $\partial_G(X)$ ($\partial(X)$ for short). If H is a subgraph of G, then define $\partial(H) = \partial(V(H))$.

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A k-*cycle* is a cycle of length k. The girth of a graph G, denoted by $g(G)$, is the length of its shortest cycle. The set of neighbors of v in G is denoted by $N_G(v)$, or simply $N(v)$. For two distinct vertices u and v, we denote by $d_G(u,v)$ ($d(u,v)$ for short) the shortest distance between u and v. A k-independent vertices is a set of k vertices that no two of its elements are adjacent.

For any subset S of $V(G)$, $G - S$ denotes the graph obtained from G by deleting all the vertices of S together with all the edges with at least one end in S. For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops generated in this process. If H is a subgraph of G, then G/H denotes $G/E(H)$.

For a graph G, we denote by $\tau(G)$ the maximum number of edge-disjoint spanning trees of graph G. Let $\mathscr{T}$ denote a family of graphs such that a graph $G \in \mathscr{T}$ if and only if $\tau(G) \geq 2$ or G is a cycle of length 3. A graph G^* is called the T-*reduction* of G if it is obtained from G by repeatedly contracting nontrivial subgraphs of G in $\mathscr{T}$ until no subgraph in $\mathscr{T}$ left. It is obvious that if G cannot be contracted to G^*, then G cannot be T-reduced to G^*.

For an orientation of a graph G and for a vertex $v \in V(G)$, denote by $E^+(v)$ (or $E^-(v)$, respectively) the set of all arcs with tail v (or head v, respectively). Let A be an abelian group where the identity of A is denoted by 0. Let A^* denote the set of nonzero elements of A. Define $F(G,A) = \{f \mid f : E(G) \to A\}$ and $F^*(G,A) = \{f \mid f : E(G) \to A^*\}$. Given a function $f \in F(G,A)$, the boundary of f is a map $\partial f : V(G) \to A$. Let

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

for all $v \in V(G)$, where " $\sum$" refers to the addition in A.

For an integer $k \geq 2$, a *nowhere-zero* k-*flow* of G is an integer-valued function f on $E(G)$ such that $0 < |f(e)| < k$ for each $e \in E(G)$, and $\partial f = 0$ for each $v \in V(G)$. A function $b : V(G) \to A$ is an A-*valued zero-sum function* on G if $\sum_{v \in V(G)} b(v) = 0$. Denoted by $Z(G,A)$ the set of all A-valued zero-sum functions on G. A graph G is A-*connected* if G has an orientation such that for any function $b \in Z(G,A)$, there is a function $f \in F^*(G,A)$ such that $\partial f = b$. It is observed in [7] that group connectivity is independent of the orientation of G, and that every A-connected graph is 2-edge-connected. Note that if a graph G is A-connected with $|A| \geq k$, then G admits a nowhere-zero k-flow.

Integer flow was originally introduced by Tutte in [13] as a dual version of graph coloring. Tutte ([13, 14], see also [7]) posed longstanding conjectures on the existence of nowhere-zero flows. The 4-flow conjecture is due to Tutte as follows.

Conjecture 1.1 *Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.*

The concept of group connectivity was introduced by Jaeger *et al.*[7] as a generalization of nowhere-zero flows. They proposed two conjectures on group connectivity of graphs: Z_3- and Z_5-connectivity conjectures, which implies the corresponding conjectures of nowhere-zero flows, and they proved the following theorem.

Theorem 1.2 *If A is an abelian group with $|A| \geq 4$, then every 4-edge-connected graph is A-connected.*

The purpose of study in group connectivity is to characterize contractible configurations for integer flow problems. Group connectivity plays an important role to study nowhere-zero flow problems. Recently, degree conditions are used to guarantee the existence of group connectivity. Fan and Zhou in [5] characterized all Z_3-connected graphs G on n vertices with degree sum of any two adjacent vertices at least $n + 2$. Zhang *et al.* in [21] further generalized the result of Fan and Zhou by lowering degree sum condition with n. Luo *et al.*[12] characterized all Z_3-connected graphs that satisfy the Ore-condition: for every $uv \notin E(G)$, $d(u) + d(v) \geq n$. This result was improved by Yang and Li in [18], who proved that if for every 3 independent vertices u, v, w, if $\max\{d(u), d(v), d(w)\} \geq \frac{3}{2}n$, then G is not Z_3-connected if and only if G is one of 12 specified graphs. For more results, the readers can be referred to [6, 16, 18] and others.

Yao *et al.* in [17] proved that for 2-edge connected simple graph on $n \geq 13$ vertices, if for every $uv \notin E(G)$, $\max\{d(u), d(v)\} \geq n/4$, then either G is A-connected with $|A| \geq 4$, or $G^* \in \{K_{2,3}, \text{4-cycle,5-cycle}\}$. Li *et al.* in [11] further showed that every graph G satisfying $\max\{d(u), d(v)\} \geq n/2$ for every $uv \notin E(G)$, G is not Z_3-connected if and only if G is isomorphic to one of twenty-two graphs or G can be Z_3-reduced to the specified graphs. Zhang and Li in [20] extended the result of Yao *et al.* and lowered the bound from $n/4$ to $n/6$, and proved the following result.

Theorem 1.3 *Let A be an abelian group with $|A| \geq 4$, and let G be a 3-edge connected simple graph on $n \geq 19$ vertices. If for every $uv \notin E(G)$, $max\{d(u), d(v)\} \geq n/6$, then G is A-connected.*

In this paper, we further relax with restriction on the degree condition in Theorem 1.3. Motivated by Conjecture 1.1 and above results, we present the theorem as follows.

Theorem 1.4 *Let A be an abelian group with $|A| \geq 4$. Suppose that G is a 3-edge-connected simple graph on $n \geq 19$ vertices. If for every 3-independent vertices $\{x, y, z\}$, $max\{d(x), d(y), d(z)\} \geq n/6$, then either G is A-connected or G can be T-reduced to the Petersen graph.*

The bound $n \geq 19$ in Theorem 1.4 is best possible in the sense that there is an example as follows. Let B denote a Blanuša's snark of order 18. Then B satisfies the degree condition of Theorem 1.4 that for every 3-independent vertices $\{x, y, z\}$, $\max\{d(x), d(y), d(z)\} = 3 = |V(B)|/6$. Watkins and Wilson proved in [15] that neither B admits a nowhere-zero 4-flow nor B is contractible to the Petersen graph. Thus neither B is A-connected with $|A| \geq 4$ nor B can be T-reduced to the Petersen graph. This shows that Theorem 1.4 does not hold when $n = 18$.

The rest of the paper is organized as follows: In section 2, some useful lemmas are presented and in section 3, Theorem 1.4 is proved.

2 Preliminaries

We first state some basic properties and known results on group connectivity as follows.

Lemma 2.1 ([7, 9]) *Let G be a graph and A be an abelian group with $|A| \geq 3$. Then each of the following holds.*

(i) K_1 is A-connected.

(ii) Let H be a subgraph of G. If H is A-connected, then G is A-connected if and only if G/H is A-connected.

(iii) An n-cycle is A-connected if and only if $|A| \geq n + 1$.

(iv) If $\tau(G) \geq 2$, then G is A-connected with $|A| \geq 4$.

Let v be a vertex of a graph G and u, w be two neighbors of v in G. Let $G_{[vu,vw]}$ be the graph obtained from G by deleting vu and vw and adding a new edge uw. The following technique is due to Lai and Chen *et al.*

Lemma 2.2 ([4, 9]) *Let A be an abelian group with $|A| \geq 3$. Let v be a vertex of graph G with $d(v) \geq 4$ and u, w be two neighbors of v in G. If $G_{[vu,vw]}$ is A-connected, then so is G.*

For an integer $i \geq 1$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$. Let G^* be the T-reduction of G. For simplicity, we write D_i for $D_i(G^*)$. Let $F(G)$ denote the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Lemma 2.3 ([2, 17]) *Let G^* be a T-reduced graph. If G^* is nontrivial, then each of the following holds.*

(i) G^* *is simple and contains no 3-cycles.*
(ii) $2|V(G^*)| - |E(G^*)| \geq 3$ *and* $\delta(G^*) \leq 3$.
(iii) $2|D_2| + |D_3| \geq 6 + \sum_{i \geq 5}(i-4)|D_i|$.
(iv) $F(G^*) = 2|V(G^*)| - |E(G^*)| - 2$.

Lemma 2.4 ([17]) *Let G be a simple graph and let H be a subgraph of G. If* $d_G(v) \geq q$ *for every* $v \in V(H)$ *and* $\partial(H) < q$, *then* $|V(H)| > q$.

Denote by $O(G)$ the set of vertices of odd degree in G. A graph G is *collapsible* if for every even set $R \subseteq V(G)$, there is a spanning connected subgraph H_R of G such that $O(H_R) = R$.

Lemma 2.5 ([3]) *Let G be a connected graph. If* $F(G) \leq 2$, *then G is collapsible or G is contractible to a* K_2 *or a* $K_{2,t}$ *for some integer* $t \geq 1$.

Lemma 2.6 ([10]) *Let A be an abelian group with* $|A| = 4$. *If G is collapsible, then G is A-connected.*

The following two results deal with group connectivity for 3-edge-connected graphs with small order.

Lemma 2.7 ([19]) *Let G be a 3-edge-connected simple graph on* $n \leq 11$ *vertices and A an abelian group with* $|A| \geq 4$. *Then either G is the Petersen graph or G is A-connected.*

Lemma 2.8 ([8]) *Let G be a 3-edge-connected graph on n vertices. If one of the following holds:*
(i) $n \leq 15$;
(ii) $n = 16$ *and* $\Delta \geq 4$; *or*
(iii) $n = 17$ *and* $\Delta \geq 5$,
then G is A-connected with $|A| \geq 5$.

3 Proof of Theorem 1.4

In this section, we first establish the following lemmas, which are useful to prove the main theorem of this paper.

Lemma 3.1 *Let v be a vertex of graph G with* $d(v) \geq 4$ *and* u, w *be two neighbors of v in G and let* $N[v] = N(v) \cup \{v\}$. *Let* $G_1 = G_{[vu,vw]}$. *Suppose that G is k-edge connected and* H_1 *is a subgraph of* G_1 *where* $N[v] \subset V(H_1)$. *Then* G_1/H_1 *is k-edge connected.*

Proof. Let $H = H_1 + \{vu, vw\} - \{uw\}$. It follows that H is the subgraph of G. By the definition of contraction, $G_1/H_1 = G/H$. Since the edge connectivity will not decrease under contraction, G/H is k-edge connected. Then so is G_1/H_1. ■

Lemma 3.2 *Let G be a 3-edge connected graph on $17 \le n \le 20$ vertices and let $V(G) = D_3(G) \cup D_4(G)$. Suppose that $|D_3(G)| = 6$ and L is a 5-cycle of G with $V(L) \subset D_3(G)$. If $g(G) = 5$, then G is A-connected for any abelian group A with $|A| \ge 5$.*

Proof. Since $|D_3(G)| = 6$ and L is a 5-cycle with $V(L) \subset D_3(G)$, $e(D_3(G), D_4(G)) \le 3 \times 6 - 5 \times 2 = 8$. Since $17 \le n \le 20$, $11 \le |D_4(G)| \le 14$. It follows that there are at least $|D_4(G)| - 8 \ge 3$ vertices of degree 4 which are not adjacent to any vertex in $D_3(G)$. We assume that $d(u) = 4$ and $e(u, D_3(G)) = 0$. Let $N(u) = \{u_1, u_2, u_3, u_4\}$ and $N[u] = N(u) \cup \{u\}$ and $N(u_i) = \{u, u_{i1}, u_{i2}, u_{i3}\}$ where $1 \le i \le 4$. Since $g(G) = 5$, $N(u_j) \cap N(u_k) = \{u\}$ for $j \ne k, 1 \le j, k \le 4$. Let $N_{u_i} = N(u_i) - (\{u\} \cup V(L))$ where $1 \le i \le 4$. By the definition of N_{u_i}, $|N_{u_i}| \le 3$. Let $S = V(G) - (N(u_1) \cup N(u_2) \cup N(u_3) \cup N(u_4) \cup N(u))$. It is easy to see that $|S| = n - 17 \le 3$. Let H be the subgraph induced by $N_{u_1} \cup N_{u_2} \cup N_{u_3} \cup N_{u_4}$.

Claim 1. For $1 \le i \le 4$, $|N(u_i) \cap V(L)| \le 1$.
Proof of Claim 1. Suppose otherwise that $|N(u_i) \cap V(L)| \ge 2$ for some i and $v_1, v_2 \in N(u_i) \cap V(L)$. Since L is a 5-cycle, $d(v_1, v_2) \le 2$. It follows that v_1, v_2 and u_i are vertices of a cycle of length 3 or 4, contrary to that $g(G) = 5$. This proves Claim 1.

Claim 2. If $i \ne j$ and $1 \le i, j \le 4$, then $e(N_{u_i}, N_{u_j}) \le 3$. Moreover, if $|N_{u_i}| = 2$, then $e(N_{u_i}, N_{u_j}) \le 2$ where $i \ne j$ and $1 \le i, j \le 4$.
Proof of Claim 2. Suppose otherwise that there exist u_i and u_j such that $e(N_{u_i}, N_{u_j}) \ge 4$. Since $|N_{u_i}| \le 3$, there is a vertex in N_{u_i} which is adjacent to at least two vertices in N_{u_j}. This implies that G contains a 4-cycle, contrary to that $g(G) = 5$. Therefore, $e(N_{u_i}, N_{u_j}) \le 3$ where $i \ne j$. Similarly, if $|N_{u_i}| = 2$, then we can show that $e(N_{u_i}, N_{u_j}) \le 2$ where $i \ne j$. This proves Claim 2.

Claim 3. Suppose that $u_{11} \in V(L)$. If $e(N_{u_1}, N_{u_2}) = e(N_{u_1}, N_{u_i}) = 2$ and $e(N_{u_1}, N_{u_j}) \ge 1$ for $\{i, j\} = \{3, 4\}$, then G is A-connected.
Proof of Claim 3. Since $g(G) = 5$, we assume, without loss of generality, that $u_{12}u_{22}, u_{13}u_{23}, u_{12}u_{i2}, u_{13}u_{i3}$ and $u_{12}u_{j2} \in E(G)$. Let $G_1 = G_{[uu_1, uu_2]}$. Then G_1 has two 4-cycles: $u_1u_{12}u_{22}u_2u_1$ and $u_1u_{13}u_{23}u_2u_1$. Contracting these 4-cycles into u^*, we denote by G_2 the resulting graph. Then $|V(G_2)| \le 15$ and G_2 contains a 4-cycle: $u_iu_{i2}u^*u_{i3}u_i$. We contract this 4-cycle into u^{**} and denote by G_3 the resulting graph. So, $|V(G_3)| \le 12$ and G_3 contains a 4-cycle: $u^{**}uu_ju_{j2}u^{**}$. Contracting this 4-cycle and repeatedly contracting all cycles of length less than 5 generated in process, we finally obtain the resulting graph G_4. Then $|V(G_4)| \le 9$. Let H_1 be the contracted subgraph of G_1, where $G_4 = G_1/H_1$ and $N[u] \subset V(H_1)$. By Lemma 3.1, G_4 is 3-edge connected. By Lemma 2.8, G_4 is A-connected.

By repeatedly applying Lemma 2.1, G_1 is A-connected. By Lemma 2.2, G is A-connected. This proves Claim 3.

By Claim 1, we conclude that $\sum_{i=1}^{4} |N(u_i) \cap V(L)| \leq 4$. Since L is a 5-cycle, $|S \cap V(L)| \geq 1$. On the other hand, since $|S| \leq 3$, we can get that $|S \cap V(L)| \leq 3$. Next we proceed our proof in three cases.

Case 1. $|S \cap V(L)| = 1$.

In this case, $\sum_{i=1}^{4} |N(u_i) \cap V(L)| = 4$. It follows by Claim 1 that $|N(u_i) \cap V(L)| = 1$ for all $1 \leq i \leq 4$. Let $\{v\} = S \cap V(L)$. We assume, without loss of generality, that $L = vu_{11}u_{21}u_{31}u_{41}v$ is a 5-cycle of G. Since $|D_3(G)| = 6$, there is at least one of u_{12} and u_{13} of degree 4. Thus, we assume, without loss of generality, that $d(u_{12}) = 4$. Since $g(G) = 5$, u_{12} is not adjacent to v. Since $|S| \leq 3$, there exists $u_{it} \in N_{u_i}$ where $t \neq 1$ such that $u_{12}u_{it} \in E(G)$. Let $G_1 = G_{[uu_1, uu_i]}$. Then G_1 contains a 4-cycle: $u_1u_{12}u_{it}u_iu_1$. Contracting this 4-cycle, we denote by G_2 the resulting graph. So $|V(G_2)| \leq 17$. Since L is a 5-cycle and $u_{11}, u_{i1} \in V(L)$, $d(u_{11}, u_{i1}) \leq 2$. It follows that u_{11} and u_{i1} are contained in a 3-cycle or a 4-cycle of G_2. Contracting this cycle, we get the resulting graph G_3. Thus, L is contracted into a 4-cycle or a 3-cycle of G_3. Contracting this cycle into u^*, we denote by G_4 the resulting graph. Thus, $|V(G_4)| \leq 12$ and G_4 contains a 4-cycle: $uu_ju^*u_ku$ where $j \neq k$ and $j, k \neq 1, i$. Contracting this 4-cycle and repeatedly contracting all cycles of length less than 5 generated in process, we finally obtain the resulting graph G_5. It follows that $|V(G_5)| \leq 9$. Let H_1 be the contracted subgraph of G_1, where $G_5 = G_1/H_1$ and $N[u] \subset V(H_1)$. By Lemma 3.1, G_5 is 3-edge connected. By Lemma 2.8, G_5 is A-connected. By repeatedly applying Lemma 2.1, G_1 is A-connected. By Lemma 2.2, G is A-connected.

Case 2. $|S \cap V(L)| = 2$.

In this case, $\sum_{i=1}^{4} |N(u_i) \cap V(L)| = 3$ and $|S| \geq 2$ and $|V(H)| = 9$. By Claim 1, we assume, without loss of generality, that $u_{11}, u_{21}, u_{31} \in V(L)$. Then, $|N_{u_1}| = |N_{u_2}| = |N_{u_3}| = 2$. By Claim 2, $e(N_{u_i}, N_{u_j}) \leq 2$ for $i \neq j$. It follows that $\partial_H(N_{u_i}) \leq 6$. Since $g(G) = 5$, $|E(H)| = \frac{1}{2}\sum_{i=1}^{4} \partial_H(N_{u_i}) \leq 12$.

Assume first that $|S| = 2$. Then $e(S, H) = 2$. So, $\partial(H) = 9 + 2 = 11$. Since $|D_3| = 6$ and $|V(L)| = 5$, there is one vertex of degree 3 in H. Since $g(G) = 5$, $|E(H)| = \frac{1}{2}(\sum_{v \in V(H)} d(v) - \partial(H)) = \frac{1}{2}(4 \times 8 + 3 - 11) = 12$, which implies that $e(N_{u_i}, N_{u_j}) = 2$ for $i \neq j$. By Claim 3, G is A-connected.

Next, we assume that $|S| = 3$. Let $\{w\} = S - V(L)$. Since $g(G) = 5$, $e(w, S - w) \leq 1$. We consider three cases: $d(w) = 3$; $d(w) = 4$ and $e(w, S - w) = 1$; $d(w) = 4$ and $e(w, S - w) = 0$.

Suppose that $d(w) = 3$ or $d(w) = 4$ and $e(w, S - w) = 1$. We first show that $|E(H)| \geq 11$. If $d(w) = 3$, then each vertex in H is of degree

4. It is not difficult to see that $e(S,H) \le 5$. So, $\partial(H) \le 9+5=14$. Therefore, $|E(H)| = \frac{1}{2}(\sum_{v\in V(H)} d(v) - \partial(H)) \ge \frac{1}{2}(4\times 9 - 14) = 11$. If $d(w)=4$ and $e(w,S-w)=1$, then there is one vertex of degree 3 in H. It is easy to see that $e(S,H)\le 4$. Thus, $\partial(H) \le 9+4=13$. Then, $|E(H)| = \frac{1}{2}(\sum_{v\in V(H)} d(v) - \partial(H)) \ge \frac{1}{2}(4\times 8+3-13) = 11$. Next, we show that $\partial_H(N_{u_1}) \ge 5$. Suppose otherwise that $\partial_H(N_{u_1}) \le 4$. Since $e(N_{u_i}, N_{u_j}) \le 2$ for $i \ne j$, $|E(H)| \le 2\times 3+4 = 10$, contrary to that $|E(H)| \ge 11$. Therefore, $\partial_H(N_{u_1}) \ge 5$. We claim that either $e(N_{u_1}, N_{u_2}) = 2$ or $e(N_{u_1}, N_{u_3}) = 2$. Suppose otherwise that $e(N_{u_1}, N_{u_2}) \le 1$ and $e(N_{u_1}, N_{u_3}) \le 1$. Then $e(N_{u_1}, N_{u_4}) \ge 3$, contrary to Claim 2. By symmetry, we assume that $e(N_{u_1}, N_{u_2}) = 2$. It follows that $e(N_{u_1}, N_{u_i}) = 2$ and $e(N_{u_1}, N_{u_j}) \ge 1$ where $\{i,j\} = \{3,4\}$. By Claim 3, G is A-connected.

Suppose that $d(w)=4$ and $e(w,S-w)=0$. Since $g(G)=5$, $e(w,N_{u_i}) = 1$ for $1\le i\le 4$. We assume, without loss of generality, that wu_{12}, wu_{22}, wu_{32}, $wu_{42} \in E(G)$. Since L is a 5-cycle, there is at least one edge in u_{11}, u_{21}, u_{31}. Thus, we assume that $u_{11}u_{21} \in E(L)$. Define $G_1 = G_{[uu_1, uu_2]}$. Then G_1 contains a 4-cycle: $u_1u_{11}u_{21}u_2u_1$. Contracting this 4-cycle into u^*, we denote by G_2 the resulting graph. Thus, L is contracted into a 4-cycle of G_2 and $u^*u_{12}wu_{22}u^*$ is also a 4-cycle of G_2. Contracting these two 4-cycles into u^{**}, we get the resulting graph G_3. Thus, $|V(G_3)| \le 11$ and G_3 contains a 3-cycle: $u^{**}u_3u_{32}u^{**}$. Contracting this cycle, we obtain the resulting graph G_4. Then $|V(G_4)| \le 9$ and u, u_4, u_{42} are contained in a 4-cycle of G_4. Contracting this 4-cycle and repeatedly contracting all cycles of length less than 5 generated in process, we finally get the resulting graph G_5. It follows that $|V(G_5)| \le 6$. Let H_1 be the contracted subgraph of G_1, where $G_5 = G_1/H_1$ and $N[u] \subset V(H_1)$. By Lemma 3.1, G_5 is 3-edge connected. By Lemma 2.8, G_5 is A-connected. By repeatedly applying Lemma 2.1, G_1 is A-connected. By Lemma 2.2, G is A-connected.

Case 3. $|S\cap V(L)| = 3$.

In this case, $\sum_{i=1}^4 |N(u_i)\cap V(L)| = 2$ and $|V(H)| = 10$. By Claim 1, we assume, without loss of generality, that $u_{11}, u_{21} \in V(L)$. Then, $|N_{u_1}| = |N_{u_2}| = 2$ and $|N_{u_3}| = |N_{u_4}| = 3$. By Claim 2, $\partial_H(N_{u_i}) \le 6$ for $i\in\{1,2\}$ and $\partial_H(N_{u_j}) \le 7$ for $j \in \{3,4\}$. Thus, $|E(H)| = \frac{1}{2}\sum_{i=1}^4 \partial_H(N_{u_i}) \le \frac{1}{2}(6\times 2+7\times 2) = 13$. On the other hand, it is easy to see that $e(S,H)=3$. Thus, $\partial(H) = 10+3 = 13$. Since $|D_3| = 6$ and $|V(L)| = 5$, there is one vertex of degree 3 in H. So, $|E(H)| = \frac{1}{2}(\sum_{v\in V(H)} d(v) - \partial(H)) = \frac{1}{2}(4\times 9+3-13) = 13$, which implies that $\partial_H(N_{u_i}) = 6$ and $\partial_H(N_{u_j}) = 7$ where $i\in\{1,2\}$ and $j\in\{3,4\}$. It follows that $e(N_{u_1}, N_{u_k}) = 2$ where $2\le k\le 4$. By Claim 3, G is A-connected. ■

Let c be an integer. Define $U = \{u \in V(G) : d_G(u) < n/c\}$ and $W = \{v\in V(G^*) : d_{G^*}(v) < n/c\}$. Let H_v be preimage of v. Let $W_1 = \{v \in W$

and preimage H_v of v has a vertex in $U\}$, and let $W_2 = W - W_1$.

Lemma 3.3 *Let G be a simple graph on n vertices. If for every 3-independent vertices $\{x, y, z\}$,*

$$max\{d(x), d(y), d(z)\} \geq n/c, \quad (1)$$

then $|W_1| \leq 5$. In particular, if $|W_1| = 5$, then the subgraph induced by W_1 is a 5-cycle in G^*.

Proof. Suppose that $W_1 = \{v_1, v_2, \cdots, v_t\}$. By the definition of W_1, we pick $x_i \in V(H_{v_i})$ such that $d_G(x_i) < n/c$ for $1 \leq i \leq t$. Let $X = \{x_1, \cdots, x_t\}$.

Claim. If $t \geq 5$, then $e(x_i, X - x_i) = 2$ for $1 \leq i \leq t$.
Proof of Claim. Assume first that there exists some i_0 such that $e(x_{i_0}, X - x_{i_0}) \geq 3$. We assume, without loss of generality, that $i_0 = 1$ and $x_2, x_3, x_4 \in N_G(x_1)$. By (1), there is at least one edge in x_2, x_3, x_4. This implies that G^* contains a 3-cycle, contrary to Lemma 2.3(i). Therefore, $e(x_i, X - x_i) \leq 2$ for $1 \leq i \leq t$. Next, we assume that there exists some j_0 such that $e(x_{j_0}, X - x_{j_0}) \leq 1$. We assume, without loss of generality, that $j_0 = 1$ and $x_3, x_4, x_5 \notin N_G(x_1)$. By (1), $x_3x_4, x_4x_5, x_3x_5 \in E(G)$. This means that G^* contains a 3-cycle, contrary to Lemma 2.3(i). Therefore, $e(x_i, X - x_i) = 2$ for $1 \leq i \leq t$. Claim is proved.

Assume that $t \geq 6$. By Claim, $e(x_1, X - x_1) = 2$. We assume, without loss of generality, that $x_4, x_5, x_6 \notin N_G(x_1)$. By (1), $x_4x_5, x_4x_6, x_5x_6 \in E(G)$. This means that G^* contains a 3-cycle, contrary to Lemma 2.3(i). This proves that $|W_1| \leq 5$.

Assume that $t = 5$. By Lemma 2.3(i), $g(G^*) \geq 4$. So by Claim, the subgraph induced by W_1 is a 5-cycle in G^*. ■

Lemma 3.4 *Let G be a simple graph on n vertices. If for every 3-independent vertices $\{x, y, z\}$, $max\{d(x), d(y), d(z)\} \geq n/c$, then $|W| \leq c + 4$.*

Proof. Let $|W| = t$. By Lemma 3.3, $|W_1| = l \leq 5$. Let $W_1 = \{u_1, u_2, \cdots, u_l\}$ and $W_2 = \{u_{l+1}, u_{l+2}, \cdots, u_t\}$. By the definition of W_2, none of the preimages of $u_{l+1}, u_{l+2}, \cdots, u_t$ has a vertex in U. It follows that for each $v \in V(H_{u_i})$, $d_G(v) \geq n/c$ where $l+1 \leq i \leq t$. On the other hand, $\partial(H_{u_i}) = d_{G^*}(u_i) < n/c$. By Lemma 2.4, $|V(H_{u_i})| > n/c$ where $l+1 \leq i \leq t$. Therefore,

$$n \geq \sum_{i=1}^{t} |V(H_{u_i})| > l + (t - l)n/c.$$

Evaluating this inequality, we get that $t < c + l - cl/n$. Since l and n are both integers, $t \leq c + l - 1 \leq c + 4$. ■

Lemma 3.5 *Let G be a 3-edge-connected simple graph on $19 \leq n \leq 24$ vertices. If for every 3 independent vertices $\{x, y, z\}$, $max\{d(x), d(y), d(z)\} \geq n/6$, then $n \geq \sum_{i \geq 4} |D_i| + 5|D_3| - 4|W_1|$. Furthermore, $n \geq \sum_{i \geq 4} |D_i| + 5|D_3| - 20$.*

Proof. Since $19 \le n \le 24$, $3 < n/6 \le 4$. Let $c = 6$. By the definition of W, $W = \{v \in V(G^*) : d_{G^*}(v) < n/6 \le 4\}$. Since the edge connectivity will not decrease under contraction, G^* is still 3-edge-connected. It follows that $W = D_3$ and $W_2 = D_3 - W_1$. By Lemma 3.3, $|W_1| \le 5$. Let $v \in W_2$. By the definition of W_2, $d_G(u) \ge 4$ for every $u \in V(H_v)$ and $\partial(H_v) = d_{G^*}(v) = 3 < 4$. By Lemma 2.4, $|V(H_v)| \ge 5$. It follows that $n \ge \sum_{i \ge 4} |D_i| + |W_1| + 5|W_2| = \sum_{i \ge 4} |D_i| + 5|D_3| - 4|W_1| \ge \sum_{i \ge 4} |D_i| + 5|D_3| - 20$. ■

Lemma 3.6 *Let G be a 3-edge-connected simple graph on $19 \le n \le 24$ vertices. If for every 3-independent vertices $\{x, y, z\}$, $max\{d(x), d(y), d(z)\} \ge n/6$, then G is A-connected for any A with $|A| \ge 4$.*

Proof. Suppose otherwise that G is not A-connected with $|A| \ge 4$. By Lemma 2.1, G^* is not A-connected with $|A| \ge 4$. Let $n^* = |V(G^*)|$, and Let $c = 6$. Then $W = D_3$ and $W_1 \subset D_3$. Next we show the following claims.

Claim 1. $6 \le |D_3| \le 8$.

Proof of Claim 1. Since the edge connectivity will not decrease under contraction, G^* is still 3-edge-connected. So, $|D_2| = 0$. By Lemma 2.3(*iii*), $|D_3| \ge 6$. It is sufficient to show that $|D_3| \le 8$. We suppose, to the contrary, that $|D_3| \ge 9$. By Lemma 3.5, $n \ge \sum_{i \ge 4} |D_i| + 5|D_3| - 20 \ge 5 \times 9 - 20 = 25$, contrary to that $n \le 24$. Therefore, $6 \le |D_3| \le 8$. This proves Claim 1.

Claim 2. $\Delta(G^*) = 4$.

Proof of Claim 2. If $\Delta(G^*) \ge 7$, then by Lemma 2.3(*iii*), $|D_3| \ge 6 + \sum_{i \ge 5}(i-4)|D_i| \ge 6 + 3 = 9$, contrary to Claim 1. If $\Delta(G^*) = 6$, then by Claim 1 and Lemma 2.3(*iii*), $8 \ge |D_3| \ge 6 + |D_5| + 2|D_6| \ge 8 + |D_5|$. It means that $|D_3| = 8, |D_5| = 0$ and $|D_6| = 1$. By Lemma 3.5, $n \ge \sum_{i \ge 4} |D_i| + 5|D_3| - 20 = |D_4| + 21$. Since $n \le 24$, $|D_4| \le 3$. Thus, $n^* \le 12$. By Lemma 2.8, G^* is A-connected with $|A| \ge 5$. Note that $2|E(G^*)| = 3 \times 8 + 6 + 4(n^* - 9) = 4n^* - 6$. By Lemma 2.3(*iv*), $F(G^*) = 2|V(G^*)| - |E(G^*)| - 2 = 1$. Since G^* is 3-edge-connected, G^* isn't contractible to a K_2 or a $K_{2,t}$ for some integer $t \ge 1$. By Lemma 2.5, G^* is collapsible. By Lemma 2.6, G^* is A-connected with $|A| = 4$. We conclude that G^* is A-connected with $|A| \ge 4$, a contradiction. Therefore, $\Delta(G^*) \le 5$.

If $\Delta(G^*) = 5$, then by Lemma 2.3(*iii*), $|D_3| \ge 6 + |D_5|$. By Claim 1, $6 \le |D_3| \le 8$. Assume that $|D_3| = 6$. This means that $|D_5| = 0$, contrary to that $\Delta(G^*) = 5$. Therefore , $7 \le |D_3| \le 8$. By Lemma 3.5, $n \ge \sum_{i \ge 4} |D_i| + 5|D_3| - 20 \ge |D_4| + |D_5| + 15$. Since $n \le 24$, $|D_4| + |D_5| \le 9$. Then $n^* \le 17$. It follows by Lemma 2.8 that G^* is A-connected with $|A| \ge 5$. Since $|D_5| \ge 1$ and $|D_3| \le 8$, $2|E(G^*)| = 3|D_3| + 5|D_5| + 4(n^* - |D_3| - |D_5|) = 4n^* + |D_5| - |D_3| \ge 4n^* - 7$. By Lemma 2.3(*iv*), $F(G^*) = 2|V(G^*)| - |E(G^*)| - 2 \le 3/2 < 2$. Since G^* is 3-edge-connected, G^* isn't contractible to a K_2 or a $K_{2,t}$ for some integer $t \geq 1$. By Lemma 2.5, G^* is collapsible. So we conclude by Lemma 2.6 that G^* is A-connected with $|A| \geq 4$, a contradiction. Therefore, $\Delta(G^*) \leq 4$.

If $\Delta(G^*) = 3$, then $n^* = |D_3|$. By Claim 1, $6 \leq |D_3| \leq 8$. It follows that G^* is 3-edge connected with $6 \leq n^* \leq 8$. By Theorem 2.7, G^* is A-connected with $|A| \geq 4$, a contradiction. Claim 2 is proved.

Claim 3. G^* is A-connected with $|A| = 4$.
Proof of Claim 3. By Claim 2, $V(G^*) = D_3 \cup D_4$. By Claim 1, $|D_3| = l \leq 8$. Then $2|E(G^*)| = 3l + 4(n^* - l) = 4n^* - l \geq 4n^* - 8$. So by Lemma 2.3($iv$), $F(G^*) = 2|V(G^*)| - |E(G^*)| - 2 \leq 2$. Since G^* is 3-edge-connected, G^* isn't contractible to a K_2 or a $K_{2,t}$ for some integer $t \geq 1$. By Lemma 2.5, G^* is collapsible. By Lemma 2.6, G^* is A-connected with $|A| = 4$. This proves Claim 3.

Claim 4. $|D_3| = 6$.
Proof of Claim 4. By Claim 2, $V(G^*) = D_3 \cup D_4$. Since there are even number of the vertices of odd degree, D_3 is even. Thus by Claim 1, $|D_3| = 6$ or 8. Assume that $|D_3| = 8$. By Lemma 3.5, $n \geq \sum_{i \geq 4} |D_i| + 5|D_3| - 20 = |D_4| + 20$. Since $n \leq 24$, $|D_4| \leq 4$. It follows that $n^* \leq 12$. By Lemma 2.8, G^* is A-connected with $|A| \geq 5$. We conclude by Claim 3, that G^* is A-connected with $|A| \geq 4$, a contradiction. Therefore, $|D_3| = 6$. This proves Claim 4.

Claim 5. $17 \leq n^* \leq 20$ and $|W_1| = 5$.
Proof of Claim 5. By Claim 4, $|D_3| = 6$. By Claim 2, $n^* = |D_3| + |D_4|$. Suppose that $|D_4| \leq 10$. Then $n^* \leq 16$. By Claim 2, $\Delta(G^*) = 4$. Then by Lemma 2.8, G^* is A-connected for every abelian group A with $|A| \geq 5$. By Claim 3, G^* is A-connected with $|A| = 4$. So G^* is A-connected with $|A| \geq 4$. a contradiction. Therefore, $|D_4| \geq 11$. By Lemma 3.5, $n \geq \sum_{i \geq 4} |D_i| + 5|D_3| - 20 = |D_4| + 10$. Thus, $|D_4| \leq 14$. It follows that $17 \leq n^* \leq 20$.

By Lemma 3.3, $|W_1| \leq 5$. Suppose that $|W_1| \leq 4$. By Lemma 3.5, $n \geq \sum_{i \geq 4} |D_i| + 5|D_3| - 4|W_1| \geq 11 + 5 \times 6 - 4 \times 4 = 25$, contrary to that $n \leq 24$. Therefore, $|W_1| = 5$. This proves Claim 5.

Claim 6. $g(G^*) = 5$.
Proof of Claim 6. By Lemma 2.3(i), $g(G^*) \geq 4$. Assume that $g(G^*) \geq 6$. By Claim 2, $\Delta(G^*) = 4$. We suppose that $d_{G^*}(u) = 4$. Assume first that there are at least three vertices of degree 3 in $N_{G^*}(u)$. By Claims 4 and 5, $|D_3| = 6$ and $|W_1| = 5$. This implies that there is only one vertex of degree 3 not in W_1. Then there are at least two vertices in $N_{G^*}(u) \cap W_1$. By Claim 5, $|W_1| = 5$. By Lemma 3.3, the subgraph induced by W_1 is a 5-cycle. Then the distance between any two vertices in W_1 is at most 2. It follows that $g(G^*) \leq 5$, contrary to that $g(G^*) \geq 6$. Therefore, there are at most two

vertices of degree 3 and at least two vertices of degree 4 in $N_{G^*}(u)$. Let v be a neighbor of u with $d_{G^*}(v) = 4$. Define $N_k^*(u) = \{w \mid d_{G-uv}(u,w) = k\}$ and $N_k^*(v) = \{w \mid d_{G-uv}(v,w) = k\}$. Since $g(G^*) \geq 6$, we can obtain that $|N_1^*(u)| = 3$ and $|N_1^*(v)| = 3$ and $N_i^*(u) \cap N_j^*(v) = \emptyset$ for $1 \leq i,j \leq 2$ and $|N_2^*(u)| \geq 2 \times 2 + 3 = 7$ and $|N_2^*(v)| \geq 3 \times 2 = 6$. Therefore, $n^* \geq 2 + |N_1^*(u)| + |N_1^*(v)| + |N_2^*(u)| + |N_2^*(v)| \geq 21$, contrary to Claim 5. Therefore, $4 \leq g(G^*) \leq 5$.

Assume that $g(G^*) = 4$. Let L_1 be a 4-cycle in G^*. Suppose first that there is at least one vertex of degree 4 in L_1. Let $G_1 = G^*/L_1$. Then, $\Delta(G_1) \geq 5$. By Claim 5, $|V(G_1)| \leq 20 - 3 = 17$. If G_1 is a simple graph, then by Lemma 2.8 , G_1 is A-connected with $|A| \geq 5$. If G_1 is multigraph, then we repeatedly contract these multiple edges such that the resulting graph G_2 is a simple graph. It follows that either $|V(G_2)| = 16$ and $\Delta(G_2) \geq 4$ or $|V(G_2)| \leq 15$. By Lemma 2.8, G_2 is A-connected with $|A| \geq 5$. By Lemma 2.1, G_1 is A-connected with $|A| \geq 5$. It follows by Lemma 2.1 that G^* is A-connected with $|A| \geq 5$. By Claim 3, G^* is A-connected with $|A| \geq 4$, a contradiction. Thus, each vertex in L_1 is of degree 3. Since the subgraph induced by W_1 is a 5-cycle in G^*, there are three vertices in $W_1 \cap V(L_1)$. Let $G^1 = G^*/L_1$. Then this 5-cycle is contracted into a 3-cycle in G^1. Contracting this 3-cycle and repeatedly contracting all cycles of length less than 5 generated in process, we obtain the resulting graph G^2. By Claim 5, $|V(G^2)| \leq 20 - 5 = 15$. Since the edge connectivity will not decrease under contraction, G^2 is still 3-edge-connected. By Lemma 2.8, G^2 is A-connected with $|A| \geq 5$. By repeatedly using Lemma 2.1, G^* is A-connected with $|A| \geq 5$. It follows by Claim 3 that G^* is A-connected with $|A| \geq 4$, a contradiction. Therefore, $g(G^*) = 5$. Claim 6 is proved.

By Claims 5 and 6, G^* satisfies the hypothesis of Lemma 3.2. Thus, G^* is A-connected with $|A| \geq 5$. By Claim 3 that G^* is A-connected with $|A| = 4$. Therefore, G^* is A-connected with $|A| \geq 4$, a contradiction. ■

Proof of Theorem 1.4 Let $c = 6$. In the case when $19 \leq n \leq 24$, G is A-connected with $|A| \geq 4$ by Lemma 3.6. It sufficient to show our theorem for the case when $n \geq 25$. In this case, $n/6 > 4$. By Lemma 3.4, $|D_3| + |D_4| \leq 10$. Let $H^* = G^* - D_3 - D_4$. Then $|V(G^*)| = |D_3| + |D_4| + |V(H^*)|$ and $2|E(G^*)| = \sum_{v \in V(G^*)} d(v) \geq 3|D_3| + 4|D_4| + 5|V(H^*)|$. Thus, $2|V(G^*)| - |E(G^*)| \leq \frac{1}{2}(|D_3| - |V(H^*)|)$. By Lemma 2.3, $2|V(G^*)| - |E(G^*)| \geq 3$. It follows that $|V(H^*)| \leq |D_3| - 6 \leq 4$. So, $|V(G^*)| \leq 14$.

Assume that $|V(G^*)| \leq 11$. By Theorem 2.7, G^* is A-connected with $|A| \geq 4$ or G^* is the Petersen graph. Thus by Lemma 2.1, G is A-connected with $|A| \geq 4$ or G can be T-reduced to the Petersen graph.

Therefore, we assume that $|V(G^*)| \geq 12$. This implies that $|V(H^*)| \geq 2$. So $F(G^*) = 2|V(G^*)| - |E(G^*)| - 2 \leq \frac{1}{2}(|D_3| - |V(H^*)|) - 2 \leq \frac{1}{2}(10-2) - 2 = 2$. Since the edge connectivity will not decrease under contraction, G^* is still 3-edge-connected. Therefore, G^* isn't contractible to a K_2 or a $K_{2,t}$ for some integer $t \geq 1$. By Lemma 2.5, G^* is collapsible. By Lemma 2.6, G^* is A-connected with $|A| = 4$. By Lemma 2.8, G^* is A-connected with $|A| \geq 5$. It follows that G^* is A-connected with $|A| \geq 4$. So far, we complete our proof. ■

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