Barycentric and zero-sum Ramsey numbers

Yair Caro* Leida González† Luz Elimar Marchan‡
Oscar Ordaz§

Abstract

Let G be a finite abelian group of order n. The *barycentric Ramsey number* $BR(H, G)$ is the minimum positive integer r such that any coloring of the edges of the complete graph K_r by elements of G contains a subgraph H whose assigned edge colors constitute a barycentric sequence, i.e., there exists one edge whose color is the "average" of the colors of all edges in H. When the number of edges $e(H) = 0 \pmod{exp(G)}$, $BR(H, G)$ are the well known *zero-sum Ramsey numbers* $R(H, G)$. In this work, these Ramsey numbers are determined for some graphs, in particular, for graphs with five edges without isolated vertices using $G = \mathbb{Z}_n$, where $2 \leq n \leq 4$, and for some graphs H with $e(H) = 0 \pmod 2$ using $G = \mathbb{Z}_2^s$.

Keywords: k-barycentric sequences, barycentric sequences, barycentric Ramsey numbers, zero-sum Ramsey numbers, barycentric-sum, zero-sum, classical Ramsey numbers.
MSC (2000): 11B50; 11P70; 11B75

1 Introduction

Let G be an abelian group of order n and let $exp(G)$ be the exponent of G, i.e., the least positive integer such that $exp(G)g = 0$ for each $g \in G$. It is clear that $exp(G) \mid n$. The focus of this work are barycentric sequences in G, i.e., sequences that contain one element which is the "average" of all terms in the sequence. Formally, a barycentric sequence is defined as follows:

*Department of Mathematics. University of Haifa-Oranim. Tivon-36006. Israel
†Departamento de Matemáticas and Laboratorio MoST Centro ISYS, Facultad de Ciencias, Universidad Central de Venezuela, Ap. 47567, Caracas 1041-A, Venezuela.
‡Departamento de Matemáticas. Decanato de Ciencias y Tecnologías, Universidad Centroccidental Lisandro Alvarado, Barquisimeto, Venezuela.
§Departamento de Matemáticas and Laboratorio MoST Centro ISYS, Facultad de Ciencias, Universidad Central de Venezuela, Ap. 47567, Caracas 1041-A, Venezuela. Corresponding author. E-mail address: oscarordaz55@gmail.com (O. Ordaz).

Definition 1 ([12]). Let $a_1, a_2, \cdots, a_k$, where $a_i \in G$ are elements not necessarily distinct, be a k-sequence in G. This sequence is *k-barycentric* or has a *k-barycentric-sum* if there exists a_j such that $a_1 + a_2 + \ldots + a_j + \ldots + a_k = ka_j$. The element a_j is called a barycenter. Moreover, when $a_1, a_2, \cdots, a_k$ is a set, the term *k-barycentric set* is used instead of k-barycentric sequence. When the length of the sequence is unconstrained, we will simply speak of a barycentric sequence or a barycentric sum.

Notice that when $k = 0 \pmod{exp(G)}$, k-barycentric sequences are zero-sum sequences of length k. Zero-sum problem consider the existence of a zero-sum k-subsequence in a given sequence; they were introduced by Erdős, Ginzburg and Ziv in [13] where they showed that every sequence of length $2n-1$ in a finite abelian group of order n contains a zero-sum n-subsequence with zero-sum. The state of the art for results, problems and conjectures regarding zero-sum problems is covered in the nice and well structured surveys of Caro [8] and Gao and Geroldinger [16]. Barycentric sequences are introduced in [11, 12] inspired by a result of Hamidoune in 1995 [17]: any sequence of length $n+k-1$ in a finite abelian group of order n contains a k-barycentric sequence. The barycentric-sum problem involves finding the smallest integer t such that any t-sequence contains a k-barycentric sequence for some given k. In [19] we give a survey on results in barycentric-sum problems related to the group of integers modulo n. Barycentric and zero-sum problems are an area of combinatorial number theory.

Let $H = (V(H), E(H))$ be a graph with $e(H)$ edges. The *classical Ramsey number* $R(H, n)$ is the smallest integer t such that in any coloring of the edges of K_t with n colors there exists a monochromatic copy of H.

The *barycentric Ramsey number* $BR(H, G)$, for the pair (H, G), is introduced in [11]. It is defined as the minimum positive integer r such that any coloring $f : E(K_r) \to G$ of the edges of K_r by elements of G contains a subgraph H with an edge e_0 such that $\sum_{e \in E(H)} f(e) = e(H)f(e_0)$. In this case, H is called *a barycentric graph.* That $BR(H, G)$ always exists is clear since $BR(H, G) \leq R(H, |G|)$. In the case that $e(H) = 0 \pmod{exp(G)}$, the barycentric Ramsey numbers concide with the *zero-sum Ramsey numbers* $R(H, G)$, i.e., the minimal positive integer s such that any coloring $f : E(K_s) \to G$ of the edges of K_s by elements of G contains a subgraph H with $\sum_{e \in E(H)} f(e) = 0$, where 0 is the zero element of G. The necessity of the condition $e(H) = 0 \pmod{exp(G)}$ for the existence of $R(H, G)$ is clear; it comes from the monochromatic coloration of the edges of H. The zero-sum Ramsey numbers were introduced by Bialostocki and Dierker in [2] when $e(H) = n$, and the concept was extended to $e(H) = 0 \pmod n$ by Caro in [9]. It is clear that $BR(H, G) \geq |V(H)|$ and, as indicated above, that $BR(H,G) = R(H,G)$ when $e(H) = 0$ (mod $exp(G)$). Notice that when $e(H) = 0$ (mod $exp(G)$), then $R(H,G) \leq R(H,|G|)$; moreover when $e(H) = n$, then $R(H,2) \leq R(H,Z_n)$. If a graph H cannot be barycentric with exactly two colors, then $R(H,2) \leq BR(H,G)$.

In [8] Caro gives Table 1 as a survey of the known results for $R(H,\mathbb{Z}_n)$ and $R(H,2)$ with H having at most four edges, based on results given in [1, 2, 5, 6, 7, 9].

The notation for Table 1 is the following: $K_{1,k}$ are stars with k edges, $MK_{1,k}$ are modified $k-$stars, defined as the tree with $k+1$ vertices, k edges and degree sequences $k-1, 2, 1, \ldots, 1$, P_k are paths with k vertices and $k-1$ edges, C_k are circuits with k vertices, mK_2 is an m matching and $C_3 + e$ is a graph with vertices a, b, c, d and edges ab, bc, ca, bd. The graph unions are disjoint.

Table 1: Classical Ramsey and zero-sum Ramsey numbers for graphs with at most four edges

H	$R(H,2)$	$R(H,\mathbb{Z}_2)$	$R(H,\mathbb{Z}_3)$	$R(H,\mathbb{Z}_4)$
$K_{1,2}$	3	3		
$2K_2$	5	5		
C_3	6		11	
P_4	5		5	
$K_{1,3}$	6		6	
$K_{1,2} \cup K_2$	6		6	
$3K_2$	8		8	
C_4	6	4		6
$K_{1,4}$	7	5		7
P_5	6	5		6
$C_3 \cup K_2$	7	6		8
$2K_{1,2}$	7	6		7
$P_4 \cup K_2$	8	6		8
$K_{1,3} \cup K_2$	7	7		8
$K_{1,2} \cup 2K_2$	9	7		9
$4K_2$	11	9		11
$MK_{1,4}$	6	5		6
$C_3 + e$	7	4		7

In [11] the barycentric Ramsey numbers for stars $BR(K_{1,k},\mathbb{Z}_p)$ are studied and some values and bounds are given. Table 2 [11] summarizes the values known at present for $BR(K_{1,k},\mathbb{Z}_p)$ with p prime. In this table, n denotes the order of G.

In [14] the authors give Table 3 with the barycentric Ramsey numbers for graphs with at most four edges and $\mathbb{Z}_n$ with $2 \leq n \leq 5$.

In [8] Caro gives the following open problem:

Table 2: Barycentric Ramsey numbers for stars

k	G	$BR(K_{1,k}, G)$
2	odd order	$n+2$
	even order	$n+1$
k	$\mathbb{Z}_2$	$k+1$
$k=0 \pmod 3$	$\mathbb{Z}_3$	$k+3$
$k \neq 0 \pmod 3$		$k+2$
3	$\mathbb{Z}_p$, p prime ≥ 5	$\leq 2\lceil \frac{p}{3} \rceil + 2$
	$\mathbb{Z}_5$	6
	$\mathbb{Z}_7$	8
	$\mathbb{Z}_{11}$	10
	$\mathbb{Z}_{13}$	10
k	$\mathbb{Z}_p$	$\leq p+k$
$4 \leq k \leq p-1$	$\mathbb{Z}_p$	$\leq p+k-1$
$p-1$	$\mathbb{Z}_p$, p prime ≥ 5	$2p-2$
4	$\mathbb{Z}_7$	9
$tp+4 \leq k \leq tp+p-1$	$\mathbb{Z}_p$, p prime ≥ 5	$\leq p+k-1$
9	$\mathbb{Z}_5$	13
$tp+1$, $t>0$	$\mathbb{Z}_p$	$(t+1)p$
$5t+2$	$\mathbb{Z}_5$	$5(t+1)$

Problem 2 ([8]). Let H be a graph and suppose $n < m$ are integers which both divide $e(H)$. Is it true that $R(H, \mathbb{Z}_n) < R(H, \mathbb{Z}_m)$?.

For the classical Ramsey numbers, the monotonicity property is true in the following two aspects:

- If G is a subgraph of H, then $R(G,k) \leq R(H,k)$.
- If $k < m$, then $R(G,k) < R(G,m)$.

There are examples of graphs G and H such that $2 \mid e(G), 2 \mid e(H)$, G is a subgraph of H and yet $R(H, \mathbb{Z}_2) < R(G, \mathbb{Z}_2)$ [5]. It is clear that the Caro open problem is true in the case $n \mid m$. For the barycentric Ramsey numbers, Table 3 shows several examples contradicting the monotonicity property.

2 Results

The objective of our paper is to compute $R(H, \mathbb{Z}_2^s)$ for some values of s and graphs with $e(H) = 0 \pmod 2$ and to compute $BR(H, \mathbb{Z}_n)$ for the following graphs with $e(H) = 5$ and $2 \leq n \leq 4$.

Graphs with five edges: $5K_2$; $K_{1,5}$; $K_{1,2} \cup 3K_2$; $K_{1,3} \cup 2K_2$; $2K_{1,2} \cup K_2$; $P_5 \cup K_2$; $C_3 \cup 2K_2$; $C_3 \cup K_{1,2}$; $C_4 \cup K_2$; $K_{1,4} \cup K_2$; $C_3 + P_3$: vertices a, b, c, d, e and edges ab, bc, ac, cd, de;

Table 3: Barycentric Ramsey numbers for graphs with at most four edges

H	$BR(H,\mathbb{Z}_2)$	$BR(H,\mathbb{Z}_3)$	$BR(H,\mathbb{Z}_4)$	$BR(H,\mathbb{Z}_5)$
$K_{1,2}$	3	5	5	7
$2K_2$	5	6	7	8
C_3	3	11	6	$[51, 126]$
P_4	4	5	5	5
$K_{1,3}$	4	6	6	6
$K_{1,2} \cup K_2$	5	6	6	6
$3K_2$	6	8	8	8
C_4	4	5	6	7
$K_{1,4}$	5	6	7	8
P_5	6	5	6	6
$C_3 \cup K_2$	6	5	8	7
$2K_{1,2}$	6	6	7	7
$P_4 \cup K_2$	6	6	8	8
$K_{1,3} \cup K_2$	7	6	8	7
$K_{1,2} \cup 2K_2$	7	7	9	9
$4K_2$	9	8	11	11
$MK_{1,4}$	5	5	6	6
$C_3 + e$	4	4	7	7

$C_3 + K_{1,2}$: vertices a, b, c, d, e and edges ab, bc, ac, cd, ce;
$C_4 + e$: vertices a, b, c, d, e and edges ab, bc, cd, da, bd;
$C_3 + 2e$: vertices a, b, c, d, e and edges ab, bc, ca, bd, ce;
$K_{1,3} + K_{1,2}$: vertices a, b, c, d, e, f and edges ab, bc, bd, de, df;
$K_{1,3} + P_3$: vertices a, b, c, d, e, f and edges ab, bc, bd, de, ef;
$MK_{1,4} \cup K_2$; P_6; C_5; $P_4 \cup 2K_2$; $P_4 \cup K_{1,2}$; $K_{1,3} \cup K_{1,2}$;
$P_5 + e$: vertices a, b, c, d, e, f and edges ab, bc, cd, de, cf;
$K_4 - e$: is the complete graph K_4 where some $e \in E(K_4)$ is deleted;
$(C_3 + e) \cup K_2$ and tH denotes t disjoint copies of H.

The results are summarized in the following two tables. Table 4 contains the computed values of $BR(H, G)$, for $G = \mathbb{Z}_n$, $2 \le n \le 4$ and the graphs listing in **Graphs with five edges.**

Some values of the table 4 are obtained directly from $R(H, \mathbb{Z}_n)$, due to the fact that $BR(H, \mathbb{Z}_n) = R(H, \mathbb{Z}_n)$ when $e(H) = 0 \pmod{exp(G)}$ or from $BR(H, \mathbb{Z}_2)$ using Remark 3 and Theorem 4. Table 5 contains the values of $R(H, G)$, with $G = \mathbb{Z}_2^s$ and for some graphs H with $e(H) = 0$ (mod 2). The upper bounds were computed manually by cases. Each case with its particular degree of difficulty was treated using the lemmas and Remarks given in Section 3. For the lower bounds, we use ad hoc decomposition of a complete graph to color its edges.

Table 4: Barycentric and zero-sum Ramsey numbers for graphs with $e(H) = 0 \pmod 2$

H	$BR(H,\mathbb{Z}_2)$	$BR(H,\mathbb{Z}_3)$	$BR(H,\mathbb{Z}_4)$
$5K_2$	10	11	11
P_6	6	6	6
C_5	5	5	6
$P_4 \cup K_{1,2}$	7	7	7
$C_3 \cup K_{1,2}$	6	6	6
$C_3 + 2e$	5	5	5
$C_3 + K_{1,2}$	5	5	6
$C_3 + P_3$	5	5	6
$P_5 + e$	6	6	6
$K_{1,2} \cup 3K_2$	9	9	9
$2K_{1,2} \cup K_2$	8	8	9
$P_5 \cup K_2$	7	7	7
$P_4 \cup 2K_2$	8	8	8
$C_3 \cup 2K_2$	7	7	7
$C_4 \cup K_2$	6	6	6
$MK_{1,5}$	6	6	6
$K_{1,5}$	6	7	7
$K_{1,3} \cup 2K_2$	8	8	8
$C_4 + e$	5	5	5
$(C_3 + e) \cup K_2$	6	6	7
$MK_{1,4} \cup K_2$	7	7	7
$K_{1,4} \cup K_2$	7	7	7
$K_{1,3} \cup K_{1,2}$	7	7	7
$K_{1,3} + P_3$	6	5	6
$K_{1,3} + K_{1,2}$	6	6	6
$K_4 - e$	4	4	6

3 Tools

In this section, the more relevant processes used to compute some of the values of Table 4 and Table 5 are summarized.

The following remark and theorem are used.

Remark 3. Let H be a graph and $e(H)$ the number of its edges. Then
$$BR(H,\mathbb{Z}_2) = \begin{cases} |V(H)| & \text{if } e(H) \text{ is odd} \\ R(H,\mathbb{Z}_2) & \text{if e(H) is even} \end{cases}$$

Theorem 4 (Caro [5]). *Let H be a graph on h vertices and an even number of edges. Then*
$$R(H,\mathbb{Z}_2) = \begin{cases} h+2 & \text{if } H = K_h, h = 0,1 \pmod 4 \\ h+1 & \text{if } H = K_p \cup K_q, \binom{p}{2} + \binom{q}{2} = 0 \pmod 2 \\ h+1 & \text{if all the degrees in } H \text{ are odd} \\ h & \text{otherwise.} \end{cases}$$

We have the following results for stars and matchings.

Theorem 5 (Bialostocki and Dierker [2]).

Table 5: Barycentric and zero-sum Ramsey numbers for graphs with five edges

H	$BR(H,\mathbb{Z}_2)$	$BR(H,\mathbb{Z}_2^2)$	$BR(H,\mathbb{Z}_2^s)$
$2tK_2$	$4t+1$	$4t+2,\ t \geq 2$	
$2K_2$	5	7	2^s+3
C_4	4	5	
P_5	5	5	
$K_{1,2n}$	$2n+1$	$2n+3$	
$K_{1,2}$	3	5	2^s+1
C_3+e	5	5	
$tK_{1,2}$	$3t$	$3t,\ t \geq 2$	
$t(K_{1,3} \cup K_2)$	$6t+1$	$6t+2$	
$t(P_4 \cup K_2)$	$6t$	$6t$	
K_n	$n+2$	$\geq n+4$	
$K_a \cup K_b, a \geq b \geq 2$	$a+b+1$	$\geq a+b+3$	

1. $R(K_{1,m},\mathbb{Z}_m) = R(K_{1,m},2) = \begin{cases} 2m & \textit{if } m \textit{ is odd} \\ 2m-1 & \textit{if } m \textit{ is even} \end{cases}$
2. $R(mK_2,\mathbb{Z}_m) = R(mK_2,2) = 3m-1.$

Theorem 6 (Caro [9]). *Let $K_{1,m}$ be a star on m edges with $m = 0$ (mod n). Then*

$$\begin{aligned} BR(K_{1,m},\mathbb{Z}_n) &= R(K_{1,m},\mathbb{Z}_n) \\ &= \begin{cases} m+n-1 & \textit{if } m=n=0 \pmod 2 \\ m+n & \textit{otherwise} \end{cases} \end{aligned}$$

Theorem 7 (Bialostocki and Dierker [3]). *Let mK_2 be a matching on m edges with $m = 0$ (mod n). Then*

$$BR(mK_2,\mathbb{Z}_n) = R(mK_2,\mathbb{Z}_n) = 2m+n-1.$$

We use the following lemmas:

Lemma 8 ([2]). *If the edges of K_n, where $n \geq 5$, are colored by at least three colors, then there exists a path on three edges each one colored differently.*

Lemma 9 ([4]). *If the edges of K_5 are colored with any number of colors, then K_5 contains either a path of length 3 using only one color or a path of length 3 using 3 different colors.*

Remark 10. Recalling the definition of Ramsey numbers, and since $R(C_4, 2) = 6$, if the edges of K_n, $n \geq 6$, are colored with exactly two colors, then there exists a monochromatic C_4.

Throughout the paper, we use the convention that the letters a, b, c and d always represent *distinct* colors.

Lemma 11. *If the edges of K_5 are colored with at least three different colors and contain a C_4 using two colors, one of them repeated three times, then there exists a C_4 using exactly three different colors.*

Proof. Set $f : E(K_5) \to \{a, b, c\}$ and set $C_4 = v_1v_2v_3v_4v_1$ in K_4 with $f(v_1v_2) = b$ and $f(v_2v_3) = f(v_3v_4) = f(v_4v_1) = a$. If $f(v_2v_4) \notin \{a, b\}$, then there will be a C_4 with exactly 3 colors. Therefore $f(v_2v_4) \in \{a, b\}$, and now, if $f(v_3v_1) \notin \{a, b\}$, there will again be a C_4 with exactly 3 colors. So we see $f(v_3v_1) \in \{a, b\}$. Let v_5 be the fifth vertex. If $f(v_5v_1) \notin \{a, b\}$, then we once more obtain a C_4 with exactly 3 colors, so $f(v_5v_1) \in \{a, b\}$. By symmetry, this also shows $f(v_5v_2) \in \{a, b\}$, and now, since f uses at least three colors, we see that w.l.o.g (by symmetry) $f(v_5v_4) \notin \{a, b\}$, in which case $v_2v_4v_5v_1$ gives a C_4 with exactly 3 colors.

□

Lemma 12 ([10]). *If the edges of K_5 are colored with three or four colors, then K_5 contains a C_4 with at least three colors.*

Proof. Consider any K_4 in K_5. If the edges of this K_4 are covered by at least three colors and do not contain a C_4 with one or three colors, then each two opposite edges have the same color. Notice that any P_3 in this K_4 has two different colors. Hence, we can connected a new vertex v_5 to K_4 with an edge colored, say c, in order to form a P_4 with three different colors. Therefore from this P_4, we can derive a C_4 with three or four colors. Notice that there exists a K_4 colored with at least three colors.

□

The following lemma is useful to establish the barycentric Ramsey numbers for small graphs. Notice that barycentric sequences are invariant of affine transformation and this greatly reduces the numbers of cases to be checked in Table 6.

Lemma 13. *Let H be a graph with $2 \leq e(H) \leq 5$ colored by elements of $\mathbb{Z}_n$ $(2 \leq n \leq 5)$. Table 6 can be seen as a matrix $M_{4\times4}$ where each entry $M(e(H), \mathbb{Z}_n)$ is a set where its elements show all the different coloring for $E(H)$ in order to obtain a barycentric H. For example, in case $e(H) = 3$ and edges are colored by elements of $\mathbb{Z}_4$, we have the following:*
$M(3, \mathbb{Z}_4) = \{a, b, c\ ;\ a, a, a+2\ ;\ a, a, a\}$, *i.e., H is barycentric when edges*

Table 6: Barycentric graph colorings

$e(H)$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$
2	a,a	a,a	a,a	a,a
3	any coloring	a,a,a	a,a,a	a,a,a
		a,b,c	$a,a,a+2$	a,b,c
			a,b,c	
4	a,a,a,a	a,a,a,a	a,a,a,a	a,a,a,a
	a,a,b,b	a,a,a,b	$a,a,a+1,a+3$	a,a,b,c
		a,a,b,c	$a,a,a+2,a+2$	
5	any coloring	a,a,a,a,a	a,a,a,a,a	a,a,a,a,a
		a,a,a,b,b	a,a,a,a,b	$a,a,a+1,a+2,a+2$
		a,a,a,b,c	$a,a,a,a+2,a+2$	$a,a,a+1,a+1,a+3$
		a,a,b,b,c	a,a,b,b,c	$a,a+1,a+1,a+1,a+2$
			$a,a+1,a+1,a+1,a+2$	$a,a+2,a+2,a+2,a+4$
			a,a,b,c,d	a,b,c,d,e

are colored with three different colors a,b,c, or edges are colored by $a,a,a+2$ for any color a, or edges are colored by a,a,a for any color a.

From Lemma 13, we can formulate the following remark.

Remark 14.

1. Any 3-sequence with zero-sum in $\mathbb{Z}_3$ is formed by *aaa* or *abc*.

2. In $\mathbb{Z}_3$, any 3-sequence with zero-sum can be extended to a 5-barycentric sequence adding any two elements, except for case *aaa* where *ab* cannot be added.

3. In $\mathbb{Z}_4$, any 4-barycentric sequence can be extended to a 5-barycentric sequence by adding any element of $\mathbb{Z}_4$.

4. Let H be a graph such that $e(H) = 5$ and let H_1 be a subgraph of H with $e(H_1) = 4$. Then $BR(H, \mathbb{Z}_4) \leq \max\{BR(H_1, \mathbb{Z}_4), |V(H)|\}$.

5. Any 5-barycentric sequence in $\mathbb{Z}_5$, except *abcde*, contains a 4-barycentric subsequence.

6. $BR(H, \mathbb{Z}_n) \leq BR(H, \mathbb{Z}_m)$, when $n < m$ and $M(e(H), \mathbb{Z}_m) \subseteq M(e(H), \mathbb{Z}_n)$. For example $BR(H, \mathbb{Z}_3) \leq BR(H, \mathbb{Z}_5)$ for every H with $e(H) = 4$.

In the next theorem, we give a general method to obtain lower bounds for $R(H, \mathbb{Z}_2^2)$ for certain graphs H.

Theorem 15.

1. *If $e(K_n) = 0 \pmod 2$, then $R(K_n, \mathbb{Z}_2^2) \geq n + 4$.*

2. *If $a \geq b \geq 2$ and $e(K_a \cup K_b) = 0 \pmod 2$, then $R(K_a \cup K_b, \mathbb{Z}_2^2) \geq a + b + 3$.*

Proof.

1. Take $A \subseteq V(K_{n+3})$ with $|A| = n-2$ and $B \subseteq V(K_{n+3}) \setminus A$ such that $B = \{v_1, v_2, v_3, v_4, v_5\}$. Color all edges inside A and edges from A to B with $(0,0)$. Color all edges from v_1 to $\{v_2, v_3, v_4, v_5\}$ by $(1,1)$. Color all edges from v_2 to $\{v_3, v_4, v_5\}$ by $(1,0)$. Color all edges from v_3 to $\{v_4, v_5\}$ by $(0,1)$ and color the edge from v_4 to v_5 by $(0,1)$. It is easy to check that no K_2, K_3, K_4, nor K_5 in B is zero-sum. Hence there is no zero-sum K_n and $R(K_n, \mathbb{Z}_2^2) \geq n+4$.

2. Set $n = a+b$. Take $A \subseteq V(K_{n+2})$ and $B \subseteq V(K_{n+2}) \setminus A$ such that $|A| = n-3$ and $B = \{v_1, v_2, v_3, v_4, v_5\}$. Color all edges inside A and edges from A to B with $(0,0)$. Color all edges from v_1 to $\{v_2, v_3, v_4, v_5\}$ by $(1,1)$. Color all edges from v_2 to $\{v_3, v_4, v_5\}$ by $(1,0)$. Color all edges from v_3 to $\{v_4, v_5\}$ by $(0,1)$, color the edges from v_4 to v_5 by $(0,1)$. It is easy to see that no K_2, K_3, K_4, K_5, $K_3 \cup K_2$, nor $K_2 \cup K_2$ in B is zero-sum. Hence $R(K_a \cup K_b, \mathbb{Z}_2^2) \geq a+b+3$.

□

For the next theorem, we need the following definition

Definition 16. A sequence $S = a_1, \cdots, a_k$ in $\mathbb{Z}_m$ is called a *non-vanishing k-sequence* in $\mathbb{Z}_m$ if, for every subsequence A of S , $|A| = j \in \{1, 2, \cdots, k\}$, and for every r with $0 \leq r \leq \frac{j(j-1)}{2}$, we have $\sum_{a_i \in A} a_i \neq r \pmod{m}$.

Remark 17. The concept of non-vanishing k-sequence is a hereditary property for any shorter subsequence. That is to say, a non-vanishing k-sequence in $\mathbb{Z}_m$ is by definition a non-vanishing j-sequence in $\mathbb{Z}_m$ for any $1 \leq j \leq k$.

Examples:
1) Every non-zero element is a non-vanishing 1-sequence in $\mathbb{Z}_m$.
2) $S = 1, 1$ is a non-vanishing 2-sequence in $\mathbb{Z}_3$.
3) $S = 2, 3, 4$ is a non-vanishing 3-sequence in $\mathbb{Z}_m$ for every $m \geq 10$. However, this is not true in $\mathbb{Z}_9$ and $\mathbb{Z}_8$, since $2+3+4 = 9$ and $2+3+4-1 = 8$ ($r = 1$ in the Definition 16).
We need the following theorem.

Theorem 18. *Let D be an abelian group and let $G = D \oplus \mathbb{Z}_m$. Let H be a graph and let k be a positive integer such that every $k-$subset of the degree sequence of H forms a non-vanishing k-sequence in $\mathbb{Z}_m$. Then we have $R(H, G) \geq R(H, D) + k$. In particular, if all degrees of H are odd, then $R(H, \mathbb{Z}_2^n) \geq |V(H)| + n$.*

Proof. We consider the elements of G as pairs (x, y) with $x \in D$ and $y \in \mathbb{Z}_m$. Consider the extremal coloring f of $E(K_{R(H,D)-1})$ that avoids a zero-sum copy of H (mod D), and add k vertices $u_1, \cdots, u_k$ so that we have $R(H, D) + k - 1$ vertices. Color the edges between $u_1, \cdots, u_k$ and all the edges from $u_1, \cdots, u_k$ to $K_{R(H,D)-1}$ by the vector $(0, 1)$, and color the rest by the coloring f in the first coordinate and 0 in the second coordinate. Clearly no zero-sum copy of H exists because, if some of $\{u_1, \cdots, u_k\}$ are vertices of H, then, by hypothesis, the sum of the second coordinate is not 0, and if we do not use any of $\{u_1, \cdots, u_k\}$, then the sum of the first coordinate on any copy of H in $K_{R(H,D)-1}$ avoids a zero-sum copy of H by the definition of f. Hence $R(H, G) \geq R(H, D) + k$.

If all degrees of H are odd (and $e(H) = 0 \pmod 2$), then this is just a case of a non-vanishing 1-sequence in $\mathbb{Z}_2$, and then $R(H, \mathbb{Z}_2) \geq |V(H)| + 1$ (by Theorem 4); the rest follows by induction on n using the first part of the theorem.

□

4 Some proofs and comments

The computation of the Ramsey number of a graph is not an easy task since there are no specific techniques. In consequence, ad hoc methods must be developed for each case. One of the most used methods is the proof by case, being also the one that consumes the most computational effort.

Nevertheless, several techniques are shown in the Tools Section that help to compute Tables 4 and 5. Most of the values of Table 4 have been computed using known Ramsey numbers and Remark 14.

- From Remark 14(2), we obtain the following:

1.1. $BR(C_3 \cup K_{1,2}, \mathbb{Z}_3) = 6$: Set $f : E(K_6) \to \mathbb{Z}_3$ and $V(K_6) = \{v_1, \cdots, v_6\}$. Since $BR(K_{1,2} \cup K_2, \mathbb{Z}_3) = 6$, there exists in K_6 a $K_{1,2} \cup K_2$ $= v_1v_2v_3 \cup v_4v_5$ that is barycentric, i.e., a-monochromatic or with three colors. Hence, by Remark 14(2), we can derive in K_6 a barycentric $C_3 \cup K_{1,2}$, except when $K_{1,2} \cup K_2$ is a-monochromatic and $f(v_3v_1)$ $= b$ and $f(v_5v_6) = a$ (or vice versa). If $f(v_4v_6) = a$, then $C_3 \cup K_{1,2} = v_4v_5v_6v_4 \cup v_1v_2v_3$ is a-monochromatic.
If $f(v_4v_6) \in \{b, c\}$, then $C_3 \cup K_{1,2} = v_4v_5v_6v_4 \cup v_2v_1v_3$ is barycentric.

1.2. $BR(C_3{+}2e, \mathbb{Z}_3) = 5$: Follows directly from the fact that $BR(P_4, \mathbb{Z}_3)$ $= 5$ and Remark 14(2).

1.3. $BR(K_{1,2} \cup 3K_2, \mathbb{Z}_3) = 9$: $f : E(K_9) \to \mathbb{Z}_3$. Let us consider $V(K_9) = \{v_1, \cdots, v_9\}$. Since $R(K_{1,2} \cup K_2, \mathbb{Z}_3) = 6$, there exists in

K_9 a zero-sum $K_{1,2} \cup K_2 = v_1v_2v_3 \cup v_4v_5$. If $K_{1,2} \cup K_2$ is a-monochromatic, then by Remark 14(2) we have the theorem, except when $f(v_6v_7) = a$ and $f(v_8v_9) = b$; in this case, for any values of $f(v_1v_9)$ and $f(v_7v_8)$, we have the theorem except when $f(v_1v_9) = a$ and $f(v_7v_8)$ and vice versa. If $f(v_6v_8) = a$, we have a $K_{1,2} \cup 3K_2$ that is a-monochromatic and, when $f(v_6v_8) \in \{b, c\}$, we also have the theorem.

1.4. $BR(2K_{1,2} \cup K_2, \mathbb{Z}_3) = 8$: Set $f : E(K_8) \to \mathbb{Z}_3$ and consider now $V(K_8) = \{v_1, \cdots, v_8\}$. Since $R(K_{1,2} \cup K_2, \mathbb{Z}_3) = 6$, there exists in K_8 a $K_{1,2} \cup K_2 = v_1v_2v_3 \cup v_4v_5$ that is a-monochromatic or with three different colors. By Remark 14(2), there exists a $2K_{1,2} \cup K_2$ barycentric, except when $K_{1,2} \cup K_2$ is a-monochromatic and $f(v_6v_7) = b$ and $f(v_7v_8) = a$ (or vice versa). Then, for any color of $f(v_6v_8) = a$, we can derive a barycentric $2K_{1,2} \cup K_2$.

1.5. $BR(P_5 \cup K_2, \mathbb{Z}_3) = 7$: Follows directly from the fact that $BR(P_4, \mathbb{Z}_3) = 5$ and Remark 14(2).

1.6. $BR(P_4 \cup 2K_2, \mathbb{Z}_3) = 8$: Follows directly from the fact that $BR(P_4, \mathbb{Z}_3) = 5$ and Remark 14(2).

1.7. $BR(MK_{1,5}, \mathbb{Z}_3) = 6$: Since $R(P_4, \mathbb{Z}_3) = 5$, for any $f : E(K_6) \to \mathbb{Z}_3$, there exists a zero-sum $P_4 = v_1v_2v_3v_4$ in K_6 with $V(K_6) = \{v_1, \cdots, v_6\}$. Assuming P_4 is a-monochromatic, $f(v_3v_5) \neq a$ and $f(v_3v_6) \neq a$. Hence by Remark 14(2), we have the theorem. If $f(v_3v_5) = f(v_3v_6) = a$, we have the theorem. Now let $f(v_3v_5) = a$ and $f(v_3v_6) = b$. If $f(v_1v_6) = f(v_1v_5) = f(v_1v_4) = a$ then $M_{1,5} = v_1v_5, v_1v_6, v_1v_4, v_1v_2v_3$ is barycentric. Else, assume that $f(v_1v_5) = b$ then $M_{1,5} = v_3v_2, v_3v_4, v_3v_6, v_3v_5v_1$ is barycentric.

- From Remark 14(3), we obtain the following:

2.1. $BR(P_6, \mathbb{Z}_4) = 6$: Follows directly using the fact that $BR(P_5, \mathbb{Z}_4) = 6$ and Remark 14(3).

2.2. $BR(P_4 \cup K_{1,2}, \mathbb{Z}_4) = 7$: Follows directly using the fact that $BR(2K_{1,2}, \mathbb{Z}_4) = 7$ and Remark 14(2).

2.3. $BR(K_{1,2} \cup 3K_2, \mathbb{Z}_4) = 9$: Follows directly using the fact that $BR(K_{1,2} \cup 2K_2, \mathbb{Z}_4) = 9$ and Remark 14(3).

2.4. $BR(2K_{1,2} \cup K_2, \mathbb{Z}_4) = 9$: Follows directly using the fact that $BR(2K_{1,2}, \mathbb{Z}_4) = 7$ and Remark 14(3).

- From Remark 14(4), we obtain the following:

3.1. $BR(P_6, \mathbb{Z}_3) = \max\{BR(P_4, \mathbb{Z}_3), |V(P_6)|\} = 6$.

3.2. $BR(P_5 \cup K_2, \mathbb{Z}_4) = 7$: It follows by Remark 14(4) that we have $BR(MK_{1,5}, \mathbb{Z}_4) = 6$.

3.3. $BR(MK_{1,5}, \mathbb{Z}_4) = 6$: Since $BR(MK_{1,4}, \mathbb{Z}_4) = 6$ and $MK_{1,4}$ is a subgraph of $MK_{1,5}$ with 4 edges then by Remark 14 (4) we have that $BR(MK_{1,5}, \mathbb{Z}_4) = 6$.

Theorem 19. $BR(C_3 \cup K_{1,2}, \mathbb{Z}_4) = 6$.

Proof. It is clear that $BR(C_3 \cup K_{1,2}, \mathbb{Z}_4) \geq 6$. Since $BR(C_3, \mathbb{Z}_4) = 6$, for any $f : E(K_6) \to \mathbb{Z}_4$, there exists a barycentric C_3. Then its edges are colored aaa, $aa\,a+2$ or abc. Let v_1, v_2, v_3, v_4, v_5 and v_6 be vertices of K_6, and let $C_3 = v_1v_2v_3v_1$ and $V(K_3) = \{v_4, v_5, v_6\}$. We have the following cases:

1. C_3 is a-monochromatic. If there exists an edge in $E(K_3)$ colored by a, then we have the theorem. Suppose that no side of $E(K_3)$ is colored by a. If there exists in $E(K_3)$ exactly two edges with the same color or the edges of $E(K_3)$ are colored with three different colors, or K_3 is $a+2$-monochromatic , we have the theorem. Assume now that K_3 is b-monochromatic with $b \in \{a+1, a+3\}$. Let $K_{3,3}$ be the bipartite graph from $V(C_3)$ to $V(K_3)$. Then we have the following subcases:

   1.1 There exists some edge in $K_{3,3}$ colored by $a+2$ or $a+3$. If $K_{3,3}$ is $a+2$-monochromatic or $a+3$-monochromatic we have the theorem. Assume w.l.o.g that $f(v_1v_4) = a+2$. If we have $f(v_1v_5) \neq f(v_3v_6)$, we are done, else we have

   - $f(v_1v_5) = f(v_3v_6) = a$. If $f(v_2v_4) \neq a$, we are done. If $f(v_2v_4) = a$, then for any color of v_3v_5 we have the theorem.
   - $f(v_1v_5) = f(v_3v_6) = a+1$. Then, for any color of $f(v_2v_4)$, we are done.
   - $f(v_1v_5) = f(v_3v_6) = a+2$. Hence $f(v_2v_4) = a+2$, else we have the theorem. Therefore, for any color of $f(v_5v_3)$, we have the theorem.
   - $f(v_1v_5) = f(v_3v_6) = a+3$. In this case, we also have the theorem.

1.2 All edges of $K_{3,3}$ are colored by a or $a+1$, with $a \in \mathbb{Z}_4$. In this case, K_6, is colored with two colors. If $K_{3,3}$ is a-monochromatic or $a+1$-monochromatic we have the theorem. Assume w.l.o.g that $f(v_1v_4) = a$. Then

- If $f(v_1v_5) = f(v_3v_6) = a$, we are done.
- $f(v_1v_5) = f(v_3v_6) = a+1$. If $f(v_2v_4) \in \{a, a+1\}$, then $f(v_5v_3) = a+1$, otherwise we have the theorem; hence $f(v_2v_6) = a+1$. Therefore, for any color of v_2v_5 we are done.
- $f(v_1v_5) = a$ and $f(v_3v_6) = a+1$. If $f(v_2v_4) = a$, then $f(v_5v_3) = a+1$. In consequence, $f(v_2v_6) = a+1$ and $f(v_1v_6) = a$, else we have the theorem. Hence, for any color of v_2v_5, we have the theorem. If $f(v_2v_4) = a+1$, then $f(v_5v_3) = a$ and hence $f(v_2v_6) = a+1$, else we are done. Hence, for any color of edge v_2v_5, we have the theorem.
- $f(v_1v_5) = a+1$ and $f(v_3v_6) = a$. Then $f(v_2v_4) = a+1$, else we are done, and now, for any color of edge v_3v_5, we have the theorem.

2. C_3 is colored by $a, a, a+2$ or a, b, c. Hence, for any color of edges of K_3, we have the theorem.

□

The computation of the values of Table 5 are more complex, and most of the details are described in this section.

Theorem 20. *Let G be an abelian group of order $n \geq 2$. Then $BR(2K_2, G) = n+3$. In particular $BR(2K_2, \mathbb{Z}_2^s) = R(2K_2, \mathbb{Z}_2^s) = 2^s + 3$.*

Proof. The complete graph K_{n+2} can be decomposed into the edge-disjoint union of $n-1$ stars and one triangle K_3, i.e., a complete graph with three vertices. For the lower bound, we color each star $K_{1,i}$ in K_{n+2} by c_{i-1} for $3 \leq i \leq n+1$ and the edges of K_3 by c_1.

For the upper bound, notice that the stars and triangles are the only graphs that have no two independent edges. Hence, it is sufficient to show that we cannot cover the edges of K_{n+3} with n colors using q stars and p triangles such that $p+q=n$. The q stars cover completely at most a K_q and the remaining edges of the bipartite $K_{q,n+3-q}$. So we have to cover the edges of K_{n+3-q} by $n-q$ triangles. Counting edges, we find that $\frac{(n+3-q)(n+2-q)}{2}$ edges must be covered by $3(n-q)$ edges of the triangle, namely we must have the following expression: $3(n-q) \geq \frac{(n+3-q)(n+2-q)}{2}$ from which we get $n^2 - (1+2q)n + q^2 + q + 6 \leq 0$. But this quadratic equation has no real root, showing that the left side is always greater than 0, a contradiction.

□

Theorem 21. $BR(5K_2, \mathbb{Z}_3) = 11$.

Proof. Since $R(3K_2, \mathbb{Z}_3) = 8$, for any $f : E(K_{11}) \to \mathbb{Z}_3$, there exists in K_{11} a zero-sum $3K_2$. Let $v_1 \cdots v_{11}$ be the vertices of K_{11} and set $3K_2 = v_1v_2, v_3v_4, v_5v_6$. By Remark 14(2), it is sufficient to study the case when $3K_2$ is a-monochromatic and one of the values $f(v_7v_8)$ or $f(v_9v_{10})$ is a and the other one is b. Assume that $f(v_7v_8) = a$ and $f(v_9v_{10}) = b$. Then, in order to forbid a barycentric $5K_2$ in K_{11}, we must have w.l.o.g. that $f(v_7v_9) = a$, $f(v_8v_{10}) = b$, $f(v_8v_9) = a$ and $f(v_7v_{10}) = b$. Hence, for any color of v_8v_{11}, we have a barycentric $5K_2$. Therefore $BR(5K_2, \mathbb{Z}_3) \leq 11$.

For the lower bound, let $v_1, \cdots, v_{10}$ be the vertices of K_{10}. Let K_8 be a complete graph in K_{10} with vertices $v_1, \cdots, v_8$. Let $K_{8,2}$ be the bipartite complete graph with edges v_iv_j with $i \in \{1, 2, \cdots, 8\}$ and $j \in \{9, 10\}$. Set $K_2 = v_9v_{10}$. We color the edges of K_8 by a, $f(v_iv_9) = a$ and $f(v_iv_{10}) = b$ for $i \in \{1, \cdots, 8\}$ and $f(v_9v_{10}) = b$. Therefore any $5K_2$ in K_{10} is colored by a, a, a, a, b and hence not barycentric in $\mathbb{Z}_3$. Therefore $BR(5K_2, \mathbb{Z}_3) \geq 11$. □

Theorem 22. $BR(5K_2, \mathbb{Z}_4) = 11$.

Proof. Since $BR(4K_2, \mathbb{Z}_4) = 11$, it follows that, for any $f : E(K_{11}) \to \mathbb{Z}_4$, there exists in K_{11} a barycentric $4K_2$. Let $v_1, \cdots, v_{11}$ be the vertices of K_{11} and let $4K_2 = v_1v_2, v_3v_4, v_5v_6, v_7v_8$ be the barycentric graph. Hence, by Remark 14(3), we find that $5K_2 = v_1v_2, v_3v_4, v_5v_6, v_7v_8, v_9v_{10}$ is barycentric, so that $BR(5K_2, \mathbb{Z}_4) \leq 11$.

For the lower bound, let $v_1, \cdots, v_{10}$ be the vertices of K_{10} and let K_7 and K_3 be two complementary complete graphs in K_{10} with vertices $\{v_1, \cdots, v_7\}$ and $\{v_8, v_9, v_{10}\}$, respectively. Let $K_{7,2}$ be the bipartite complete graph with edges v_iv_j with $i \in \{1, 2, \cdots, 7\}$ and $j \in \{9, 10\}$. We color the edges of K_7 by a, and the edges of K_3 and $K_{7,2}$ by $a+1$, respectively. It is easy to see that with this coloration a barycentric $5K_2$ does not exist in K_{10}. Therefore $BR(5K_2, \mathbb{Z}_4) \geq 11$.

□

Theorem 23. $BR(2tK_2, \mathbb{Z}_2^2) = R(2tK_2, \mathbb{Z}_2^2) = 4t + 2$ *for* $t \geq 2$.

Proof. By Theorem 20, we have $R(2K_2, \mathbb{Z}_2^2) = 7$. Notice that we have the following:

1. $2tK_2$ is a graph on $4t$ vertices all with degree 1 (odd). Hence, by the general lower bound given by Theorem 18, we have the following $R(2tK_2, \mathbb{Z}_2^2) \geq 4t + 2$.

2. Suppose we have proved $R(4K_2, \mathbb{Z}_2^2) = 10$. Then, by induction for $t = 2$, this is true (i.e. the case $R(4K_2, \mathbb{Z}_2^2) = 10$). Assuming it holds for t, let us prove it now for $t + 1$ and consider a coloring by $\mathbb{Z}_2^2$ of $E(K_{4(t+1)+2})$. Since $4(t+1)+2 > 7$ and $R(2K_2, \mathbb{Z}_2^2) = 7$, we know that a zero-sum copy of $2K_2$ exists. Take this copy of $2K_2$ and consider the complete graph that remains by deleting the 4 vertices of this $2K_2$. Now we have a coloring of $E(K_{4t+2})$ and, by induction, there exists a zero-sum copy of $2tK_2$, which, together with the $2K_2$ zero-sum components we had before, gives a zero-sum copy of $2(t + 1)K_2$ proving the induction step.

To show that $R(4K_2, \mathbb{Z}_2^2) = 10$, we need the following claim:

Claim: Consider a coloring of the edges of $E(K_6)$ using at most 4 colors such that there exist no two independent edges of the same color.

1. We must use 4 colors.

2. Three colors form a star and one color a triangle, and there are precisely two possible coloring up to change of colors:

   Coloring A.

   Color 1: $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = f(v_1v_6)$.

   Color 2: $f(v_2v_3) = f(v_2v_4) = f(v_2v_5) = f(v_2v_6)$.

   Color 3: $f(v_3v_4) = f(v_3v_5) = f(v_3v_6)$.

   Color 4: $f(v_4v_5) = f(v_4v_6) = f(v_5v_6)$.

   In this case, we have three stars of orders $5, 4$ and 3 respectively and K_3

   Coloring B:

   Color 1: $f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = f(v_1v_6)$.

   Color 2: $f(v_2v_1) = f(v_2v_4) = f(v_2v_5) = f(v_2v_6)$.

   Color 3: $f(v_2v_3) = f(v_3v_4) = f(v_3v_5) = f(v_3v_6)$.

   Color 4: $f(v_4v_5) = f(v_4v_6) = f(v_5v_6)$.

   In this case, we have three stars of orders $4, 4$ and 4 respectively and K_3

3. In coloring A, there is a $3K_2$ colored with three colors for the triple of colors $(1,2,3),(1,2,4),(1,3,4)$ but no $3K_2$ with three colors for $(2,3,4)$. In coloring B, there is a $3K_2$ colored with three colors for all the possible triples of colors $(1,2,3)$, $(1,2,4),(1,3,4)$, $(2,3,4)$.

Proof of the claim

Observe that the only graphs not having two independent edges are stars of various orders and K_3. So we must have all colors forming stars and triangles, that is to say, K_3. Consider a coloring of the 15 edges of K_6.

If all colors are stars, we have four vertices as the centers of these stars, say v_1, v_2, v_3, v_4, but then v_5 and v_6 are not a center of star and edge v_5v_6 is not covered. So we need at least one triangle.

Suppose we have exactly two triangles, hence we have two cases.

The triangles are vertex-disjoint or the triangles have a vertex in common. In both cases, the graphs left to be covered by two stars $E(K_6 \setminus 2K_3)$ or $E(K_6 \setminus 2 * K_3)$, where $2 * K_3$ denotes two triangles having a vertex in common, have three independent edges and hence cannot be covered by two stars.

Hence we finally get that we have precisely three stars and one triangle. It is now routine to check that if there is a star of order 5, coloring A is forced, and if we have three stars of degree 4, then coloring B is forced

Part 3 of the claim is trivially checked.

Hence the claim is proved.

Now, we show that $R(4K_2, \mathbb{Z}_2^2) = 10$.

The lower bound 10 comes from the fact that all degrees are odd and by the general lower bound in Theorem 18.

Consider a coloring by $\mathbb{Z}_2^2$ of $E(K_{10})$.

Case 1: if there is $4K_2$ with all edges of distinct colors, we are done.

Case 2: there is no $3K_2$ with all edges of distinct colors.

But then in K_{10} there is a $5K_2$ colored by just one or two colors.

There are three possibilities: a, a, a, a, a ; a, a, a, a, b; and a, a, a, b, b. In all cases, we have a zero-sum $4K_2$.

So the last case remains:

Case 3: by Case 1, all the edges between A are colored by a, b or c. If there is a pair of independent edges in A colored distinctly, say by a and b, then these edges, along with v_1v_2 and v_3v_4, give the desired zero-sum subgraph. So we may assume otherwise, and w.l.o.g. assume a occurs as the color of two opposite edges. This gives a perfect matching M of K_{10} with three edges having color a, say x_1x_2, x_3x_4 and x_5x_6, and the other two of colors b and c, respectively. Remove the vertices x_3, x_4, x_5 and x_6 (corresponding to the vertices of two of the a-colored edges in M) and apply the claim to the remaining 6 vertices. If there are two independent edges of the same color, then taking them along with the removed edges gives a zero-sum $4K_2$; if there are three independent edges colored with exactly three colors, none of which is a, then taking these three edges along with one of the two removed edges gives a zero-sum $4K_2$. Therefore, we see by the claim that the proof is complete unless we have Coloring A with the 5-star being in color a, and thus the center of the 5-star is one of the two vertices of the third a-colored edge from the matching M not removed, say w.l.o.g. x_1. We conclude that the same holds whenever any two a-colored edges of M are removed, and so assume w.l.o.g. x_3 and x_5 are the resulting centers after each application. But now, regardless of the color of x_2x_4, we find a zero-sum $4K_2$.

□

Theorem 24. $R(P_5, \mathbb{Z}_2^2) = BR(P_5, \mathbb{Z}_2^2) = 5$.

Proof. Since $BR(C_4, \mathbb{Z}_2^2) = R(C_4, \mathbb{Z}_2^2) = 5$, it follows that, for any $f : E(K_5) \to \mathbb{Z}_2^2$, there exists a zero-sum $C_4 \subseteq K_5$. Then its edges are colored $a, a, a, a,$ or $a, a, b, b,$ or a, b, c, d with $a, b, c, d \in \mathbb{Z}_2^2$. For the upper bound, we consider the following cases:

1. C_4 is colored by a, a, a, a. Set $f(v_1v_2) = f(v_2v_3) = f(v_3v_4) = f(v_1v_4) = a$. Then $f(v_1v_5) \in \{b, c, d\}$, else we have a zero-sum P_5. Without loss of generality, suppose that $f(v_1v_5) = b$. Then $f(v_2v_5) \in \{c, d\}$, as otherwise we have a zero-sum P_5 . Without loss of generality, we may assume $f(v_2v_5) = c$. By a similar argument, $f(v_3v_5) \in \{b, d\}$. If $f(v_3v_5) = b$, then $v_2v_3v_5v_1v_4$ is a zero-sum P_5. Thus $f(v_3v_5) = d$, and now we see that $f(v_5v_4) = c$ (since, as seen above, any edge from v_5 to this C_4 must be colored by one of b, c or d and, moreover, one of which is not shared by either adjacent edge from v_5 to C_4). But now $v_3v_5v_4v_1$ is a zero-sum P_5.

2. C_4 is colored by a, a, b, b. Then we have the following subcases:

    2.1. Set $f(v_1v_2) = f(v_2v_3) = a$; $f(v_3v_4) = f(v_1v_4) = b$. Then $f(v_1v_5) \in \{c, d\}$, else we have a zero-sum P_5. Without loss of generality, assume $f(v_1v_5) = c$. Then $f(v_2v_5) = b$, else we have a zero-sum P_5, whence $f(v_3v_5) = c$, and now, regardless of the color of v_1v_4, we still find a zero-sum P_5.

    2.2. Set $f(v_1v_2) = f(v_3v_4) = a$; $f(v_2v_3) = f(v_1v_4) = b$. Then $f(v_1v_5) \in \{c, d\}$, else we have a zero-sum P_5. Without loss of generality, set $f(v_1v_5) = c$. Then $f(v_5v_2) = f(v_5v_3) = f(v_5v_4) = c$ (as, for instance, $f(v_5v_2) = d$ would mean $v_3v_4v_1v_5v_2$ is a zero-sum P_5.

3. C_4 is colored by a, b, c, d. Set $f(v_1v_2) = a$; $f(v_2v_3) = b$; $f(v_3v_4) = c$; $f(v_1v_4) = d$. Then $f(v_1v_5) \in \{b, c\}$, otherwise we get a zero-sum P_5. Set $f(v_1v_5) = b$. Then, with any color of v_2v_5, we have a zero-sum P_5. Set $f(v_1v_5) = c$. Then, for any color of v_4v_5, a zero-sum P_5 is obtained.

The lower bound is obvious.

□

Theorem 25. *$BR(K_{1,2n}, \mathbb{Z}_2^2) = R(K_{1,2n}, \mathbb{Z}_2^2) = 2n + 3$. In particular, $R(K_{1,4}, \mathbb{Z}_2^2) = 7$.*

Proof. Lower bound: We shall show a coloring of the edges of K_{2n+2} by $\mathbb{Z}_2^2$ without a zero-sum $K_{1,2n}$. The chromatic index of K_{2n+2} is $2n+1$ [18] and the edges can be decomposed into $(2n + 1)$ 1-factors. We take $(2n - 1)$ 1-factors colored by $(0, 0)$, one 1-factor colored $(1, 0)$ and one 1-factor colored $(0, 1)$. Clearly no zero-sum $K_{1,2n}$ exists.

Upper bound: For $n = 1$, we have $R(K_{1,2}, \mathbb{Z}_2^2) = R(K_{1,2}, 4) = 5$. So assume $n > 2$. By a result of Gao [15], every sequence of length 6 in $\mathbb{Z}_2^2$ contains a zero-sum subsequence of length 4. Thus, given a sequence from $\mathbb{Z}_2^2$ of length $2n+2 \geq 6$, we can repeatedly remove zero-sums of length four until we are left with a subsequence T of size less than 6. Since $2n + 2$ is even, this subsequence either has size 2 or 4. If $|T| = 2$, then we will have again found a zero-sum of length $2n$. We proceed to show a zero-sum subsequence of length $2n$ also exists in the case $|T| = 4$. Indeed, if $|T| = 4$, then we will have found again a zero-sum of length $2n$ unless T consists of the four distinct elements from $\mathbb{Z}_2^2$, in which case it is itself a zero-sum subsequence. Thus the entire sequence has sum zero, and as it has length $2n + 2 > 4$, the pigeonhole principle guarantees a 2-term zero-sum, and, consequently, a $2n$-term zero-sum. In summary, we see that any sequence of length $2n + 2$ from $\mathbb{Z}_2^2$ contains a $2n$-term zero-sum.

Hence if any vertex v of K_{2n+3} is considered, its degree is $2n + 2$ and it is the center of a zero-sum $K_{1,2n}$. This completes the proof showing that indeed $R(K_{1,2n}, \mathbb{Z}_2^2) = 2n + 3$ and, in particular, $R(K_{1,4}, \mathbb{Z}_2^2) = 7$.

□

Theorem 26. *$BR(K_{1,2}, \mathbb{Z}_2^s) = R(K_{1,2}, \mathbb{Z}_2^s) = 2^s + 1$. In particular, $R(K_{1,2}, \mathbb{Z}_2^2) = 5$.*

Proof. The chromatic index of K_{2m} is $2m - 1$ (well known). Hence, in particular, K_t $(t = 2^s)$ can be colored using $t - 1$ distinct colors, each of the colors forming a matching; in particular, no $K_{1,2}$ exists with both edges of the same color, showing that $R(K_{1,2}, \mathbb{Z}_2^s) > 2^s$.

On the other hand, the chromatic index of K_{2m+1} is $2m + 1$ (well known Vizing class 2 graphs). Hence, in particular, K_t $(t = 2^s - 1)$ cannot be colored using t distinct colors each forming a matching, so a $K_{1,2}$ exists with both edges of the same color, which implies a zero-sum in $\mathbb{Z}_2^s$ (for every element x in $\mathbb{Z}_2^s$ we have $2x = 0$.)

□

Theorem 27. *$BR(tK_{1,2}, \mathbb{Z}_2^2) = R(tK_{1,2}, \mathbb{Z}_2^2) = 3t$ unless $t = 1$, in which case $BR(K_{1,2}, \mathbb{Z}_2^2) = R(K_{1,2}, \mathbb{Z}_2^2) = 5$.*

Proof. We already proved $R(K_{1,2}, \mathbb{Z}_2^2) = 5$. We will prove $R(2K_{1,2}, \mathbb{Z}_2^2) = 6$. Notice that:

1. $tK_{1,2}$ is a graph on $3t$ vertices, and hence $3t$ is a trivial lower bound.

2. Suppose we proved $R(2K_{1,2}, \mathbb{Z}_2^2) = 6$. Then, by induction for $t = 2$, this is true, i.e., $R(2K_{1,2}, \mathbb{Z}_2^2) = 6$. Assuming it holds for t, let us prove it now for $t + 1$. Consider a coloring by $\mathbb{Z}_2^2$ of $E(K_{3(t+1)})$. Since $3(t + 1) > 5$, we know that a zero-sum copy of $K_{1,2}$ exists by the result $R(K_{1,2}, \mathbb{Z}_2^2) = 5$. Take this copy of $K_{1,2}$ and consider the complete graph that remains by deleting the 3 vertices of this $K_{1,2}$. Now we have a coloring of $E(K_{3t})$ and, by the induction hypothesis, there exists a zero-sum copy of $tK_{1,2}$ which, together with the $K_{1,2}$ zero-sum components obtained before, gives a zero-sum copy of $(t + 1)K_{1,2}$, proving the induction step.

Now, we show that: $R(2K_{1,2}, \mathbb{Z}_2^2) = 6$. Consider a $\mathbb{Z}_2^2$ coloring of $E(K_6)$. We have several observations

A: First let us show that if v_1v_2 and v_1v_3, are edges of the same color, say a, then v_2v_3 is also of color a. To see this, let v_4, v_5 and v_6 be the remaining three vertices in K_6. The triangle they form must be colored by exactly 3 colors, else we obtain a zero-sum $2K_{1,2}$. But now, if v_2v_3 is not of color a, then we can find a zero-sum $2K_{1,2}$ from among the edges of these two triangles. This shows the claim.

B: Since $R(K_{1,2}, \mathbb{Z}_2^2) = 5 < 6$, there exists a monochromatic triangle in view of the above Observation A, say $v_1v_2v_3$ with all edges colored a. Let v_4, v_5 and v_6 be the three remaining vertices of K_6. Note that all three edges of the triangle $v_4v_5v_6$ must be colored by three distinct colors, else we find a zero-sum $2K_{1,2}$ among the edges of both triangles.

By Observation A, we see that every connected, monochromatic component of K_6 is a clique. Let H be a maximal cardinality connected, monochromatic component. Thus H has at least three vertices as seen above. If H has more than three vertices, then it must be the entire K_6, as otherwise, taking any vertex v outside H, the pigeonhole principle would guarantee two edges from v to H of the same color (as the maximality of H ensures that no edge from v to H is the same as that of an edge in H), and now these two edges, along with a disjoint $K_{1,2}$ from H, give a zero-sum $2K_{1,2}$. However, if K_6 were monochromatic, then we would also clearly have a zero-sum $2K_{1,2}$. Consequently, we conclude that a maximal cardinality monochromatic clique has size 3.

C: In particular, no edge between $\{v_4, v_5, v_6\}$ and $\{v_1, v_2, v_3\}$ is of color a.

We divide the proof into two cases based on whether or not a occurs as a color in the triangle $v_4v_5v_6$.

Case 1: $f(v_4v_5) = b$, $f(v_5v_6) = c$ and $f(v_6v_4) = d$ are all distinct from a: if $f(v_5v_1) = b$, then $f(v_4v_1) = b$ and $f(v_6v_1) = c$ (by Observations A, B and C). Then $f(v_6v_1) = f(v_6v_4) = c$ and Observation A imply $f(v_1v_4) = c$, a contradiction. Thus $f(v_5v_1) \neq b$, and by symmetry $f(v_5v_i) \neq b$ for $i = 1, 2, 3$. Hence, in view of Observation C, we conclude that $f(v_5v_1) = c$. By symmetry, this argument also shows $f(v_5v_2) = c$, whence Observation A implies $f(v_2v_1) = c \neq a$, a contradiction.

Case 2: w.l.o.g. $f(v_4v_5) = b$, $f(v_5v_6) = c$ and $f(v_6v_4) = a$. By Observation A and C, each of the three edges from v_5 to the triangle $v_1v_2v_3$ must be of a distinct color and none of them is equal to a. Thus w.l.o.g. $f(v_5v_1) = b$, $f(v_5v_2) = c$ and $f(v_5v_3) = d$. Then $f(v_4v_5) = b$, $f(v_5v_1) = b$ and Observation A together imply $f(v_1v_4) = b$. But now $v_4v_1v_2$ and $v_3v_5v_6$ form a zero-sum $2K_{1,2}$, completing the proof.

□

Theorem 28. $BR(C_3 + e, \mathbb{Z}_2^2) = R(K_3 + e, \mathbb{Z}_2^2) = 5$.

Proof. Lower bound: color $E(K_4)$ with three colors (elements of $\mathbb{Z}_2^2$) such that each color appears on two independent edges. No zero-sum $C_3 + e$ exists.

Upper bound:

Case 1 There is a C_3 colored with three colors, say $f(v_1v_2) = a$, $f(v_2v_3) = b$, $f(v_1v_3) = c$. Consider the edges (v_1v_4), (v_2v_4), (v_3v_4). The only possible way to extend the C_3 to this K_4 without a zero-sum $C_3 + e$ is by coloring $f(v_1v_4) = b, f(v_2v_4) = c, f(v_3v_4) = a$. Now consider the edges $(v_1v_5), (v_2v_5), (v_3v_5)$. As before, the only way to extend $(v_1, v_2, v_3) \cup (v_5)$ to a K_4 without zero-sum C_3+e is by coloring $f(v_1v_5) = b, f(v_2, v_5) = c, f(v_3v_5) = a$. Now consider the edge (v_4v_5). No matter how it is colored, a zero-sum $C_3 + e$ appears.

Case 2 : No C_3 has three colors but there is a C_3 with 2 colors, say $f(v_1v_2) = f(v_2v_3) = a$, $f(v_1v_3) = b$. Consider the edges (v_1v_4), (v_2v_4), (v_3v_4). The only way to color them without a zero-sum $C_3 + e$ or without forming a C_3 with three colors is by coloring $f(v_1v_4) = a$, $f(v_2v_4) = c$ or d , $f(v_3v_4) = a$. Consider the edges (v_1v_5), (v_2v_5), (v_3v_5). As before, the only way to color them without a zero-sum $C_3 + e$ or without forming a C_3 with three colors is by coloring $f(v_1v_5) = a$, $f(v_2v_5) = c$ or $d, f(v_3v_5) = a$. But observe that if $f(v_2v_4) = f(v_2v_5)$, then a zero-sum $C_3 + e$ is formed by $(v_4, v_2, v_5, v_3, v_2)$, hence assume $f(v_2v_4) = c$ and $f(v_2v_5) = d$. Now consider $f(v_4v_5)$: no matter how it is colored, a zero-sum $C_3 + e$ appears.

Case 3 : All C_3 are monochromatic, but then all edges are colored the same and clearly every copy of $C_3 + e$ is a zero-sum.

□

Theorem 29. $BR(t(K_{1,3} \cup K_2), \mathbb{Z}_2^2) = R(t(K_{1,3} \cup K_2), \mathbb{Z}_2^2) = 6t + 2$.

Proof. Lower bound: this graph has all degrees are odd, and hence by the general lower bound Theorem 18, we have that $R(t(K_{1,3} \cup K_2), \mathbb{Z}_2^2) \geq 6t+2$.

Upper bound: it is sufficient to prove it for $K_{1,3} \cup K_2$ as the rest follows by simple induction. So consider a coloring of $E(K_8)$ with elements of $\mathbb{Z}_2^2$. Then we have the following:

1. If there is a star of 4 edges with 4 distinct colors, say $f(v_1v_2) = a$, $f(v_1v_3) = b$, $f(v_1v_4) = c$, $f(v_1v_5) = d$, then consider $f(v_6v_7)$. If $f(v_6v_7) = f(v_1v_i)$ for some $i = 2, 3, 4, 5$ (and it must hold for one of these possibilities), then delete edge (v_1v_i) and add edge (v_6v_7), and the resulting $K_{1,3} \cup K_2$ is zero-sum.

2. There is a vertex, say v_1, incident with exactly three colors. Then, for any coloring of $f(v_7v_8)$, a zero-sum $K_{1,3} \cup K_2$ is obtained.

3. Every vertex is adjacent with at most two colors. If every vertex is adjacent with one color, then this is a monochromatic coloring, and every copy of $K_{1,3} \cup K_2$ is zero-sum. So assume we have a vertex with exactly two colors on the edges incident with it. We have three basic possibilities for the distribution of the colors:

   3.1. $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = f(v_1v_6) = f(v_1v_7) = a$, $f(v_1v_8) = b$: If we color $f(v_2v_3)$ either a or b, we have a zero-sum $K_{1,3} \cup K_2$. So assume without loss of generality that $f(v_2v_3) = c$. Since no vertex has three colors and there is no zero-sum in $K_{1,3} \cup K_2$, it follows that $f(v_3v_4) = f(v_4v_5) = f(v_5v_6)$

$= f(v_6v_7) = f(v_7v_8) = c$ is forced. Now no matter how $f(v_2v_4)$ is colored, a zero-sum $K_{1,3} \cup K_2$ is obtained.

3.2. $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = f(v_1v_6) = a$, $f(v_1v_7) = f(v_1v_8) = b$: If we color $f(v_2v_3)$ either a or b, we have a zero-sum $K_{1,3} \cup K_2$. So assume without loss of generality that $f(v_2v_3) = c$. Since no vertex has three colors and there is no zero-sum in $K_{1,3} \cup K_2$, it follows that $f(v_3v_4) = f(v_4, v_5) = f(v_5v_6) = f(v_6v_7) = f(v_7v_8) = c$ is forced. Now, no matter how $f(v_2v_4)$ is colored, you get a zero-sum $K_{1,3} \cup K_2$.

3.3. $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = f(v_1v_5) = a$, $f(v_1v_6) = f(v_1v_7) = f(v_1v_8) = b$. If we color $f(v_2, v_3)$ either a or b, we have a zero-sum $K_{1,3} \cup K_2$. So assume without loss of generality that $f(v_2v_3) = c$. Since no vertex has three colors and there is no zero-sum in $K_{1,3} \cup K_2$, it follows that $f(v_3v_4) = f(v_4v_5) = f(v_5v_6) = f(v_6v_7) = f(v_7v_8) = c$ is forced. Now, no matter how $f(v_2v_4)$ is colored, a zero-sum $K_{1,3} \cup K_2$ is obtained.

Hence $R(t(K_{1,3} \cup K_2), \mathbb{Z}_2^2) = 6t + 2$. □

Theorem 30. $BR(t(P_4 \cup K_2), \mathbb{Z}_2^2) = R(t(P_4 \cup K_2), \mathbb{Z}_2^2) = 6t$.

Proof. Clearly, it is enough to prove $R(P_4 \cup K_2, \mathbb{Z}_2^2) = 6$ (the rest follows by a simple induction). Let $f : E(K_6) \to \mathbb{Z}_2^2$ be any coloring of edges of K_6 with $\{a, b, c, d\}$.

We consider two main cases.

Case 1: There is a path of length three colored with three colors.
We have two basic subcases :
Case A: $f(v_1v_2) = a, f(v_2v_3) = b, f(v_3v_4) = c$ and $f(v_5v_6) = a$.
Case B: $f(v_1v_2) = a, f(v_2v_3) = b, f(v_3v_4) = c$ and $f(v_5v_6) = b$.
Case A:
We must have $f(v_1v_4) = a$, and then there are just two possibilities : $f(v_1v_3) = a$ or $f(v_1v_3) = d$. All other possibilities create a zero-sum $P_4 \cup K_2$.

Suppose $f(v_1v_3) = a$. Then we are forced to have $f(v_2v_4) = d$, and in consequence, we have $f(v_2v_5) = a$, but then no matter how we color $f(v_3v_6)$, we have a zero-sum $P_4 \cup K_2$.

Suppose now that $f(v_1v_3) = d$. Then we have $f(v_2v_4) = a$, and in consequence, also $f(v_2v_5) = a$, but then no matter what the color $f(v_3v_6)$ is, we have a zero-sum $P_4 \cup K_2$.

Case B:
We must have $f(v_1v_4) = b$. Then we have two basic cases: B1, where $f(v_1v_3) = a$, and B2, where $f(v_1v_3) = c$.

Case B1:

We must have $f(v_2v_4) = c$, and now either $f(v_2v_5) = a$ and $f(v_3v_6) = c$, or $f(v_2v_5) = c$ and $f(v_3v_6) = a$.

In both cases, it follows that $f(v_1v_5) = c$ and $f(v_4v_6) = a$ (by symmetry), and now there is a zero-sum $P_4 \cup K_2$.

Case B2:

We must have $f(v_2v_4) = a$. Then we have four possible cases:

1) $f(v_2v_5) = a$ and $f(v_3v_6) = a$, but then $v_5v_3v_2v_6$ and v_1v_4 give a zero-sum $P_4 \cup K_2$.

2) $f(v_2v_5) = a$ and $f(v_3v_6) = c$; hence $f(v_1v_6) = c$ and $f(v_4v_5) = a$, and now there is a zero-sum $P_4 \cup K_2$.

Note that 1) and 2) show $f(v_2v_5) = c$. By symmetry, this also shows $f(v_3v_6) = a$.

3) $f(v_2v_5) = c$ and $f(v_3v_6) = a$, but then regardless of the color $f(v_4v_6)$, a zero-sum $P_4 \cup K_2$ is obtained.

Hence Case 1 is completed.

If no path of length two has two colors, then clearly the coloring is monochromatic and every copy of $P_4 \cup K_2$ is zero-sum. Hence it remains to consider Case 2 below.

Case 2: there is no path of three edges colored with three colors but there is a path of two edges colored with two colors.

We have two basic possibilities.

Case A: $f(v_1v_2) = f(v_2v_3) = a$ and $f(v_3v_4) = b$.

Case B: $f(v_1v_2) = a, f(v_2v_3) = b$ and $f(v_3v_4) = a$.

Case A: We must have either $f(v_4v_5) = a$ or $f(v_4v_5) = b$.

If $f(v_4v_5) = a$, then we are forced to have $f(v_5v_6) = a$, but then no matter what the color $f(v_3v_6)$ is, we have a zero-sum $P_4 \cup K_2$. If $f(v_4v_5) = b$, then $f(v_3v_5) = a$ or b, but in both cases, regardless of the color $f(v_5v_6)$, we have a zero-sum $P_4 \cup K_2$.

Case B:

If $f(v_4v_5) = a$, then $f(v_1v_6) = a$, implying $f(v_2v_6) = a$, and now, for any color $f(v_5v_6)$, we have a zero-sum $P_4 \cup K_2$.

If $f(v_4v_5) = b$, then $f(v_1v_6) = b$, implying $f(v_5v_6) = a$ (else either Case A completes the proof or we find a 3-color P_4), and now, regardless of $f(v_2v_6)$, we find a zero-sum $P_4 \cup K_2$. Hence the proof is complete.

□

5 Conclusion

The combinatorial arguments used in this paper are really elementary. Having in mind a possible automation, we have detailed some of them. The method used most often in filling up Tables 4 and 5 was an exhaustive enumeration of alternative cases, driving the manual computation of each one of the values. Nevertheless, in the Tools Section, the identification and categorization of the most used techniques is one of the contributions of this work. We believe that this will facilitate future work. The computation of $BR(H, \mathbb{Z}_n)$, $n \geq 5$, for the graphs given in Table 4, and the completion of Table 5 are open problems; we expect that their degree of difficulty will increase, as in case of the computation of the classic Ramsey numbers and the zero-sum Ramsey numbers involving many colors.

Acknowledgements. The authors wish to thank Irene Santos for the careful review of the computations for the values in Table 4, Francisca Losavio for the review of the English text, and Wolfgang Schmid for the final edition. We are grateful for the referee's remarks, that helped to improve the overall quality of the paper.

References

[1] N. Alon and Y. Caro, On three zero-sum Ramsey-type problems, *J. of Graph Theory* **29A** (1993) 177–192.

[2] A. Bialostocki and P. Dierker, On zero-sum Ramsey numbers — small graphs. *Ars Combinatoria* **29A** (1990) 193–198.

[3] A. Bialostocki and P. Dierker, On the Erdős, Ginzburg and Ziv theorem and the Ramsey numbers for stars and matchings *Discrete Math.* **110** (1992)1–8.

[4] A. Cammaroto, Estrellas con cero sumas en grafos *MSc. UCV. Caracas Venezuela* (1996).

[5] Y. Caro, A complete characterization of the zero-sum (mod 2) Ramsey numbers, *J. Combin. Theory* **A 68** (1994)205–211.

[6] Y. Caro, On zero-sum subsequences in abelian non-cyclic groups, *Israel J. of Mathematics* **92** (1995)221–233.

[7] Y. Caro, Exact cuts and a simple proof of the zero graphs theorem, *Ars Combinatoria* **41** (1995)123–128.

[8] Y. Caro. Zero-sum problems: a survey. *Discrete Math.* **152** (1996) 93–113.

[9] Y. Caro. On zero-sum Ramsey numbers-stars, *Discrete Math.* **104**(1992)1–6.

[10] L. Cordero, Determinación del número Ramsey baricéntrico de algunos grafos, *MSc. UCV. Caracas Venezuela* (1999).

[11] C. Delorme, S. González, O. Ordaz and M.T. Varela . Barycentric sequences and barycentric Ramsey numbers stars, *Discrete Math.* **277**(2004)45–56.

[12] C. Delorme, I. Márquez, O. Ordaz and A. Ortuño. Existence condition for barycentric sequences, *Discrete Math.* **281**(2004)163–172.

[13] P. Erdős, A. Ginzburg and A. Ziv. Theorem in the additive number theory, *Bull. Res. Council Israel* **10F** (1961) 41–43.

[14] S. González, L. González and O. Ordaz. Barycentric Ramsey numbers for small graphs, Bulletin of the Malaysan Mathematical Sciences Society. **(2)** **32(1)**(2009) 1–17.

[15] W. Gao, Addition theorems for finite abelian groups, *J. Number Theory*, **53** (1995), 241–246.

[16] W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: A survey. *Expositiones Mathematicae* **24** (2006), n. 4, 337–369.

[17] Y. O. Hamidoune. On weighted sums in abelian groups, *Discrete Math.* **162** (1996) 127–132.

[18] Jensen Tommy R. and Toft Bjarne (1995), Graph Coloring Problems, New York: Wiley-Interscience.

[19] O. Ordaz and D. Quiroz, Barycentric-sum problem: a survey. *Divulgaciones Matemáticas* **15** **No. 2** (2007)193–206.