The Hamiltonian Numbers in Graphs

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Abstract

A *Hamiltonian walk* of a connected graph G is a closed spanning walk of minimum length in G. The length of a hamiltonian walk in G is called the *hamiltonian number*, denoted by $h(G)$. An *Eulerian walk* of a connected graph G is a closed walk of minimum length which contains all edges of G. In this paper, we improve some results in [5] and give a necessary and sufficient condition for $h(G) < e(G)$. Then we prove that if two nonadjacent vertices u and v satisfying that $deg(u) + deg(v) \geqslant |V(G)|$, then $h(G) = h(G + uv)$. This result generalizes a theorem of Bondy and Chvátal for the hamiltonian property. Finally, we show that if $0 \leq k \leq n-2$ and G is a 2-connected graph of order n satisfying $deg(u) + deg(v) + deg(v) \geqslant \frac{3n-k-2}{2}$ for every independent set $\{u, v, w\}$ of three vertices in G, then $h(G) \leqslant n + k$. It is a generalization of a Bondy's result.

1 Introduction

Graphs considered in this paper are finite, without loop, or multiple edges. For a graph $G = (V, E)$ with vertex set V and edge set E, we use $|G|$ and $\|G\|$ to denote the cardinalities of V and E, respectively. Let G be a connected graph. A *Hamiltonian cycle* of G is a spanning cycle (in which

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all vertices are encountered). A graph G is called *hamiltonian* if G contains hamiltonian cycle. Discuss the existence of a hamiltonian cycle is an important topic in graph theory.(See [2],[3],[6]) A *walk* $W = (v_1, v_2, ..., v_t)$ of a graph G is a sequence of vertices such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, ..., t-1$ and its length $l(W)$ is $t-1$. A *hamiltonian walk* of G is a closed spanning walk of minimum length. Not all graphs have a hamiltonian cycle, however, there always exists a Hamiltonian walk in a connected graph. The length of a Hamiltonian walk in G is called the *hamiltonian number*, denoted by $h(G)$. The value of $h(G)$ measures how far of G from being hamiltonian. If $h(G) = |G|$, then G is hamiltonian.

In [4], Chartrand et al. gave an alternating interpretation of hamiltonian number of a graph G. Let $|V(G)| = n$, $V(G) = \{v_1, ..., v_n\}$, and $\mathcal{P}(G)$ be the set of all permutations on $V(G)$. In a graph G, $d(u, v)$ denotes the length of a shortest path between vertices u and v. Define $d(x_1, x_2, ..., x_n) = d(x_1, x_2) + d(x_2, x_3) + ... + d(x_{n-1}, x_n) + d(x_n, x_1)$ for $(x_1, x_2, ..., x_n) \in \mathcal{P}(G)$. In [4], they prove that $h(G) = \min\{d(p) : p \in \mathcal{P}(G)\}$. Thus a hamiltonian walk also can be expressed as a permutation $(x_1, x_2, ..., x_n)$ of $V(G)$ with $d(x_1, x_2, ..., x_n) = h(G)$.

A closed covering walk of G is a closed walk which contains all edges of G. An *Eulerian walk* of G is a closed covering walk of minimum length. The length of an Eulerian walk is denoted by $e(G)$. It's trivial that if G is Eulerian, then $e(G) = \|G\|$.

In this paper, we improve some results in [5] and give a necessary and sufficient conditions for $h(G) < e(G)$. And we prove that if u and v are distinct nonadjacent vertices of a connected graph G of order n with $deg(u) + deg(v) \geqslant n$, then $h(G)$ is equal $h(G + uv)$. Finally, we show that if $0 \leqslant k \leqslant n-2$ and G is a 2-connected graph of order n satisfying $deg(u) + deg(v) + deg(w) \geqslant \frac{3n-k-2}{2}$ for every independent set $\{u, v, w\}$ of three distinct vertices of G, then the hamiltonian number of G is at most $n + k$.

2 Relations Between Hamiltonian and Eulerian Walks

In [5] Goodman and Hedetniemi introduced the concepts of Hamiltonian walks and Eulerian walks in a connected graph G. They determined some bounds on $h(G)$ and present several relationships between Hamiltonian and Eulerian walks. The following three theorems are obtained by Goodman and Hedetniemi in [5].

Theorem 1 *If a graph G is connected, then $|G| \leqslant h(G) \leqslant e(G) \leqslant 2\|G\|$.*

They also show that every edge of G appears at most twice in every Hamiltonian (Eulerian) walk of G.

Theorem 2 *If $B_1, B_2, ..., B_k$ are the blocks of a connected graph G, then $h(G) = \sum_{i=1}^{k} h(B_i)$.*

So, the union of the edges in a Hamiltonian walk for each of the block B_i forms a Hamiltonian walk for G. Conversely, the edges in a Hamiltonian walk of G that belong to B_i forms a Hamiltonian walk in B_i. Theorem 2 implies that the topic of Hamiltonian walks can be restricted to 2-connected graphs.

Theorem 3 *If a graph G is Eulerian, then $h(G) < e(G)$ if and only if G contains a cycle which more than half the edges can be removed without disconnecting G.*

In this section, we will improve Theorem 1 and give a necessary and sufficient conditions for $h(G) < e(G)$.

Let $v_1, v_2, .., v_{2k}$ be all vertices with odd degree in a graph G. Define the *minimum distance sum* $s(v_1, v_2, ..., v_{2k})$ is the minimum of $d_G(v_{\pi(1)}, v_{\pi(2)}) + d_G(v_{\pi(3)}, v_{\pi(4)}) ... + d_G(v_{\pi(2k-1)}, v_{\pi(2k)})$ among all permutations π on $\{1, 2, ..., k\}$.

Proposition 4 *If $v_1, v_2, ..., v_{2k}$ are all odd degree vertices of a connected graph G, then $e(G) \leq \|G\| + s(v_1, v_2, ..., v_{2k})$.*

Proof. Let $v_1, v_2, ..., v_{2k}$ be all vertices with odd degree in G and let W_i be a shortest path from v_i to v_{i+1} in G for $i = 1, 2, ..., 2k-1$. Define H as a multigraph obtained from G by adding the paths $W_1, W_3, ..., W_{2k-1}$. Then H has an Eulerian walk with length $\|G\| + l(W_1) + l(W_3) + ... + l(W_{2k-1})$. By the definition of $e(G)$, we have that $e(G) \leq \|G)\| + s(v_1, v_2, ..., v_{2k})$. ∎

Theorem 5 *If a graph G is connected, then $|G| \leqslant h(G) \leqslant e(G) \leqslant \|G\| + \frac{h(G)}{2} \leqslant |G| + \|G\| - 1$.*

Proof. If G is Eulerian, then $e(G) = \|G\|$.
Assume that G is not Eulerian. Thus we can find some vertices with odd degree in G. Let $v_1, v_2, ..., v_{2k}$ be those vertices in a Hamiltonian walk W. Let W_i and W_{2k} be the $v_i - v_{i+1}$ and $v_{2k} - v_1$ subwalks of W, respectively. It is clear that $\sum_{i=1}^{2k} |W_i| = h(G)$ and $l(W_i) \leq d(v_i, v_{i+1})$, $l(W_{2k}) \leq d(v_{2k}, v_1)$ for $i = 1, 2, ..., 2k-1$.

Without loss of generality, suppose that $\sum_{i=1}^{k} |W_{2i}| \leqslant \frac{h(G)}{2}$. By Proposition 4, we have that $e(G) \leqslant \|G\| + \frac{h(G)}{2}$. It is easy to see that if G is connected, then $h(G) \leq 2|G| - 2$. Thus, $e(G) \leqslant \|G\| + \frac{h(G)}{2} \leqslant |G| + \|G\| - 1$.
∎

Lemma 6 *In a 2-edge-connected graph G, if $h(G) = e(G)$, then G is Eulerian.*

Proof. Let W be an Eulerian walk in G and let G_W be the multigraph induced by W. If G_W has no multiple edges, then G is Eulerian. Assume that there exists an edge e occurs twice in W. Let G'_W be a multigraph obtained from G_W by deleting two multipe edges with respect to e. Since G has no bridge, G'_W is Eulerian. This implies that there is a closed spanning walk of length $< e(G)$ in G; it contradicts that $h(G) = e(G)$. Hence G_W has no multiple edges; that is, G must be Eulerian. ∎

The following is a consequence of Theorem 3 and Lemma 6.

Corollary 7 *Suppose that G is a 2-edge-connected graph. Then $h(G) < e(G)$ if and only if either G contains a cycle which more than half the edges can be removed without disconnecting G or G is not Eulerian.*

3 Existence of Hamiltonian Walks

There are several sufficient conditions which had been established for a graph being hamiltonian. The following results are well-known.

A connected graph G of order n is hamiltonian if :

3.1 Ore : $deg(u) + deg(v) \geqslant n$ for all distinct nonadjacent vertices u and v.

3.2 Pósa : $|\{u : deg(u) \leqslant j\}| < j$ for $j < \frac{n-1}{2}$ and $|\{u : deg(u) \leqslant \frac{n-1}{2}\}| \leqslant \frac{n-1}{2}$ for n is odd.

3.3 Bondy and Chvátal : If $uv \notin E(G)$ and $deg(u) + deg(v) \geqslant n$, then $G + uv$ is hamiltonian.

3.4 Bondy : G is 2-connected and $deg(u) + deg(v) + deg(w) \geqslant \frac{3n}{2}$ for every set $\{u, v, w\}$ of three independent vertices of G.

In [1], Bermond proved the generalizations of Ore's (3.1) and Pósa's (3.2) theorems.

Theorem 8 *Let G be a connected graph of order $n \geqslant 3$ and let k be an integer with $0 \leqslant k \leqslant n-2$. If $deg(u) + deg(v) \geqslant n - k$ for every pair u, v of nonadjacent vertices of G, then $h(G) \leqslant n + k$.*

Theorem 9 *Let G be a connected graph of order $n \geqslant 3$ and let k be an integer with $0 \leq k \leqslant n$. If $|\{u : deg(u) \leqslant j\}| < j$ for $j < \frac{n-k-1}{2}$ and $|\{u : deg(u) \leqslant \frac{n-k-1}{2}\}| \leqslant \frac{n-k-1}{2}$ for k is odd, then $h(G) \leqslant n + k$.*

We are going to give the generalizations of results 3.3 and 3.4, respectively.

Theorem 10 *Let G be a connected graph of order $n \geqslant 3$. If u and v are nonadjacent vertices with $deg(u) + deg(v) \geqslant n$, then $h(G) = h(G + uv)$.*

Proof. Since G is a spanning connected subgraph of $G + uv$, $h(G) \geqslant h(G + uv)$. In the following we prove that $h(G) \leqslant h(G + uv)$. Let W be a hamiltonian walk in $G + uv$ passing through the edge uv k times in which k as small as possible. If $k = 0$, then W is also a hamiltonian walk in G. This implies that $h(G + uv) \geqslant h(G)$. If $k \geq 1$, then exists a permutation $(v_1, v_2, ..., v_n)$ such that $v_1 = u$, $v_2 = v$ and $d(v_1, v_2, ..., v_n) = h(G + uv)$.

We observe that if $uv_i \in E(G)$, then $v_{i-1}v \notin E(G)$. (Otherwise, we can find a hamiltonian walk W' with respect to $(v_2, ..., v_{i-1}, v_n, v_{n-1}, ..., v_i, v_1)$ and W' passes through uv at most $k-1$ times, a contradiction.) Then we have that $deg(v) \leqslant n-1-deg(u)$. Thus, $deg(u) + ndeg(v) \leqslant n-1$, a contradiction. Hence, we have that W is a hamiltonian walk without passing through uv in $G+uv$; that is, $h(G) \leqslant h(G+uv)$. The proof is complete. ∎

The closure $C(G)$ of a graph G of order n is the graph obtained from G by recursively joining pairs of nonadjacent vertices u and v with deg(u)+deg(v) $\geq n$ (in the resulting graph at each stage) until no such pair remains. By Theorem 10, we have the following corollary.

Corollary 11 *Let G be a connected graph of order $n \geqslant 3$. Then $h(G) = h(C(G))$.*

Next, we discuss a generalization of the result 3.4. Let G be a graph and $v \in V(G)$. Define that $N(v) = \{u : uv \in E(G)\}$.

Theorem 12 *Let G be a 2-connected graph of order n. If u and v are nonadjacent vertices with $|N(u) \cup N(v)| = n-2$, then $h(G) = h(G+uv)$.*

Proof. Since G is a spanning connected subgraph of $G+uv$, it is immediately that $h(G) \geqslant h(G+uv)$. Then we prove that $h(G) \leqslant h(G+uv)$ in the following.

Let W be a hamiltonian walk of $G+uv$ which passes through the edge uv k times with k as small as as possible. If $k=0$, then it is trivial that $h(G+uv) \geqslant h(G)$. Assume that $k \geq 1$. Then exists a permutation $(v_1, v_2, ..., v_n)$ on V(G) such that $v_1 = u$, $v_n = v$ and $d(v_1, v_2, ..., v_n) = h(G+uv)$. We can observe that if $uv_i \in E(G)$, then $v_{i-1}v \notin E(G)$. (Otherwise, we can find a hamiltonian walk of $G+uv$ passes through edge uv at most $k-1$ times. It contradicts the assumption.) Since $|N(u) \cup N(v)| = n-2$, $N(v_1) = \{v_2, ..., v_k\}$ and $N(v_n) = \{v_{k+1}, ..., v_{n-1}\}$ for some $2 \leq k \leq n-1$. Since G is 2-connected, there exist i, j with $i < k < j$ such that $v_iv_j \in E(G)$. Then we can find a hamiltonian walk W' in $G+uv$ with respect to the permutation $(v_1, v_{i+1}, ..., v_{j-1}, v_n, v_{n-1}, ..., v_j, v_i, v_{i-1}, ..., v_2)$ and W' passes the edge uv k' times with $k' < k$; it is impossible. So, k must be 0; that is, $h(G) \leqslant h(G+uv)$. Therefore, $h(G) = h(G+uv)$. ∎

The 2-closure $C_2(G)$ of a 2-connected graph G of order n is the graph obtained from G by recursively joining a pair of nonadjacent vertices u and v satisfying either deg(u)+deg(v) $\geq n$ or $|N(u) \cup N(v)| = n-2$ (in the resulting graph at each stage) until no such pair remains.

Corollary 13 *Let G be a 2-connected graph of order n. Then $h(G) = h(C_2(G))$.*

Finally, we give a generalization of result 3.4.

Lemma 14 *If a 2-connected graph G is not hamiltonian, then there are three independent vertices in G.*

Proof. Since G is 2-connected, G contains a cycle. Let C be a cycle with maximum length in G. By G being not hamiltonian, there exists a vertex v_0 in $V(G) - V(C)$. By Manger's Theorem, there are two paths $P_1 = (v_0, x_1, x_2, ..., x_s, u_1)$ and $P_2 = (v_0, y_1, y_2, ..., y_t, u_2)$ in G such that $x_i, y_j \notin V(C)$ and $u_1, u_2 \in V(C)$ for all i, j. Without loss of generality, let $C = (u_1, v_1, ..., u_2, v_2, ..., u_1)$. Then we can observe that v_1 and v_2 are nonadjacent with v_0 (if not, we can get a longer cycle in G by P_i and v_0v_i instead of u_iv_i for $i = 1$ or 2). Also, we have that $v_1v_2 \notin E(G)$ (if v_1 and v_2 are adjacent in G, then we could replace the edge u_1v_1 and u_2v_2 in C by the paths P_1 and P_2 together with the edge v_1v_2 and obtain a longer cycle, a contradiction). Thus we have three independent vertices v_0, v_1, v_2 in G. ∎

Theorem 15 *Let G be a 2-connected graph of order n and $0 \leqslant k \leqslant n-2$. If deg($u$) + deg($v$) + deg($v$) $\geqslant \frac{3n-k-2}{2}$ for every independent set $\{u, v, w\}$ of three vertices in G, then $h(G) \leqslant n + k$.*

Proof. By Corollary 13, we only consider the 2-closure $C_2(G) = G'$ of G.
Case 1. $k = 0$.

Suppose that G' is not hamiltonian. By Lemma 14, there are three independent vertices u, v, w of G'. Since the degree sum of each pair of two nonadjacent vertices of G' is at most $n-1$, $deg(u)+deg(v)+deg(w) \leqslant \frac{3n-3}{2}$. It is a contradiction.

Case 2. $k = 1$.

Suppose that $h(G') > n+1$. By Theorem 8, there exist two nonadjacent vertices u and v of G' such that $deg(u)+deg(v) \leqslant n-2$. By Theorem 12, we get that $|N(u) \cup N(v)| < n-2$. Thus there exists a vertex w such that $\{u, v, w\}$ is an independent set of G'. Hence, $deg(u)+deg(v)+deg(w) \leqslant \frac{3n-4}{2}$. This contradicts the assumption.

Case 3. $2 \leqslant k \leqslant n-2$.

Suppose $h(G') > n+k$. By Theorem 8, there exist two nonadjacent vertices u and v of G' such that $deg(u)+deg(v) \leqslant n-k-1 < n-2$. Then there exists a vertex w such that u, v, w are independent in G'. Thus $deg(u)+deg(v)+deg(w) \leqslant \frac{3n-k-3}{2}$. This also contradicts the assumption.

This completes the proof. ∎

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