A necessary and sufficient condition for a graph to be E_2-cordial

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Abstract In this paper, E_2-cordiality of a graph G is considered. Suppose G contains no isolated vertex, it is shown that G is E_2-cordial if and only if G is not of order $4n+2$.
Keywords E_2-cordial, maximal matching

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For an integer k, an edge labelling $f : E \to \{1, 2, \ldots, k-1\}$ induces the vertex labelling $f : V \to \{1, 2, \ldots, k-1\}$ defined by $f(v) \equiv \sum_{x \in N(v)} f(vx) (\text{mod } k)$, where $N(v)$ is the set of vertices adjacent to v and $f(v) \in \{0, 1, \ldots, k-1\}$. The number of vertices(resp. edges) of G labelled with i under f will be denoted by $v_i(G)$(resp. $e_i(G)$), $i = 0, 1, \ldots, k-1$. A labelling f of G is said to be E_k-cordial if $|v_i(G) - v_j(G)| \leq 1$ and $|e_i(G) - e_j(G)| \leq 1$, $\forall i, j \in \{0, 1, 2, \ldots, k-1\}$.

Definition. A graph G is said to be E_k-cordial if it admits an E_k-cordial labelling.

The notion of an E_k-cordial labelling was first introduced by Cahit and Yilmaz[1], who showed that the following graphs are E_3-cordial: $P_n (n \geq 3)$, Stars S_n if and only if $n \not\equiv 1(\text{mod}3)$, $K_n (n \geq 3)$, $C_n (n \geq 3)$, friendship graphs and fans$(n \geq 3)$. They also proved $S_n (n \geq 2)$ is E_k-cordial if and only if $n \not\equiv 1(\text{mod } k)$ when k is odd or $n \not\equiv 1(\text{mod}2k)$ when k is even and $k \neq 2$. Some further results see [2]-[5]. In this paper, the E_2-cordiality of a graph G is considered.

Suppose S is a vertex set or an edge set, we denote by $|S|$ the number of elements of S and write $v_i(S)(e_i(S))$ to indicate the number of vertices(edges) labelled with i in S , $i = 0, 1$. Sometimes vertices(edges) labelled with i are called $i - vertices(i - edges)$ for simplicity. We first introduce a necessary condition for a graph to be E_2-cordial.

Lemma 1. If $|V(G)| \equiv 2 \pmod 4$, then G is not E_2-cordial.

Proof. Suppose f is an E_2-cordial labelling of G. Since $|V(G)| \equiv 2 (\mathrm{mod}\ 4)$, we have $v_1(G) = v_0(G) = \frac{|V(G)|}{2}$. So $v_1(G)$ is odd. However, by $\sum_{x \in V(G)} f(x) \equiv 2 \sum_{uv \in E(G)} f(uv) \pmod 2$, we have $v_1(G) = \sum_{x \in V(G)} f(x) \equiv 2 \sum_{uv \in E(G)} f(uv) \pmod 2$, which implies that $v_1(G)$ is even, a contradiction.

In the following discussion, we assume G contains no isolated vertex. We shall prove that G is E_2-cordial if $|V(G)| \not\equiv 2 \pmod 4$.

2 M-W structure of a graph

Suppose $M = \{x_1y_1, x_2y_2, \ldots, x_my_m\}$ is a maximal matching of a graph G, W is the set of vertices unsaturated. Then we have a partition $\{X, Y, W\}$ of $V(G)$, where $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_m\}$. Define $E_* = \{uv | uv \in E(G), \{u, v\} \subseteq X \cup Y\} - M$, $E^* = \{wu | wu \in E(G), w \in W, u \in X \cup Y\}$. Since M is a maximal matching, then $\{E_*, E^*, M\}$ is a partition of $E(G)$. If an edge labelling $h : E_* \cup E^* \to \{0, 1\}$ induces a vertex labelling $h : V(G) \to \{0, 1\}$ by $h(v) \equiv \sum_{u \in N(v), uv \notin M} h(uv) \pmod 2$, then edges in M fall into three groups, i.e., $E_{ii} = \{xy | xy \in M, h(x) = h(y) = i\}$, $i = 0, 1$ and $E_{01} = \{xy | xy \in M, |h(x) - h(y)| = 1\}$. For simplicity, let $e_{ij} = |E_{ij}|$, $0 \le i \le j \le 1$. Note that the preceding notations will be adopted throughout the following discussion.

Lemma 2. Suppose M is a maximal matching of G. For any h defined above, $v_1(W)$ and e_{01} are of the same parity.

Proof. For any $h : E_* \cup E^* \to \{0, 1\}$, we have $v_1(G) \equiv 2 \sum_{uv \in E_* \cup E^*} h(uv) \pmod 2$. So $v_1(G)$ is even. It follows that $v_1(W)$ and $v_1(X \cup Y)$ are of the same parity. Since the number of $1 - vertices$ generated by endpoints of edges in E_{00} and E_{11} is always even, then $v_1(X \cup Y)$ and e_{01} are of the same parity. Hence the conclusion follows.

3 G has a perfect matching

Theorem 1. If M is a perfect matching of G, then G is E_2-cordial if and only if $|V(G)| \not\equiv 2 \pmod 4$.

Proof. By Lemma 1, we need only to prove the sufficiency. Since M is a perfect matching and $|V(G)| \not\equiv 2 \pmod 4$, then $|M|$ is even. We distinguish three cases.

Case 1. $|E_*| = 0$. In this case, $G = (2n)K_2$. By assigning 0 to n edges and 1 to the other, we obtain an E_2-cordial labelling of G.

Case 2. $|E_*| = 1$. In this case, $G = P_4 \cup (2n-2)K_2$. Edges of P_4 are labelled in order with 0, 0, 1; one half of the rest edges are labelled with 0 and the other with 1. One can check that the labelling of G defined above is E_2-cordial.

Case 3. $|E_*| \geq 2$. We first label edges in E_* and then those in M. Once a labelling $h : E_* \to \{0, 1\}$ is given, each $v \in V(G)$ gets a temporary label induced by $h(v) \equiv \sum_{u \in N(v), uv \in E_*} h(uv) \pmod 2$ and M is partitioned into E_{00}, E_{11} and E_{01}. In order to obtain a labelling of G, we need further to label edges in M. Note that if an edge in M is labelled with 0, labels of its endpoints stay the same as their temporary labels; if an edge in M is labelled with 1, labels of its endpoints are both changed.

Subcase 3.1. $|E_*| = 2k$. Assign 0 to k edges in E_* and 1 to the other. It follows from Lemma 2 that e_{01} is even and e_{00} and e_{11} are of the same parity.

If $e_{01} > 0$, we can get an E_2-cordial labelling of G as follows. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ edges in E_{11} and $\frac{e_{01}}{2}$ edges in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ edges in E_{11} and $\frac{e_{01}}{2} - 1$ edges in E_{01} are labelled with 0 and others in M with 1. Simple calculation shows that $e_0(M) = e_1(M)$, $e_0(G) = e_0(M) + k = e_1(M) + k = e_1(G)$, $v_0(G) = v_1(G)$. Therefore G is E_2-cordial.

If $e_{01} = 0$, by changing a $0-edge$ in E_* into a $1-edge$, we get another labelling $h' : E_* \to \{0, 1\}$ under which $e_0(E_*) = e_1(E_*) - 2$ and $e_{01} = 2$. Next we shall label edges in M based on h'. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ edges in E_{11} and 2 edges in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ edges in E_{11} and 1 edge in E_{01} are labelled with 0 and others in M with 1. Similar discussion shows that $e_0(G) = e_1(G)$, $v_0(G) = v_1(G)$.

Subcase 3.2. $|E_*| = 2k+1$. Assign 0 to $k+1$ edges in E_* and 1 to others.

If $e_{01} > 0$, we can get an E_2-cordial labelling of G as follows. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ edges in E_{11} and $\frac{e_{01}}{2} - 1$ edges in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}-1}{2}$ edges in E_{00}, $\frac{e_{11}-1}{2}$ edges in E_{11} and $\frac{e_{01}}{2}$ edges in E_{01} are labelled with 0 and others in M with 1. In either case, we have $e_0(G) = e_1(G) - 1$, $v_0(G) = v_1(G)$.

If $e_{01} = 0$, by changing a $0-edge$ in E_* into a $1-edge$, we get another labelling $h' : E_* \to \{0,1\}$ under which $e_0(E_*) = e_1(E_*) - 1$ and $e_{01} = 2$. Next we shall label edges in M based on h'. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ in E_{11} and 2 in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}-1}{2}$ edges in E_{00}, $\frac{e_{11}-1}{2}$ in E_{11} and 1 in E_{01} are labelled with 0 and others in M with 1. In either case, we have $e_0(G) = e_1(G) + 1$, $v_0(G) = v_1(G)$. Hence G is E_2-cordial.

4 G has no perfect matching but $E_* \neq \Phi$

We give the E_2-cordial labelling of G when $|W| = 4m > 0, |W| = 4m+1, |W| = 4m+2$ and $|W| = 4m+3$ respectively. First we introduce a lemma.

Lemma 3. Suppose M is a maximal matching of G, $E_* \neq \Phi$ and $|W| = 4m > 0$. Then there is a labelling $h : E_* \cup E^* \to \{0,1\}$ under which $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2,-1,0\}$ and $v_1(W) = 2m$.

Proof. Assume $W = \{w_1, w_2, \ldots, w_{4m}\}$. We take four steps to get the labelling desired.

Step 1. Assign 1 to $2m$ edges incident with different w_i's and 0 to others in $E_* \cup E^*$. Now we have $v_0(W) = v_1(W) = 2m$.

Step 2. If there is a w_i with two $0-edges$ incident with it, change the two $0-edges$ into $1-edges$. Then we also have $v_1(W) = 2m$ but $e_1(E_* \cup E^*) - e_0(E_* \cup E^*)$ increases by 4. Repeat the process until each w_i has at most one $0-edge$ incident with it.

Step 3. If there exist a $0-vertex$ and a $1-vertex$ in W each has a $0-edge$ incident with it, change the two $0-edges$ into $1-edges$. Then we also have $v_1(W) = 2m$ and $e_1(E_* \cup E^*) - e_0(E_* \cup E^*)$ increases by 4. Repeat the process until that there are at most $2m$ w_i's each has one $0-edge$ incident with it. In this case, $e_1(E^*) \geq e_0(E^*)$ and $e_0(E_*) = |E_*|$.

Step 4. If $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2,-1,0\}$ occurs during the process of step 2 or step 3, then the conclusion follows. If $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) = -3$ occurs during the process of step 2 and step 3, since $E_* \neq \Phi$, change a $0-edge$ in E_* into a $1-edge$ and we get a labelling $h: E_* \cup E^* \to \{0,1\}$ under which $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) = -1$ and $v_1(W) = 2m$. If $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-3,-2,-1,0\}$ has not occurred after step 3, we now have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) < -3$. Since all the edges in E_* are labelled with 0 until now, change these $0-edges$ into $1-edges$ one by one and eventually we have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \geq 1$. Observe that when a $0-edge$ in E_* is changed into a $1-edge$, $e_1(E_* \cup E^*) - e_0(E_* \cup E^*)$ increases by 2. Then $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2,-1\}$ occurs inevitably during the process. Hence the conclusion follows.

Theorem 2. Under the assumption of Lemma 3, G is E_2-cordial if and only if $|V(G)| \not\equiv 2 \pmod 4$.

Proof. By Lemma 1, we shall prove the sufficiency. It follows from Lemma 2 and Lemma 3 that e_{01} is even under the labelling h. Since $|W| = 4m > 0$ and $|V(G)| \not\equiv 2 \pmod 4$, then $|M|$ is even. So e_{00} and e_{11} are of the same parity. In order to get a labelling of G, we need further to label edges in M. According to Lemma 3, we distinguish three cases.

Case 1. $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) = -1$. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ in E_{11} and $\frac{e_{01}}{2}$ in E_{01} are labelled with 0 and others in M with 1. It is easy to show that $e_0(G) = e_1(G) + 1$, $v_0(G) = v_1(G)$. When e_{00} and e_{11} are both odd, then $\frac{e_{00}-1}{2}$ edges in E_{00}, $\frac{e_{11}-1}{2}$ edges in E_{11} and $\frac{e_{01}}{2}$ edges in E_{01} are labelled with 0 and others in M with 1. Similarly we have $e_0(G) = e_1(G) - 1$, $v_0(G) = v_1(G)$. In either case, G is E_2-cordial.

Case 2. $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) = -2$.

Subcase 2.1. $e_{01} > 0$. We can get an E_2-cordial labelling of G as follows. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ in E_{11} and $\frac{e_{01}}{2} + 1$ in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ in E_{11} and $\frac{e_{01}}{2}$ in E_{01} are labelled with 1 and others in M with 0. In either case, we have $e_0(G) = e_1(G)$, $v_0(G) = v_1(G)$.

Subcase 2.2. $e_{01} = 0$. In this case, there exists a $0-edge$ in E_*. If not, we have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \geq 1$, which is a contradiction. By changing a $0-edge$ in E_* into a $1-edge$, we get another labelling $h': E_* \cup E^* \to \{0,1\}$ under which $e_0(E_* \cup E^*) = e_1(E_* \cup E^*)$ and $e_{01} = 2$. Then we label edges in M based on h'. When e_{00} and e_{11} are both even, then one half of edges in E_{00}, E_{11} and E_{01} are labelled with 0 and the other in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}-1}{2}$ edges in E_{00}, $\frac{e_{11}-1}{2}$

in E_{11} and 2 in E_{01} are labelled with 0 and others in M with 1. In either case, we have $e_0(G) = e_1(G)$, $v_0(G) = v_1(G)$.

Case 3. $e_1(E_* \cup E^*) = e_0(E_* \cup E^*)$.

Subcase 3.1. $e_{01} > 0$. When e_{00} and e_{11} are both even, then one half of edges in E_{00}, E_{11} and E_{01} are labelled with 0 and the other in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}-1}{2}$ edges in E_{00}, $\frac{e_{11}-1}{2}$ in E_{11} and $\frac{e_{01}}{2}+1$ in E_{01} are labelled with 0 and others in M with 1. In either case, we have $e_0(G) = e_1(G)$, $v_0(G) = v_1(G)$.

Subcase 3.2. $e_{01} = 0$. By changing a $0-edge$ in E_* into a $1-edge$, we get another labelling $h' : E_* \cup E^* \to \{0,1\}$ under which $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) = 2$ and $e_{01} = 2$. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ in E_{11} and 2 in E_{01} are labelled with 0 and the other in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ in E_{11} and 1 in E_{01} are labelled with 0 and others in M with 1. In either case, we have $e_0(G) = e_1(G)$, $v_0(G) = v_1(G)$.

Lemma 4. Suppose M is a maximal matching of G, $E_* \neq \Phi$ and $|W| = 4m+1$. Then there are two labelling $h_1, h_2 : E_* \cup E^* \to \{0,1\}$ satisfying $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2,-1,0\}$ and $v_1(W) = 2m+1$ under h_1; $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2,-1,0\}$ and $v_1(W) = 2m$ under h_2.

Proof. Assume $W = \{w_1, w_2, \ldots, w_{4m+1}\}$. The proof is much similar to that of Lemma 3. We first take four steps to get h_1.

Step 1. Assign 1 to $2m+1$ edges incident with different w_i's and 0 to others in $E_* \cup E^*$.

Step 2. If there is a w_i with two $0-edges$ incident with it, change the two $0-edges$ into $1-edges$. Then repeat the process until each w_i has at most one $0-edge$ incident with it.

Step 3. If there exist a $0-vertex$ and a $1-vertex$ in W each has a $0-edge$ incident with it, change the two $0-edges$ into $1-edges$. Repeat the process until that there are at most $2m+1$ w_i's each has one $0-edge$ incident with it.

Step 4. If $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2,-1,0\}$ occurs during the process of step 2 or step 3, then the labelling desired is obtained; if $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) = -3$ occurs during the process of step 2 and step 3, change a $0-edge$ in E_* into a $1-edge$; if $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-3,-2,-1,0\}$ has not occurred after step 3, change $0-edges$ in E_* into $1-edges$ one by one until we get the labelling desired.

The four steps we take to get h_2 are the same as above except that $2m+1$ is replaced by $2m$. So details are omitted.

Theorem 3. Under the assumption of Lemma 4, G is E_2-cordial.

Proof. There are two cases.

Case 1. $|M|$ is odd. In this case, we label edges in $E_* \cup E^*$ with h_1. According to Lemma 4, we have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2, -1, 0\}$ and $v_1(W) = 2m + 1$. It follows that e_{01} is odd and e_{00} and e_{11} are of the same parity. No matter which value $e_1(E_* \cup E^*) - e_0(E_* \cup E^*)$ takes, we can label edges in M as follows. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ in E_{11} and $\frac{e_{01}-1}{2}$ in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ in E_{11} and $\frac{e_{01}-1}{2}$ in E_{01} are labelled with 1 and others in M with 0. In any case above, we have $|e_0(G) = e_1(G)| \leq 1$, $v_1(G) = v_0(G) + 1$. Hence G is E_2-cordial.

Case 2. $|M|$ is even. In this case, we label edges in $E_* \cup E^*$ with h_2. According to Lemma 4, we have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2, -1, 0\}$ and $v_1(W) = 2m$. It follows that e_{01} is even and e_{00} and e_{11} are of the same parity. Observe that conditions above are the same as those in Theorem 2. So G can be proved to be E_2-cordial if edges in M are labelled identically with Theorem 2. In fact, corresponding to cases of Theorem 2, equations of $e_1(G)$ and $e_0(G)$ stay the same but $v_1(G) = v_0(G) + 1$.

Theorem 4. Suppose M is a maximal matching of G, $E_* \neq \Phi$ and $|W| = 4m + 2$. Then G is E_2-cordial if and only if $|V(G)| \not\equiv 2 (\text{mod } 4)$.

Proof. We shall prove the sufficiency. It follows from $|W| = 4m + 2$ and $|V(G)| \not\equiv 2(\text{mod } 4)$ that $|M|$ is odd. Suppose $W = \{w_1, w_2, \ldots, w_{4m+2}\}$. By labelling edges in $E_* \cup E^*$ with h_1 in Lemma 4, we have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2, -1, 0\}$ and $v_1(W) = 2m+1$. Then e_{01} is odd and e_{00} and e_{11} are of the same parity. No matter which value $e_1(E_* \cup E^*) - e_0(E_* \cup E^*)$ takes, we can label edges in M as follows. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00}, $\frac{e_{11}}{2}$ in E_{11} and $\frac{e_{01}-1}{2}$ in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ in E_{11} and $\frac{e_{01}-1}{2}$ in E_{01} are labelled with 1 and others in M with 0. In any case above, we have $|e_0(G) = e_1(G)| \leq 1$, $v_1(G) = v_0(G)$.

Theorem 5. Suppose M is a maximal matching of G, $E_* \neq \Phi$ and $|W| = 4m + 3$, then G is E_2-cordial.

Proof. We shall prove the sufficiency. Suppose $W = \{w_1, w_2, \ldots, w_{4m+3}\}$.

Case 1. $|M|$ is odd. By labelling edges in $E_* \cup E^*$ with h_1 in Lemma 4, we have $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2, -1, 0\}$ and $v_1(W) = 2m + 1$. In order to get a labelling of G, we need further to label edges in M. No matter which value $e_1(E_* \cup E^*) - e_0(E_* \cup E^*)$ takes, edges in M can be labelled as follows. When e_{00} and e_{11} are both even, then $\frac{e_{00}}{2}$ edges in E_{00},

$\frac{e_{11}}{2}$ in E_{11} and $\frac{e_{01}-1}{2}$ in E_{01} are labelled with 0 and others in M with 1; when e_{00} and e_{11} are both odd, then $\frac{e_{00}+1}{2}$ edges in E_{00}, $\frac{e_{11}+1}{2}$ in E_{11} and $\frac{e_{01}-1}{2}$ in E_{01} are labelled with 1 and others in M with 0. In any case above, we have $|e_0(G) = e_1(G)| \leq 1$, $v_1(G) = v_0(G) - 1$. Hence G is E_2-cordial.

Case 2. $|M|$ is even. Following the same steps as Lemma 4 except that $2m+1$ is replaced by $2m+2$, we obtain a labelling on $E_* \cup E^*$ under which $e_1(E_* \cup E^*) - e_0(E_* \cup E^*) \in \{-2, -1, 0\}$ and $v_1(W) = 2m + 2$. Similar to case 2 of Theorem 3, G can be proved to be E_2-cordial if edges in M are labelled the same as Theorem 2.

5 $E_* = \Phi$

If $P_4 = P_{u_1u_2u_3u_4} \subseteq G$, then we can choose a maximal matching $M \supseteq \{u_1u_2, u_3u_4\}$, which implies that $u_2u_3 \in E_*$. Hence the condition $E_* \neq \Phi$ in Theorem 2-Theorem 5 is satisfied. If there is no $P_4 \subseteq G$, then branches of G must be C_3 or Stars. We first introduce two lemmas.

Lemma 5. Suppose G is E_2-cordial. If G^* is obtained by one of the following transformations, then G^* is also E_2-cordial. (1) $G^* = G \cup (4K_2)$; (2) $G^* = G \cup (4P_3)$; (3) $G^* = G \cup (4C_3)$; (4) $x \in V(G)$, G^* is obtained by merging four edges xu_1, xu_2, xu_3, xu_4 on x; (5) $\{x, y\} \subseteq V(G)$, G^* is obtained by merging xu_1, xu_2 on x and yu_3, yu_4 on y; (6) $\{x, y\} \subseteq V(G)$, G^* is obtained by merging xu_1, xu_2, xu_3 on x and yu_4 on y; (7) $\{x, y, z\} \subseteq V(G)$, G^* is obtained by merging xu_1, xu_2, yu_3 and zu_4.

Proof. Since the proof is trivial, we just prove (3) and (7). Suppose f is an E_2-cordial labelling of G. Then we extend the labelling f to G^*.

(3) Assume $4C_3 = \cup_{i=1}^4 C_{x_iy_iz_i}$. Next we extend f to G^* as follows. Let $f(x_iy_i) = f(x_4y_4) = f(y_4z_4) = f(z_4x_4) = 1$, $f(x_iz_i) = f(y_iz_i) = 0$, $i = 1, 2, 3$. Then f is an E_2-cordial labelling of G^*.

(7) We extend f to G^* as follows. Let $f(xu_1) = f(xu_2) = 1$, $f(yu_3) = f(zu_4) = 0$. Then f is an E_2-cordial labelling of G^*.

Lemma 6. Suppose g is an E_2-cordial labelling of G. If H satisfied one of the following conditions, then $G \cup H$ is E_2-cordial. (1) there is a labelling of H under which $v_0(H) = v_1(H), e_0(H) = e_1(H)$; (2) there are two labelling h_1, h_2 of H such that $v_0(H) = v_1(H), e_0(H) = e_1(H) - 1$ under h_1 and $v_0(H) = v_1(H), e_0(H) = e_1(H) + 1$ under h_2.

Proof. (1) is easy to prove, we now prove (2). If the labelling g satisfies $|v_0(G) - v_1(G)| \leq 1$ and $0 \leq e_0(H) - e_1(H) \leq 1$, by combining g of G and h_1 of H, we obtain a labelling of $G \cup H$ which can be proved to be E_2-cordial.

Similarly if g satisfies $|v_0(G) - v_1(G)| \leq 1$ and $e_0(G) = e_1(G) - 1$, we can get an E_2-cordial labelling of $G \cup H$ by combining g of G and h_2 of H.

Theorem 6. If $G = mC_3 \cup (\cup_i n_i S_i)$, where S_i is a Star of order i, then G is E_2-cordial if and only if $|V(G)| \not\equiv 2 \pmod 4$.

Proof. We shall make use of Lemma 5 and Lemma 6 to simplify the graph G, if the simplified graph is E_2-cordial, so is the graph G. By (3) and (4) of Lemma 5, we only have to prove G is E_2-cordial under the condition that $m \in \{0, 1, 2, 3\}$ and $i \leq 5$.

If $n_5 > 0$, by (5) and (6) of Lemma 5, we can assume $n_5 = 1$ and $n_3 = n_4 = 0$. Since $S_5 \cup C_3$ satisfies (2) of Lemma 6, it suffices to show that $S_5 \cup (kP_2), k \in \{0, 1, 2, 3\}$ are E_2-cordial, which are easy to prove. Hence the conclusion follows when $n_5 > 0$.

If $n_5 = 0$, by (6) and (7) of Lemma 5, we can assume $n_4 \leq 1$ and $n_3 \leq 1$ if $n_4 = 1$. Since $S_4 \cup (2K_2)$ satisfied (2) of Lemma 6, we can further assume $n_2 \leq 1$. In other words, if $n_4 = 1$, we have to prove $G = S_4 \cup n_3 S_3 \cup n_2 S_2 \cup mC_3$ is E_2-cordial when $n_3 \in \{0, 1\}, n_2 \in \{0, 1\}$ and $m \in \{0, 1, 2, 3\}$. There are 16 cases in all. In any case, G can be proved to be E_2-cordial.

Finally we shall prove $G = n_3 S_3 \cup n_2 S_2 \cup mC_3$ is E_2-cordial. If $n_2 > 0$, since $K_2 \cup (2S_3)$ and $K_2 \cup (2C_3)$ satisfy (2) of Lemma 6, we may assume $m, n_3 \in \{0, 1\}$. So there are 12 cases in all. If $n_2 = 0$, there are 16 cases. In any case, one can check that G is E_2-cordial.

Combining Theorem 1-Theorem 6, we have the following result.

Theorem 7. If a graph G contains no isolated vertex, then G is E_2-cordial if and only if $|V(G)| \not\equiv 2 \pmod 4$.

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