Signed k-domatic numbers of graphs

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Abstract

Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A function $f : V(G) \longrightarrow \{-1, 1\}$ is called a *signed k-dominating function* if $\sum_{u \in N[v]} f(u) \geq k$ for each vertex $v \in V(G)$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed k-dominating family* (of functions) on G. The maximum number of functions in a signed k-dominating family on G is the *signed k-domatic number* of G, denoted by $d_{kS}(G)$. In this paper we initiate the study of signed k-domatic numbers in graphs and we present some sharp upper bounds for $d_{kS}(G)$. In addition, we determine the signed k-domatic number of complete graphs.

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1 Introduction

In this paper, G is a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$. The *open neighborhood* $N(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v \in S} N(v)$, and the *closed neighborhood* $N[S]$ of S is the set $N(S) \cup S$. The minimum degree of a vertex of G is denoted by $\delta(G)$. Consult [9] for the notation and terminology which are not defined here.

For a real-valued function $f : V(G) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. So $w(f) = f(V)$. Let $k \geq 1$ be an integer and let G be a graph with minimum degree at least $k-1$. A *signed k-dominating function* (SkD function) is a function $f : V(G) \to \{-1, 1\}$ satisfying $\sum_{u \in N[v]} f(u) \geq k$ for every $v \in V(G)$. The minimum of the values of $\sum_{v \in V(G)} f(v)$ taken over all signed k-dominating functions f is called the *signed k-domination number* and is denoted by $\gamma_{kS}(G)$. As the assumption $\delta(G) \geq k-1$ is clearly necessary for a graph to have a SkD function, we will always assume that when we discuss $\gamma_{kS}(G)$, all graphs involved satisfy $\delta(G) \geq k-1$. Then the function assigning $+1$ to every vertex of G is a SkD function, called the function ϵ, of weight n. Thus $\gamma_{kS}(G) \leq n$ for every graph of order n with $\delta \geq k-1$. Moreover, the weight of every SkD function different from ϵ is at most $n-2$ and more generally, $\gamma_{kS}(G) \equiv n \pmod 2$. Hence $\gamma_{kS}(G) = n$ if and only if ϵ is the unique SkD function of G. In the special case when $k = 1$, $\gamma_{kS}(G)$ is the signed domination number investigated in [2] and has been studied by several authors (see for example [1, 3]). The signed k-domination number of graphs was introduced by Wang [8].

Observation 1. Let G be a graph of order n and minimum degree $\delta \geq k-1$. Then $\gamma_{ks}(G) = n$ if and only if for each $v \in V$, there exists a vertex $u \in N[v]$ such that $\deg(u) = k-1$ or $\deg(u) = k$ (this condition implies $\delta \leq k$).

Proof. If for each $v \in V$, there exists a vertex $u \in N[v]$ such that $\deg(u) = k-1$ or $\deg(u) = k$, then for each $v \in V$ there exists a vertex $u \in N[v]$ such that each SkD function satisfies $f(x) = +1$ for all $x \in N[u]$ and in particular $f(v) = +1$. Therefore ϵ is the unique SkD function and $\gamma_{ks}(G) = n$.

Conversely, assume that $\gamma_{kS}(G) = n$. If there exists a vertex v such that $\deg(u) \geq k+1$ for each $u \in N[v]$, then the function f defined by $f(v) = -1$ and $f(x) = 1$ for $x \neq v$ is a signed k-dominating function of weight $n-2$, a contradiction. This completes the proof. □

A set $\{f_1, f_2, \ldots, f_d\}$ of signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed k-*

dominating family on G. The maximum number of functions in a signed k-dominating family on G is the *signed k-domatic number* of G, denoted by $d_{kS}(G)$. The signed k-domatic number is well-defined and $d_{kS}(G) \geq 1$ for all graphs G with $\delta(G) \geq k-1$ since the set consisting of any one SkD function, for instance the function ϵ, forms a SkD family of G. A *d_{kS}-family* of a graph G is a SkD family containing $d_{kS}(G)$ SkD functions. The signed 1-domatic number $d_{1S}(G)$ is the usual signed domatic number $d_S(G)$ which was introduced by Volkmann and Zelinka in [7] and has been studied by several authors (see for example [4, 5, 6]).

Observation 2. Let G be a graph of order n. If $\gamma_{kS}(G) = n$, then ϵ is the unique SkD function of G and so $d_{kS}(G) = 1$.

The following two observations are consequence of Observations 1 and 2.

Observation 3. If G is a graph of order n and $k = n$, then G is the complete graph and thus $\gamma_{ks}(G) = n$ and $d_{ks}(G) = 1$.

Observation 4. If G is a graph of order n and $k = n-1$, then $\gamma_{ks}(G) = n$ and so $d_{ks}(G) = 1$.

Proof. If G is the complete graph, then Proposition A implies that $\gamma_{ks}(G) = n$ and so $d_{ks}(G) = 1$. Thus we may assume that G is not the compete graph. Let $u, v \in V(G)$ such $uv \notin E(G)$. Since $\delta(G) \geq n-2$, we have $\deg(u) = \deg(v) = n-2$, $N[u] = V(G) - \{v\}$ and $N[v] = V(G) - \{u\}$. Let $f : V(G) \to \{-1, 1\}$ be a $\gamma_{ks}(G)$-function. Since $\sum_{x \in N[u]} f(x) \geq n-1$, we must have $f(x) = 1$ for each $x \in N[u]$. Similarly, $f(x) = 1$ for each $x \in N[v]$. Thus $f(x) = 1$ for all vertices $x \in V(G)$. It follows that $\gamma_{ks}(G) = n$ and by Observation 2 we have $d_{ks}(G) = 1$. □

Corollary 5. If G is a r-regular graph and $k = r+1$ or r, then $\gamma_{ks}(G) = n$ and $d_{ks}(G) = 1$.

We first study basic properties and sharp upper bounds for the signed k-domatic number of a graph. Some of them generalize the result obtained for the signed domatic number. Then we determine the signed k-domatic number of complete graphs.

In this paper we make use of the following results.

Proposition A. [8] Let $k \geq 1$ be an integer. For any integer $n \geq k$, we have

$$\gamma_{kS}(K_n) = \begin{cases} k & \text{if } n \equiv k \pmod 2 \\ k+1 & \text{otherwise.} \end{cases} \tag{1}$$

Proposition B. [7] If $G = K_n$ is the complete graph of order n, then

$$d_S(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ p & \text{if } n = 2p \text{ and } p \text{ is odd} \\ p-1 & \text{if } n = 2p \text{ and } p \text{ is even.} \end{cases} \tag{2}$$

2 Basic properties of the signed k-domatic number

In this section we present basic properties of $d_{kS}(G)$ and sharp bounds on the signed k-domatic number of a graph.

Proposition 6. If $k > p \geq 1$ are integers, then $d_{pS}(G) \geq d_{kS}(G)$ for any graph G.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family on G such that $d = d_{kS}(D)$. Then $\{f_1, f_2, \ldots, f_d\}$ is also a SpD family on G and thus $d_{pS}(G) \geq d_{kS}(G)$. □

Theorem 7. The signed k-domatic number of a graph is an odd integer.

Proof. Let G be an arbitrary graph, and suppose that $d = d_{kS}(G)$ is even. Let $\{f_1, f_2, \ldots, f_d\}$ be a $d_{kS}(G)$−family. If $u \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(u) \leq 1$. But on the left-hand side of this inequality, a sum of an even number of odd summands occurs. Therefore it is an even number, and we obtain $\sum_{i=1}^{d} f_i(u) \leq 0$ for each $u \in V(G)$. This forces

$$\begin{aligned} d &= \textstyle\sum_{i=1}^{d} 1 \\ &\leq \textstyle\sum_{i=1}^{d}(\frac{1}{k}\sum_{u\in N[v]} f_i(u)) \\ &= \textstyle\frac{1}{k}\sum_{u\in N[v]}\sum_{i=1}^{d} f_i(u) \\ &\leq 0 \end{aligned}$$

which is a contradiction. □

Theorem 8. If G is a graph of order n and $\delta(G) \geq k-1$, then

$$1 \leq d_{kS}(G) \leq \frac{\delta(G)+1}{k} \leq \frac{n}{k}.$$

Moreover if $d_{kS}(G) = \dfrac{\delta(G)+1}{k}$, then for each function of any d_{kS}−family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of degree $\delta(G)$, $\sum_{u\in N[v]} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(u) = 1$ for every $u \in N[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family of G such that $d = d_{kS}(G)$ and let v be a vertex of minimum degree $\delta(G)$. Then $|N[v]| = \delta + 1$ and

$$\begin{aligned} 1 \leq d &= \sum_{i=1}^{d} 1 \\ &\leq \sum_{i=1}^{d} \frac{1}{k} \sum_{u \in N[v]} f_i(u) \\ &= \frac{1}{k} \sum_{u \in N[v]} \sum_{i=1}^{d} f_i(u) \\ &\leq \frac{1}{k} \sum_{u \in N[v]} 1 \\ &= \frac{\delta(G)+1}{k}. \end{aligned}$$

If $d_{kS}(G) = \frac{\delta(G)+1}{k}$, then the two inequalities occurring in the proof become equalities, which gives the two properties given in the statement. □

The special case $k = 1$ in Theorems 7 and 8 can be found in [7]. The next corollary is a consequence of Theorems 7 and 8.

Corollary 9. If G is a graph of minimum degree δ, then $d_{kS}(G) = 1$ for every integer k such that $\frac{\delta+1}{3} < k \leq \delta + 1$.

In particular for a tree T and for a cycle C_n, we have $d_{1S}(T) = d_{2S}(T) = 1$ and $d_{2S}(C_n) = d_{3S}(C_n) = 1$.

Next we improve the bound given in Theorem 8 for some special cases.

Theorem 10. Let $k \geq 1$ be an integer, and let G be a graph of order n with $\delta(G) \geq k - 1$. If k and $\delta(G)$ are odd or k and $\delta(G)$ are even, then

$$d_{kS}(G) \leq \frac{\delta(G)+1}{k+1}.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family on G such that $d = d_{kS}(G)$, and let v be a vertex of minimum degree $\delta(G)$. Assume first that k and $\deg(v) = \delta(G)$ are odd. The definition yields to $\sum_{x \in N[v]} f_i(x) \geq k$ for each $i \in \{1, 2, \ldots, d\}$. On the left hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{x \in N[v]} f_i(x) \geq k + 1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$\begin{aligned} d &= \sum_{i=1}^{d} 1 \leq \sum_{i=1}^{d} \frac{1}{k+1} \sum_{u \in N[v]} f_i(u) \\ &= \frac{1}{k+1} \sum_{u \in N[v]} \sum_{i=1}^{d} f_i(u) \\ &\leq \frac{1}{k+1} \sum_{u \in N[v]} 1 = \frac{\delta(G)+1}{k+1}, \end{aligned}$$

and the desired bound is proved. Assume next that k and $\deg(v) = \delta(G)$ are even. Note that $\sum_{x\in N[v]} f_i(x) \geq k$ for each $i \in \{1,2,\ldots,d\}$. On the left hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{x\in N[v]} f_i(x) \geq k+1$ for each $i \in \{1,2,\ldots,d\}$. Now the desired bound follows as above, and the proof is complete. □

As an application of Theorems 8 and 10, we will prove the following Nordhaus-Gaddum type results.

Proposition 11. Let G be a graph of order n, minimum degree $\delta(G)$, maximum degree $\Delta(G)$, and let $\overline{G}$ be its complementary graph. Then

$$d_{kS}(G) + d_{kS}(\overline{G}) \leq \frac{n+\delta(G)-\Delta(G)+1}{k} \leq \frac{n+1}{k} \qquad (3)$$

for every integer $k \leq \min\{\delta+1, n-\Delta\}$. The equality $d_{kS}(G) + d_{kS}(\overline{G}) = \frac{n+1}{k}$ implies that G is a regular graph.

Proof. Since $\delta(\overline{G}) = n - \Delta(G) - 1$, it follows from Theorem 8 that

$$d_{kS}(G)+d_{kS}(\overline{G}) \leq \frac{\delta(G)+1}{k}+\frac{n-\Delta(G)}{k} = \frac{n+\delta(G)-\Delta(G)+1}{k} \leq \frac{n+1}{k}.$$

If $d_{kS}(G) + d_{kS}(\overline{G}) = \frac{n+1}{k}$, then $\delta(G) = \Delta(G)$ and G is regular. □

Theorem 12. Let $k \geq 1$ be an integer, and let G be a graph of order n such that $\delta(G) \geq k-1$ and $\delta(\overline{G}) \geq k-1$. If $\Delta(G) - \delta(G) \geq 1$ or k is even or k and $\delta(G)$ are odd or k is odd and $\delta(G)$ and n are even, then

$$d_{kS}(G) + d_{kS}(\overline{G}) \leq \frac{n}{k}. \qquad (4)$$

Proof. If $\Delta(G) - \delta(G) \geq 1$, then Theorem 11 implies the desired bound. Thus assume now that G is $\delta(G)$-regular and so $\delta(G) + \delta(\overline{G}) = n - 1$.

Case 1: Assume that k is even. If $\delta(G)$ is even, then the hypothesis $\delta(G) \geq k-1$ implies that $\delta(G) \geq k$. Therefore it follows from Theorems 8 and 10 that

$$\begin{aligned} d_{kS}(G) + d_{kS}(\overline{G}) &\leq \frac{\delta(G)+1}{k+1} + \frac{\delta(\overline{G})+1}{k} \\ &= \frac{\delta(G)+1}{k+1} + \frac{n-\delta(G)}{k}. \\ &\leq \frac{n}{k}. \end{aligned}$$

If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ even. The hypothesis $\delta(\overline{G}) \geq k - 1$ implies that $\delta(\overline{G}) \geq k$, and hence it follows from Theorems 8 and 10 that

$$\begin{aligned} d_{kS}(G) + d_{kS}(\overline{G}) &\leq \frac{\delta(G)+1}{k} + \frac{\delta(\overline{G})+1}{k+1} \\ &= \frac{n-\delta(\overline{G})}{k} + \frac{\delta(\overline{G})+1}{k+1}. \\ &\leq \frac{n}{k}. \end{aligned}$$

Case 2: Assume that k is odd. If $\delta(G)$ is odd, then $\delta(G) \geq k-1$ implies that $\delta(G) \geq k$. Combining this with Theorems 8 and 10, we find that

$$d_{kS}(G) + d_{kS}(\overline{G}) \leq \frac{\delta(G)+1}{k+1} + \frac{\delta(\overline{G})+1}{k} \leq \frac{n}{k}.$$

If $\delta(G)$ and n are even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above. □

Since $d_{3S}(C_5) = 1$, we have

$$d_{3S}(C_5) + d_{3S}(\overline{C_5}) = 2d_{3S}(C_5) = 2 = \frac{n+1}{3}.$$

This example shows that Theorem 12 is not valid in general when k and n are odd and $\delta(G)$ is even.

Theorem 13. Let G be a graph of order n and $\delta(G) \geq k - 1$ with signed k-domination number $\gamma_{kS}(G)$ and signed k-domatic number $d_{kS}(G)$. Then

$$\gamma_{kS}(G) \cdot d_{kS}(G) \leq n.$$

Moreover, if $\gamma_{kS}(G) \cdot d_{kS}(G) = n$, then for each d_{kS}-family $\{f_1, f_2, \cdots, f_d\}$ on G, each function f_i is a γ_{kS}-function and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family on G such that $d = d_{kS}(G)$ and let $v \in V$. Then

$$\begin{aligned} d \cdot \gamma_{kS}(G) &= \sum_{i=1}^{d} \gamma_{kS}(G) \\ &\leq \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ &= \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \\ &\leq \sum_{v \in V} 1 \\ &= n. \end{aligned}$$

If $\gamma_{kS}(G) \cdot d_{kS}(G) = n$, then the two inequalities occurring in the proof become equalities. Hence for the d_{kS}-family $\{f_1, f_2, \cdots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{kS}(G)$, thus each function f_i is a γ_{kS}-function, and $\sum_{i=1}^{d} f_i(v) = 1$ for all v. □

Corollary 14 is a consequence of Theorems 13 and 7 and improves Observation 2.

Corollary 14. If $\gamma_{kS}(G) \geq \frac{n}{2}$, then $d_{kS}(G) = 1$.

The upper bound on the products $\gamma_{kS}(G) \cdot d_{kS}(G)$ leads to a bound on the sum.

Corollary 15. If G is a graph of order $n \geq 4$ and $k \leq \delta + 1$, then

$$\gamma_{kS}(G) + d_{kS}(G) \leq n + 1.$$

Equality $\gamma_{kS}(G) + d_{kS}(G) = n + 1$ occurs if and only if $G = K_n$ with n odd and $k = 1$, or for each $v \in V(G)$ there exists a vertex $u \in N[v]$ such that $\deg(u) = k - 1$ or $\deg(u) = k$.

Proof. According to Theorem 13, we obtain

$$\gamma_{kS}(G) + d_{kS}(G) \leq \frac{n}{d_{kS}(G)} + d_{kS}(G). \tag{5}$$

The bound results from the facts that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$ and that $1 \leq d_{kS}(G) \leq n$ by Theorem 8. Equality occurs if and only if $d_{kS}(G) = n$ and $\gamma_{kS}(G) = 1$ or $d_{kS}(G) = 1$ and $\gamma_{kS}(G) = n$. The description of the extremal graphs comes from Theorem 8 and Propositions B and A when $d_{kS}(G) = n$ and $\gamma_{kS}(G) = 1$, and from Observations 1 and 2 when $\gamma_{kS}(G) = n$ and $d_{kS}(G) = 1$. □

By Corollary 15, $\gamma_{kS}(G) + d_{kS}(G)$ can be equal to $n + 1$ if $\gamma_{kS}(G) = n$ or 1 or if $d_{kS}(G) = n$ or 1. But if $1 < \gamma_{kS}(G) < n$ or if $1 < d_{kS}(G) < n$ or if $\min\{\gamma_{kS}(G), d_{kS}(G)\} > 1$, we can lower the upper bound $n + 1$.

Corollary 16. Let G be a graph of order $n \geq 4$. If $2 \leq \gamma_{kS}(G) \leq n - 1$ or if $2 \leq d_{kS}(G) \leq n - 1$, then

$$\gamma_{kS}(G) + d_{kS}(G) \leq n - 1.$$

Proof. By Corollary 15, $\gamma_{kS}(G) + d_{kS}(G) < n + 1$. The result follows from Theorem 7 and the fact that, as seen in the introduction, $\gamma_{kS}(G) \equiv n \pmod 2$. □

Corollary 17. Let G be a graph of order n, and let $k \geq 1$ be an integer. If $\min\{\gamma_{kS}(G), d_{kS}(G)\} \geq a$, with $2 \leq a \leq \sqrt{n}$, then

$$\gamma_{kS}(G) + d_{kS}(G) \leq a + \frac{n}{a}.$$

Proof. Since $\min\{\gamma_{kS}(G), d_{kS}(G)\} \geq a \geq 2$, it follows from Theorem 13 that $2 \leq d_{ks}(G) \leq \frac{n}{a}$. Applying the inequality (5), we obtain

$$\gamma_{kS}(G) + d_{kS}(G) \leq d_{kS}(G) + \frac{n}{d_{kS}(G)} \leq g(a) = a + \frac{n}{a} .$$

□

3 Complete graphs

In this section, we use Theorem 13 to determine the signed k-domatic number for complete graphs. The next result is a generalization of Proposition B.

Theorem 18. For $n \geq 2$ and $1 \leq k \leq n$,

$$d_{kS}(K_n) = \begin{cases} \lfloor \frac{n}{k} \rfloor & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k} \rfloor - 1 & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k} \rfloor \text{ is even} \\ \lfloor \frac{n}{k+1} \rfloor & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+1} \rfloor - 1 & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is even.} \end{cases}$$

Proof. Let $V(K_n) = \{x_0, x_1, \ldots, x_{n-1}\}$ be the vertex set of K_n. We consider two cases.

Case 1. $n \equiv k \pmod 2$. By Proposition A, $\gamma_{kS}(K_n) = k$. Let $n = kq + r$, where q is a positive integer and $0 \leq r \leq k-1$. By Theorems 13 and 7, $d_{kS}(K_n) \leq \lfloor \frac{n}{k} \rfloor = q$ if q is odd and $d_{kS}(K_n) \leq q-1$ if q is even.

Subcase 1.1 q is odd. Then r is even. Define the functions $f_1, \ldots, f_q$ as follows.

$$f_1(x_i) = \begin{cases} +1 & \text{if } 0 \leq i \leq \frac{k(q+1)}{2} - 1 \\ -1 & \text{if } \frac{k(q+1)}{2} \leq i \leq kq-1 \\ (-1)^{i+1} & \text{if } r \neq 0 \text{ and } kq \leq i \leq kq+r-1 \end{cases}$$

and for $2 \leq j \leq q$,

$$f_j(x_i) = \begin{cases} f_1(x_{i+2k(j-1) \pmod{kq}}) & \text{if } 0 \leq i \leq kq-1 \\ (-1)^{i+j} & \text{if } r \neq 0 \text{ and } kq \leq i \leq kq+r-1. \end{cases}$$

Since for every j, the expression $i + 2k(j-1) \pmod{kq}$ takes the kq consecutive values $0, 1, \cdots, kq-1$, each function f_j takes, as f_1, $k\frac{q+1}{2} + \frac{r}{2}$

times the value +1 and $k\frac{q-1}{2}+\frac{r}{2}$ times the value -1. In the complete graph G, $\sum_{u\in N[v]} f_j(u) = \sum_{u\in V} f_j(u) = k\frac{q+1}{2}+\frac{r}{2}-(k\frac{q-1}{2}+\frac{r}{2}) = k$ for every vertex v. Hence every function f_j is a SkD function.

Now we show that $f_1, f_2, \cdots, f_q$ is a SkD family, that is that $\sum_{j=1}^{q} f_j(x_i) \le 1$ for each vertex x_i.
When $kq \le i \le kq+r-1$ (in the case $r \ne 0$), we have $\sum_{j=1}^{q} f_j(x_i) = \sum_{j=1}^{q}(-1)^{i+j} = \pm 1 \le 1$.
When $0 \le i \le kq-1$, $\sum_{j=1}^{q} f_j(x_i)$ is equal to the number ν_i of indices j such that $f_j(x_i) = +1$ minus the number of indices j such that $f_j(x_i) = -1$, i. e., to $2\nu_i - q$. By the definition of f_j and f_1, $f_j(x_i) = +1$ if and only if $0 \le i + 2k(j-1) \pmod{kq} < k\frac{q+1}{2}$. Hence

$$\nu_i = |\{j \mid 1 \le j \le q \text{ and } 0 \le i + 2k(j-1) (\text{mod } kq) < k\frac{q+1}{2}\}|.$$

Adding -1 to the three members of the double inequality above shows that $\nu_i = \nu_0$ for each i. Thus it is sufficient to prove that $2\nu_0 \le q+1$. Since $j \le q$, we have $2k(j-1) < 2kq$. Hence

$$\begin{aligned}\nu_0 &= |\{j \mid 0 \le 2k(j-1) < \tfrac{k(q+1)}{2}\}| + |\{j \mid kq \le 2k(j-1) < \tfrac{k(q+1)}{2} + kq\}| \\ &= |\{j \mid 0 \le 2(j-1) < \tfrac{q+1}{2}\}| + |\{j \mid q \le 2(j-1) < \tfrac{q+1}{2} + q\}| \\ &= |\{j \mid 1 \le j < \tfrac{q+5}{4}\}| + |\{j \mid \tfrac{q+2}{2} \le j < \tfrac{3q+5}{4}\}|.\end{aligned}$$

If $q \equiv 3 \pmod 4$ then $|\{j \mid 1 \le j < \frac{q+5}{4}\}| = \frac{q+1}{4}$ and $|\{j \mid \frac{q+2}{2} \le j < \frac{3q+5}{4}\}| = |\{j \mid \frac{q+3}{2} \le j \le \frac{3q+3}{4}\}| = \frac{q+1}{4}$. Thus $\nu_0 = \frac{q+1}{2}$ and $2\nu_0 = q+1$.
If $q \equiv 1 \pmod 4$ then $|\{j \mid 1 \le j < \frac{q+5}{4}\}| = |\{j \mid 1 \le j \le \frac{q+3}{4}\}| = \frac{q+3}{4}$ and $|\{j \mid \frac{q+2}{2} \le j < \frac{3q+5}{4}\}| = |\{j \mid \frac{q+3}{2} \le j \le \frac{3q+1}{4}\}| = \frac{q-1}{4}$. Thus $\nu_0 = \frac{q+1}{2}$ and $2\nu_0 = q+1$.
Hence $\{f_1, \ldots, f_q\}$ is a signed k-dominating family of K_n and $d_{kS}(K_n) \ge q$. Therefore $d_{kS}(K_n) = q$, as desired.

Subcase 1.2 q is even. Then $r+k$ is even. Define the functions $f_1, \ldots, f_{q-1}$ as follows.

$$\begin{aligned} f_1(x_i) &= 1 \quad \text{if} \quad 0 \le i \le \tfrac{k(q-2)}{2} + k - 1 \\ f_1(x_i) &= -1 \quad \text{if} \quad \tfrac{k(q-2)}{2} + k \le i \le k(q-1) - 1 \end{aligned}$$

and for $2 \le j \le q-1$ and $0 \le i \le k(q-1)-1$,

$$f_j(x_i) = f_{j-1}(x_{i+2k}),$$

where the sum is taken modulo $k(q-1)$. In addition,

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } k(q-1) \le i \le kq + r - 1.$$

It is easy to see that f_j is a signed k-dominating function of K_n for each $1 \le j \le q-1$ and $\{f_1, \ldots, f_{q-1}\}$ is a signed k-dominating family on K_n. Hence, $d_{kS}(K_n) \ge q-1$ and so $d_{kS}(K_n) = q-1$ as desired.

Case 2. $n \equiv k+1 \pmod 2$. Suppose that $n = (k+1)q + r$, where q is a positive integer and $0 \le r \le k$. By Proposition A, $\gamma_{kS}(K_n) = k+1$. Hence, by Theorems 13 and 7, $d_{kS}(K_n) \le q$ if q is odd and $d_{kS}(K_n) \le q-1$ if q is even.

Subcase 2.1 q is odd. Then r is even. Define the functions $f_1, \ldots, f_q$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \le i \le \frac{(k+1)(q-1)}{2} + k \\ f_1(x_i) = -1 & \text{if} & \frac{(k+1)(q-1)}{2} + k + 1 \le i \le (k+1)q - 1 \end{array}$$

and for $2 \le j \le q$ and $0 \le i \le (k+1)q - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)q$. In addition, if $r > 0$

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)q \le i \le (k+1)q + r - 1.$$

It is easy to see that f_j is a signed k-dominating function of G for each $1 \le j \le q$ and $\{f_1, \ldots, f_q\}$ is a signed k-dominating family of G. Hence, $d_{kS}(K_n) \ge q$. Therefore $d_{kS}(K_n) = q$, as desired.

Subcase 2.2 q is even. Then $r + k + 1$ is even. Define the functions $f_1, \ldots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = 1 & \text{if} & 0 \le i \le \frac{(k+1)(q-2)}{2} + k \\ f_1(x_i) = -1 & \text{if} & \frac{(k+1)(q-2)}{2} + k + 1 \le i \le (k+1)(q-1) - 1 \end{array}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+1)(q-1) - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)(q-1)$. In addition,

$$f_j(x_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)(q-1) \le i \le (k+1)q + r - 1.$$

It is easy to see that f_j is a signed k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, \ldots, f_{q-1}\}$ is a signed k-dominating family of K_n. Hence, $d_{kS}(K_n) \ge q-1$ and so $d_{kS}(K_n) = q-1$, as desired. □

4 The signed k-domatic number of $K_{n,n}$

In this section, we apply Theorem 13 to determine the signed k-domatic number for the complete bipartite graph $K_{n,n}$.

Lemma 19. Let $G = K_{n,n}$ be the complete bipartite graph with the partite sets X and Y . If $\{f_1, f_2, \ldots, f_d\}$ is a signed k-dominating family with $d = d_{kS}(G) \geq 3$, then, for each $i \in \{1, 2, \ldots, d\}$, there exist at least one vertex $x \in X$ and at least one vertex $y \in Y$, such that $f_i(x) = -1$ and $f_i(y) = -1$, respectively.

Proof. Let, to the contrary, there exists a function f_i, say f_1, with the property that $f_1(x) = 1$ for each $x \in X$. Since $\sum_{y \in N[x]} f_1(y) \geq k$ for each $x \in X$, we observe that there are at most $\frac{n-k-1}{2}$ vertices $y \in Y$ with $f_1(y) = -1$. Hence, the conditions $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$ and $\sum_{u \in N[v]} f_i(u) \geq k$ for each $v \in V(G)$ and each $i \in \{1, 2, \ldots, d\}$, lead to

$$\begin{aligned}
n(n+1) &= \sum_{y \in Y} \sum_{x \in N[y]} 1 \\
&\geq \sum_{y \in Y} \sum_{x \in N[y]} \sum_{i=1}^{d} f_i(x) \\
&= \sum_{i=1}^{d} \sum_{y \in Y} \sum_{x \in N[y]} f_i(x) \\
&= \sum_{y \in Y} \sum_{x \in N[y]} f_1(x) + \sum_{i=2}^{d} \sum_{y \in Y} \sum_{x \in N[y]} f_i(x) \\
&\geq \sum_{y \in Y} f_1(y) + \sum_{y \in Y} \sum_{x \in N(y)} f_1(x) + \sum_{i=2}^{d} \sum_{y \in Y} k \\
&\geq (k-1) + n^2 + n(d-1)k.
\end{aligned}$$

This implies that $d \leq 2$, a contradiction to the hypothesis. Thus for each $i \in \{1, 2, \ldots, d\}$, there exist at least one vertex $x \in X$, such that $f_i(x) = -1$. Similarly, for each $i \in \{1, 2, \ldots, d\}$, there exist at least one vertex $y \in Y$, such that $f_i(y) = -1$. This completes the proof. □

Lemma 20. Let $G = K_{n,n}$ and let k be an positive integer such that $k \leq n$. If $n = 1, 2$, then $d_{kS}(G) = 3$. If $n \geq 3$, then

$$d_{kS}(G) \leq \begin{cases} \lfloor \frac{n}{k+2} \rfloor & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k+2} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+2} \rfloor - 1 & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k+2} \rfloor \text{ is even} \\ \lfloor \frac{n}{k+1} \rfloor & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+1} \rfloor - 1 & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is even.} \end{cases}$$

Proof. Let X and Y be the partite sets of G. Furthermore, let $\{f_1, f_2, \ldots, f_d\}$ is a signed k-dominating family with $d = d_{kS}(G)$.

If $n = 1, 2$, then Observations 3 and 4 lead to the desired result.

Now let $n \geq 3$. If we assume that $d = d_{kS}(G) \geq 2$, then Theorem 7 implies $d \geq 3$. Thus, according to Lemma 19, for each $i \in \{1, 2, \ldots, d\}$, there exist at least one vertex $u \in X$ and at least one vertex $v \in Y$, such that $f_i(u) = -1$ and $f_i(v) = -1$, respectively. Because of $\sum_{x \in N[w]} f_i(x) \geq k$ for each $w \in V(G)$ and each $i \in \{1, 2, \ldots, d\}$, we conclude that there are at most $\frac{n-k-1}{2}$ vertices $x \in X$ with $f_i(x) = -1$ for each $i = 1, 2, \ldots, d$. Thus, it follows for each $i = 1, 2, \ldots, d$ that

$$\sum_{x \in X} f_i(x) \geq k + 2 \text{ if } n \equiv k \pmod 2, \tag{6}$$

and

$$\sum_{x \in X} f_i(x) \geq k + 1 \text{ if } n \equiv k + 1 \pmod 2. \tag{7}$$

In the case that $n \equiv k \pmod 2$, we deduce from (6) and the condition $\sum_{i=1}^{d} f_i(x) \leq 1$ for each $x \in V(G)$ that

$$\begin{array}{rcl} d(k+2) & = & \sum_{i=1}^{d}(k+2) \\ & \leq & \sum_{i=1}^{d} \sum_{x \in X} f_i(x) \\ & = & \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \\ & \leq & \sum_{x \in X} 1 = n. \end{array}$$

Therefore in the case $d \leq \lfloor \frac{n}{k+2} \rfloor$ if $\lfloor \frac{n}{k+2} \rfloor$ is odd and $d \leq \lfloor \frac{n}{k+2} \rfloor - 1$ if $\lfloor \frac{n}{k+2} \rfloor$ is even by Theorem 7.

In the case that $n \equiv k+1 \pmod 2$, we deduce from (7) and the condition $\sum_{i=1}^{d} f_i(x) \leq 1$ for each $x \in V(G)$ that

$$\begin{array}{rcl} d(k+1) & = & \sum_{i=1}^{d}(k+1) \\ & \leq & \sum_{i=1}^{d} \sum_{x \in X} f_i(x) \\ & = & \sum_{x \in X} \sum_{i=1}^{d} f_i(x) \\ & \leq & \sum_{x \in X} 1 = n. \end{array}$$

Thus in this case $d \leq \lfloor \frac{n}{k+1} \rfloor$ if $\lfloor \frac{n}{k+1} \rfloor$ is odd and $d \leq \lfloor \frac{n}{k+1} \rfloor - 1$ if $\lfloor \frac{n}{k+2} \rfloor$ is even by Theorem 7.

□

Theorem 21. Let $G = K_{n,n}$ and let k be an positive integer such that $k \leq n$. If $n \geq 3$, then

$$d_{kS}(G) = \begin{cases} \lfloor \frac{n}{k+2} \rfloor & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k+2} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+2} \rfloor - 1 & \text{if } n \equiv k \pmod 2 \text{ and } \lfloor \frac{n}{k+2} \rfloor \text{ is even} \\ \lfloor \frac{n}{k+1} \rfloor & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is odd} \\ \lfloor \frac{n}{k+1} \rfloor - 1 & \text{if } n \equiv k+1 \pmod 2 \text{ and } \lfloor \frac{n}{k+1} \rfloor \text{ is even.} \end{cases}$$

Proof. By Corollary 5 we may assume $k \leq n-2$. Let $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$ be the partite sets of $K_{n,n}$. We consider two cases.

Case 1. $n \equiv k \pmod 2$. Suppose that $n = (k+2)q + r$, where q is a positive integer and $0 \le r \le k+1$. By Lemma 20, $d_{kS}(K_{n,n}) \le q$ if q is odd and $d_{kS}(K_{n,n}) \le q-1$ if q is even.

Subcase 1.1 q is odd. Then r is even. Define the functions $f_1, \dots, f_q$ as follows.

$$\begin{array}{lll} f_1(x_i) = f_1(y_i) = 1 & \text{if} & 0 \le i \le \frac{(k+2)(q-1)}{2} + k + 1 \\ f_1(x_i) = f_1(y_i) = -1 & \text{if} & \frac{(k+2)(q-1)}{2} + k + 2 \le i \le (k+2)q - 1 \end{array}$$

and for $2 \le j \le q$ and $0 \le i \le (k+2)q - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2k}) \text{ and } f_i(y_i) = f_{i-1}(y_{i+2k}),$$

where the sum is taken modulo $(k+2)q$. In addition, if $r > 0$

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+2)q \le i \le (k+2)q + r - 1.$$

It is easy to see that f_j is a signed k-dominating function of G for each $1 \le j \le q$ and $\{f_1, \dots, f_q\}$ is a signed k-dominating family of G. Hence $d_{kS}(K_{n,n}) \ge q$. Therefore $d_{kS}(K_{n,n}) = q$, as desired.

Subcase 1.2 q is even. Then $r + k$ is even. Define the functions $f_1, \dots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = f_1(y_i) = 1 & \text{if} & 0 \le i \le \frac{(k+2)(q-2)}{2} + k + 1 \\ f_1(x_i) = f_1(y_i) = -1 & \text{if} & \frac{(k+2)(q-2)}{2} + k + 2 \le i \le (k+2)(q-1) - 1 \end{array}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+2)(q-1) - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2k}) \text{ and } f_i(y_i) = f_{i-1}(y_{i+2k}),$$

where the sum is taken modulo $(k+2)(q-1)$. In addition,

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+2)(q-1) \le i \le (k+2)q + r - 1.$$

It is easy to see that f_j is a signed k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, \dots, f_{q-1}\}$ is a signed k-dominating family on G. Hence, $d_{kS}(K_{n,n}) \ge q-1$ and so $d_{kS}(K_{n,n}) = q-1$ as desired.

Case 2. $n \equiv k+1 \pmod 2$. Suppose that $n = (k+1)q + r$, where q is a positive integer and $0 \le r \le k$. By Lemma 20, $d_{kS}(K_{n,n}) \le q$ if q is odd and $d^t_{kS}(K_{n,n}) \le q-1$ if q is even.

Subcase 2.1 q is odd. Then r is even. Define the functions $f_1, \dots, f_q$ as follows.

$$\begin{array}{lll} f_1(x_i) = f_1(y_i) = 1 & \text{if} & 0 \le i \le \frac{(k+1)(q-1)}{2} + k \\ f_1(x_i) = f_1(y_i) = -1 & \text{if} & \frac{(k+1)(q-1)}{2} + k + 1 \le i \le (k+1)q - 1 \end{array}$$

and for $2 \le j \le q$ and $0 \le i \le (k+1)q - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}) \text{ and } f_i(y_i) = f_{i-1}(y_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)q$. In addition, if $r > 0$

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)q \le i \le (k+1)q+r-1.$$

It is easy to see that f_j is a signed k-dominating function of G for each $1 \le j \le q$ and $\{f_1, \ldots, f_q\}$ is a signed k-dominating family of G. Hence, $d_{kS}(K_{n,n}) \ge q$. Therefore $d_{kS}(K_{n,n}) = q$, as desired.

Subcase 2.2 q is even. Then $r + k + 1$ is even. Define the functions $f_1, \ldots, f_{q-1}$ as follows.

$$\begin{array}{lll} f_1(x_i) = f_1(y_i) = 1 & \text{if} & 0 \le i \le \frac{(k+1)(q-2)}{2} + k \\ f_1(x_i) = f_1(y_i) = -1 & \text{if} & \frac{(k+1)(q-2)}{2} + k + 1 \le i \le (k+1)(q-1) - 1 \end{array}$$

and for $2 \le j \le q-1$ and $0 \le i \le (k+1)(q-1) - 1$,

$$f_j(x_i) = f_{j-1}(x_{i+2(k+1)}) \text{ and } f_i(y_i) = f_{i-1}(y_{i+2(k+1)}),$$

where the sum is taken modulo $(k+1)(q-1)$. In addition,

$$f_j(x_i) = f_j(y_i) = (-1)^{i+j} \text{ for } 1 \le j \le q \text{ and } (k+1)(q-1) \le i \le (k+1)q+r-1.$$

It is easy to see that f_j is a signed k-dominating function of G for each $1 \le j \le q-1$ and $\{f_1, \ldots, f_{q-1}\}$ is a signed k-dominating family of G. Hence, $d_{kS}(K_{n,n}) \ge q - 1$ and so $d_{kS}(K_{n,n}) = q - 1$, as desired. □

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