On the (l,ω)-domination number of the cube network *

Xin Xie[a†] Jun-Ming Xu[b]
[a]Department of Mathematics, Huangshan University
Huangshan, 245041, China
[b]Department of Mathematics, University of Science and Technology of China
Hefei, 230026, China

Abstract For an n-connected graph G, the n-wide diameter $d_n(G)$, is the minimum integer m such that there are at least n internally disjoint (di)paths of length at most m between any vertices x and y. For a given integer l, a subset S of $V(G)$ is called an (l,n)- dominating set of G if for any vertex $x \in V(G) - S$ there are at least n internally disjoint (di)paths of length at most l from S to x. The minimum cardinality among all (l,n)-dominating sets of G is called the (l,n)-domination number. In this paper, we obtain that the (l,ω)-domination number of the d-ary cube network $C(d,n)$ is 2 for $1 \le \omega \le n$ and $d_\omega(G) - f(d,n) \le l \le d_\omega(G) - 1$ if $d,n \ge 4$, where $f(d,n) = \min\{e(\lfloor n/2 \rfloor + 1), \lceil n/2 \rceil e'\}$.

Keywords: Cube network, Domination number, Wide diameter, Combinatorial problems

MR Subject Classification: 05C40 68M10 68M15 68R10

1 Introduction

This paper uses graphs to represent networks. The distance $d_G(x,y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a (di)path from x to y. The diameter $d(G)$ is the maximum distance from one vertex to another. The connectivity $k(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial network.

In order to characterize the reliability of transmission delay in a real-time parallel processing system, Hsu and Lyuu [6], Flandrin and Li [4]

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† Corresponding author: xiexin@hsu.edu.cn

independently introduced n-wide diameter. For an n-connected graph G, the distance with width n from x to y, denoted by $d_n(G;x,y)$, is the minimum number m for which there are n internally disjoint (x,y)-(di)paths in G of length at most m. The n-wide diameter of G, i.e., the n-diameter, denoted by $d_n(G)$, is the maximum of $d_n(G;x,y)$ over all pairs (x,y) of vertices of G.

Li and Xu [7] defined a new parameter (l,n)-domination number. This motives us to generalize the definition to that of the digraph. Let G be an n-connected digraph, S a nonempty and proper subset of $V(G)$, x a vertex in $G-S$. For a given positive integer l, x is (l,n)-dominated by S if there are at least n internally disjoint (S,x)-dipaths of length at most l. S is said to be an (l,n)-dominating set of G if S can (l,n)-dominate any vertex in $G-S$. The minimum cardinality among all (l,n)-dominating sets of G is called the (l,n)-domination number, denoted by $\gamma_{l,n}(G)$.

The d-ary cube network $C(d,n)$ is a digraph of d^n vertices, in which any vertex x has the form $(x_{n-1},x_{n-2},\dots,x_0)$ where $0\le x_i\le d-1$ for $0\le i\le n-1$, and x is adjacent to $(x_{n-1},\dots,x_{j+1},x_j+1,x_{j-1},\dots,x_0)$ for $0\le j\le n-1$, where additions are taken modulo d. $C(2,n)$ is the n-dimensional binary hypercube Q_n. It is clear that $C(d,n)$ is vertex-transitive and its diameter is $n(d-1)$. Hsu and Lyuu [6] proved that $d_n(C(d,n))=n(d-1)+1$. Liaw and Chang [8] showed that $d_\omega(C(d,n))=n(d-1)$ for $1\le\omega\le n-1$ and $d_n(C(d,n))=n(d-1)+1$. Since $\gamma_{l,n}(G)=1$ for $l\ge d_n(G)$ and $\gamma_{l,n}(G)\ge 2$ for $l<d_n(G)$, so it is of interest to show some general properties and values of the (l,ω)-domination numbers of n-connected graphs for $l<d_n(G)$ and $1\le\omega\le n$ (see, for example [1, 2, 5, 7, 9, 10, 11, 12]).

In this paper, we obtain $\gamma_{l,\omega}(C(d,n))=2$ for $1\le\omega\le n$ and $d_\omega(G)-f(d,n)\le l\le d_\omega(G)-1$ if $d,n\ge 4$, where $f(d,n)=\min\{e(\lfloor n/2\rfloor+1),\lceil n/2\rceil e'\}$.

Terminologies and notations not defined here are referred to [3].

2 Preliminaries

Let $c_i(x)=x_i$ denote the ith component of vertex $x=(x_{n-1},x_{n-2},\dots,x_0)$. For $0\le i\le n-1$, the ith unit vector is the vector e_i^n with $c_i(e_i^n)=1$ and $c_j(e_i^n)=0$ for $0\le j\le n-1$, with $j\ne i$. The vertex set of $C(d,n)$ can be viewed as a module over Z_d. So vertex x can also be written as $x=\sum_{i=0}^{n-1}x_ie_i^n$. Denote $\lfloor d/2\rfloor$ and $\lceil d/2\rceil$ by e and e', respectively. Let $f(d,n)=\min\{e(\lfloor n/2\rfloor+1),\lceil n/2\rceil e'\}$ in this paper.

Suppose $a_0,a_1,\cdots a_r$ are positive integers and $0\le i_0\le i_1\le\cdots\le i_r\le n-1$ for $0\le r\le n-1$. Denote by $<<a_0e_{i_0}^n(o),a_1e_{i_1}^n(o),\cdots,a_re_{i_r}^n(o)>>$

the following dipath from vertex $o=(0,0,\cdots,0)$ to vertex $\sum\limits_{j=0}^{r} a_j e^n_{i_j}$:

$$\begin{aligned} o \; & \to e^n_{i_0} \to 2e^n_{i_0} \to \cdots \to a_0 e^n_{i_0} \\ & \to a_0 e^n_{i_0} + e^n_{i_1} \to a_0 e^n_{i_0} + 2e^n_{i_1} \to \cdots \to a_0 e^n_{i_0} + a_1 e^n_{i_1} \\ & \to \cdots \\ & \to \sum_{j=0}^{r-1} a_j e^n_{i_j} + e^n_{i_r} \to \sum_{j=0}^{r-1} a_j e^n_{i_j} + 2e^n_{i_r} \to \cdots \to \sum_{j=0}^{r} a_j e^n_{i_j}, \end{aligned}$$

and by $<< a_0 e^n_{i_0}(v), a_1 e^n_{i_1}(v), \cdots, a_r e^n_{i_r}(v) >>$ the following dipath from vertex $v=(e,e,\cdots,e)$ to vertex $v+\sum\limits_{j=0}^{r} a_j e^n_{i_j}$:

$$\begin{aligned} v \; & \to v + e^n_{i_0} \to v + 2e^n_{i_0} \to \cdots \to v + a_0 e^n_{i_0} \\ & \to v + a_0 e^n_{i_0} + e^n_{i_1} \to v + a_0 e^n_{i_0} + 2e^n_{i_1} \to \cdots \to v + a_0 e^n_{i_0} + a_1 e^n_{i_1} \\ & \to \cdots \\ & \to v + \sum_{j=0}^{r-1} a_j e^n_{i_j} + e^n_{i_r} \to v + \sum_{j=0}^{r-1} a_j e^n_{i_j} + 2e^n_{i_r} \to \cdots \to v + \sum_{j=0}^{r} a_j e^n_{i_j}. \end{aligned}$$

3 Main results

Lemma 3.1 Let $S=\{o,v\}$ be a subset of $V(C(d,n))$ with $o=(0,0,\cdots,0)$ and $v=(e,e,\cdots,e)$, $d,n\geq 4$. Then there exists n internally disjoint dipaths of length at most $n(d-1)-f(n,d)+1$ from S to $x\in V(C(d,n))-S$ if vertex x has no zero components.

Proof Since the digraph is vertex-transitive, without loss of generality, we consider the following cases for any vertex x with no zero components in $V(C(d,n))-S$:

Case 1. Vertex x has no components with value e.

Assume $x=(\overbrace{x_{n-1},\ldots,x_j}^{n-j},\overbrace{x_{j-1},\ldots,x_0}^{j})$ for $e+1\leq x_{n-1},\ldots,x_j\leq d-1$ and $1\leq x_{j-1},\ldots,x_0\leq e-1$.

Subcase 1a. $\lceil n/2\rceil \leq j\leq n$. Construct n internally disjoint dipaths from o to x as follows:

$P_t :<< x_t e^n_t(o), x_{t+1}e^n_{t+1}(o), \cdots, x_{n-1}e^n_{n-1}(o), x_0 e^n_0(o), x_1 e^n_1(o), \cdots, x_{t-1}e^n_{t-1}(o) >>$ for $0\leq t\leq n-1$.

We can see that the length of each dipath is

$$\sum_{l=0}^{n-1} x_l \leq j(e-1)+(n-j)(d-1) = n(d-1)-je' \leq n(d-1)-\lceil n/2\rceil e'.$$

Subcase 1b. $0\leq j\leq \lceil n/2\rceil -1$. By vertex-transitive, we can construct n internally disjoint dipaths from v to x in the same way as in Subcase 1a, and the length of each dipath is

$$\begin{aligned}\sum_{l=0}^{j-1}(e'+x_l)+\sum_{l=j}^{n-1}(x_l-e) &\le j(d-1)+(n-j)(e'-1)\\ &= n(e'-1)+je\\ &\le n(e'-1)+e(\lceil n/2\rceil-1)\\ &= n(d-1)-e(\lfloor n/2\rfloor+1).\end{aligned}$$

Case 2. Vertex x has some components with value e.

Assume $x=(\overbrace{x_{n-1},\ldots,x_{j+k}}^{n-j-k},\overbrace{e,\ldots,e}^{k},\overbrace{x_{j-1},\ldots,x_0}^{j})$ for $e+1\le x_{n-1},\ldots,x_{j+k}\le d-1$ and $1\le x_{j-1},\ldots,x_0\le e-1$, $k\ge 1$.

Subcase 2a. $\lceil n/2\rceil-1\le j\le n$. Construct the same dipaths from o to x as in Subcase 1a. The length of each dipath is

$$\begin{aligned}\sum_{l=0}^{j-1}x_l+ke+\sum_{l=j+k}^{n-1}x_l &\le j(e-1)+ke+(n-j-k)(d-1)\\ &= n(d-1)-je'-k(e'-1)\\ &\le n(d-1)-(\lceil n/2\rceil-1)e'-(e'-1)\\ &= n(d-1)-\lceil n/2\rceil e'+1.\end{aligned}$$

Subcase 2b. $0\le j\le\lceil n/2\rceil-2$. Construct n internally disjoint dipaths from v to x as follows:

$P_t :<< (e'+x_t)e_t^n(v),(e'+x_{t+1})e_{t+1}^n(v),\cdots,(e'+x_{j-1})e_{j-1}^n(v),(x_{j+k}-e)e_{j+k}^n(v),(x_{j+k+1}-e)e_{j+k+1}^n(v),\cdots,(x_{n-1}-e)e_{n-1}^n(v),(e'+x_0)e_0^n(v),(e'+x_1)e_1^n(v),\cdots,(e'+x_{t-1})e_{t-1}^n(v) >>$ for $0\le t\le j-1$;

$P_t :<< (d-1)e_t^n(v),(x_{j+k}-e)e_{j+k}^n(v),(x_{j+k+1}-e)e_{j+k+1}^n(v),\cdots,(x_{n-1}-e)e_{n-1}^n(v),(e'+x_0)e_0^n(v),(e'+x_1)e_1^n(v),\cdots,(e'+x_{j-1})e_{j-1}^n(v),e_t^n(v) >>$ for $j\le t\le j+k-1$;

$P_t :<< (x_t-e)e_t^n(v),(x_{t+1}-e)e_{t+1}^n(v),\cdots,(x_{n-1}-e)e_{n-1}^n(v),(e'+x_0)e_0^n(v),(e'+x_1)e_1^n(v),\cdots,(e'+x_{j-1})e_{j-1}^n(v),(x_{j+k}-e)e_{j+k}^n(v),(x_{j+k+1}-e)e_{j+k+1}^n(v),\cdots,(x_{t-1}-e)e_{t-1}^n(v) >>$ for $j+k\le t\le n-1$.

The length of each dipath is at most

$$\begin{aligned}&\sum_{l=0}^{j-1}(e'+x_l)+\sum_{l=j+k}^{n-1}(x_l-e)+d\\ &\le j(d-1)+(n-j-k)(e'-1)+d\\ &= n(e'-1)+je-k(e'-1)+d\\ &\le n(e'-1)+(\lceil n/2\rceil-2)e-e'+1+d\\ &= n(d-1)-e(\lfloor n/2\rfloor+1)+1.\end{aligned}$$

Summarizing cases 1 and 2, the length of each dipath is at most $n(d-1)-f(n,d)+1$.

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Lemma 3.2 Let $S = \{o, v\}$ be a subset of $V(C(d,n))$ with $o = (0, 0, \cdots, 0)$ and $v = (e, e, \cdots, e)$, $d, n \geq 4$. Then there exists n internally disjoint dipaths of length at most $n(d-1) - f(n,d) + 1$ from S to $x \in V(C(d,n)) - S$ if vertex x has some zero components.

Proof We consider the following cases:

Case 1. Vertex x has no components with value e.

Assume $x = (\overbrace{x_{n-1}, \ldots, x_{i+j}}^{n-i-j}, \overbrace{x_{i+j-1}, \ldots, x_i}^{j}, \overbrace{0, \ldots, 0}^{i})$ for $e + 1 \leq x_{n-1}, \cdots, x_{i+j} \leq d - 1$ and $1 \leq x_{i+j-1}, \cdots, x_i \leq e - 1$, $i \geq 1$.

Subcase 1a. $\lceil n/2 \rceil + 1 \leq i + j \leq n$. Construct n internally disjoint dipaths from o to x in the samy way as in Subcase 2b of Lemma 3.1. The length of each dipath is at most

$$\begin{aligned}
\sum_{l=i}^{n-1} x_l + d &\leq j(e-1) + (n-i-j)(d-1) + d \\
&= n(d-1) - i(d-1) - je' + d \\
&\leq n(d-1) - i(d-1) - (\lceil n/2 \rceil + 1 - i)e' + d \\
&= n(d-1) - i(e-1) - (\lceil n/2 \rceil + 1)e' + d \\
&\leq n(d-1) - (e-1) - (\lceil n/2 \rceil + 1)e' + d \\
&\leq n(d-1) - \lceil n/2 \rceil e' + 1.
\end{aligned}$$

Subcase 1b. $0 \leq i + j \leq \lceil n/2 \rceil$. Construct the same dipaths from v to x as in Subcase 1b of Lemma 3.1. The length of each dipath P_t is

$$\begin{aligned}
&ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j}^{n-1} (x_l - e) \\
&\leq ie' + j(d-1) + (n-i-j)(e'-1) \\
&= n(e'-1) + i + je \\
&\leq n(e'-1) + i + (\lceil n/2 \rceil - i)e \\
&= n(e'-1) - i(e-1) + \lceil n/2 \rceil e \\
&\leq n(e'-1) - (e-1) + \lceil n/2 \rceil e \\
&= n(d-1) - e(\lfloor n/2 \rfloor + 1) + 1.
\end{aligned}$$

Case 2. Vertex has some component with value e.

Assume $x = (\overbrace{x_{n-1}, \ldots, x_{i+j+k}}^{n-i-j-k}, \overbrace{e, \ldots, e}^{k}, \overbrace{x_{i+j-1}, \ldots, x_i}^{j}, \overbrace{0, \ldots, 0}^{i})$ for $e + 1 \leq x_{n-1}, \ldots, x_{i+j+k} \leq d - 1$ and $1 \leq x_{i+j-1}, \ldots, x_i \leq e - 1$, $i, k \geq 1$.

Subcase 2a. $\lceil n/2 \rceil + 1 \leq i + j \leq n$. Construct n internally disjoint dipaths from o to x in the same way as in Subcase 2b of Lemma 3.1. So the length of each dipath is at most

$$\begin{aligned}\sum_{l=i}^{n-1} x_l + d &\le j(e-1) + ke + (n-i-j-k)(d-1) + d \\ &= n(d-1) - i(d-1) - je' - k(e'-1) + d \\ &\le n(d-1) - i(d-1) - (\lceil n/2\rceil + 1 - i)e' - k(e'-1) + d \\ &= n(d-1) - i(e-1) - (\lceil n/2\rceil + 1)e' - k(e'-1) + d \\ &\le n(d-1) - (e-1) - (\lceil n/2\rceil + 1)e' - (e'-1) + d \\ &= n(d-1) - e'(\lceil n/2\rceil + 1) + 2.\end{aligned}$$

Subcase 2b. $i+j = \lceil n/2\rceil$. Construct $n-i-j$ internally disjoint dipaths from o to x and $i+j$ internally disjoint dipaths from v to x as follows:

$P_t :<< x_i e_i^n(o), x_{i+1}e_{i+1}^n(o), \cdots, x_{i+j-1}e_{i+j-1}^n(o), x_t e_t^n(o), x_{t+1}e_{t+1}^n(o), \cdots, x_{n-1}e_{n-1}^n(o), x_{i+j}e_{i+j}^n(o), x_{i+j+1}e_{i+j+1}^n(o), \cdots, x_{t-1}e_{t-1}^n(o) >>$ for $i+j \le t \le n-1$;

$P_t :<< (e'+x_t)e_t^n(v), (e'+x_{t+1})e_{t+1}^n(v), \cdots, (e'+x_{i+j-1})e_{i+j-1}^n(v), (x_{i+j+k}-e)e_{i+j+k}^n(v), (x_{i+j+k+1}-e)e_{i+j+k+1}^n(v), \cdots, (x_{n-1}-e)e_{n-1}^n(v), (e'+x_0)e_0^n(v), (e'+x_1)e_1^n(v), \cdots, (e'+x_{t-1})e_{t-1}^n(v) >>$ for $0 \le t \le i+j-1$.

The length of dipath P_t for $i+j \le t \le n-1$ is $\sum_{l=i}^{n-1} x_l$ and the length of dipath P_t for $0 \le t \le i+j-1$ is $\sum_{l=0}^{i+j-1}(e'+x_l) + \sum_{l=i+j+k}^{n-1}(x_l - e)$. Noting $\sum_{l=i}^{n-1} x_l \le \sum_{l=0}^{i+j-1}(e'+x_l) + \sum_{l=i+j+k}^{n-1}(x_l - e)$ for $i+j = \lceil n/2\rceil$, so the length of each dipath P_t is at most

$$\begin{aligned}&\sum_{l=0}^{i+j-1}(e'+x_l) + \sum_{l=i+j+k}^{n-1}(x_l - e) \\ &\le ie' + j(d-1) + (n-i-j-k)(e'-1) \\ &= n(e'-1) + i + je - k(e'-1) \\ &= n(e'-1) + i + (\lceil n/2\rceil - i)e - k(e'-1) \\ &= n(e'-1) + \lceil n/2\rceil e - i(e-1) - k(e'-1) \\ &\le n(e'-1) + \lceil n/2\rceil e - (e-1) - (e'-1) \\ &= n(d-1) - \lfloor n/2\rfloor e - d + 2.\end{aligned}$$

Subcase 2c. $i+j = \lceil n/2\rceil - 1$. Construct the same dipaths as in Subcase 2b of Lemma 3.2. Similarly, $\sum_{l=0}^{i+j-1}(e'+x_l) + \sum_{l=i+j+k}^{n-1}(x_l - e) \le n(d-1) - \lfloor n/2\rfloor e - d - e + 2$, and

$$\begin{aligned}\sum_{l=i}^{n-1} x_l &\le j(e-1)+ke+(n-i-j-k)(d-1)\\ &= n(d-1)-i(d-1)-je'-k(e'-1)\\ &= n(d-1)-i(d-1)-(\lceil n/2\rceil-1-i)e'-k(e'-1)\\ &= n(d-1)-i(e-1)-(\lceil n/2\rceil-1)e'-k(e'-1)\\ &\le n(d-1)-(e-1)-(\lceil n/2\rceil-1)e'-(e'-1)\\ &= n(d-1)-e-\lceil n/2\rceil e'+2.\end{aligned}$$

Subcase 2d. $0 \le i+j \le \lceil n/2\rceil - 2$. Construct the same dipaths as in Subcase 2b of Lemma 3.1. So the length of dipath is at most

$$\begin{aligned}&ie' + \sum_{l=i}^{i+j-1}(e'+x_l) + \sum_{l=i+j+k}^{n-1}(x_l-e)+d\\ &\le ie'+j(d-1)+(n-i-j-k)(e'-1)+d\\ &= n(e'-1)+i+je-k(e'-1)+d\\ &\le n(e'-1)+i+(\lceil n/2\rceil-2-i)e-k(e'-1)+d\\ &= n(e'-1)-i(e-1)+(\lceil n/2\rceil-2)e-k(e'-1)+d\\ &\le n(e'-1)-(e-1)+(\lceil n/2\rceil-2)e-(e'-1)+d\\ &= n(d-1)-e(\lfloor n/2\rfloor+2)+2.\end{aligned}$$

Summarizing cases 1 and 2, the length of each dipath is at most $n(d-1)-f(n,d)+1$.

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Finally, we can see that Lemma 3.1 and 3.2 yield the following theorem.

Theorem 3.3 If $d,n \ge 4$, then $\gamma_{l,n}(C(d,n)) = 2$ for $n(d-1)-f(n,d)+1 \le l \le n(d-1)$.

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Lemma 3.4 Let $S = \{o,v\}$ be a subset of $V(C(d,n))$ with $o = (0,0,\cdots,0)$ and $v = (e,e,\cdots,e)$, $d,n \ge 4$. For $1 \le \omega \le n-1$, there exists ω internally disjoint dipaths of length at most $n(d-1)-f(n,d)$ from S to $x \in V(C(d,n)) - S$ if vertex x has no zero components.

Proof We consider the following cases:

Case 1. Vertex x has no components with value e.

From the Case 1 of Lemma 3.1, the result follows.

Case 2. Vertex x has some component with value e.

Assume $x = (\overbrace{x_{n-1},\dots,x_{j+k}}^{n-j-k},\overbrace{e,\dots,e}^{k},\overbrace{x_{j-1},\dots,x_0}^{j})$ for $e+1 \le x_{n-1}$, $\dots, x_{j+k} \le d-1$ and $1 \le x_{j-1},\dots,x_0 \le e-1$, $k \ge 1$.

Subcase 2a. $\lceil n/2\rceil \le j \le n$. Construct the same ω internally disjoint dipaths as in Subcase 1a of Lemma 3.1, and we can easily see the length of each dipath is

$$\sum_{l=0}^{j-1} x_l + ke + \sum_{l=j+k}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 1 - e'.$$

The details are omitted here.

Subcase 2b. $0 \leq j \leq \lceil n/2 \rceil - 1$.

If $k = 1$, construct the same ω internally disjoint dipaths as P_t for $0 \leq t \leq j-1$ and $j+1 \leq t \leq n-1$ in Subcase 2b of Lemma 3.1. Similarly, the length of each dipath is

$$\sum_{l=0}^{j-1}(e' + x_l) + \sum_{l=j+1}^{n-1}(x_l - e) \leq n(d-1) - e(\lfloor n/2 \rfloor + 1) + 1 - e'.$$

Otherwise, $k \geq 2$. We consider the following cases:

For $j = \lceil n/2 \rceil - 1$, the desired ω internally disjoint dipaths are similar to that in Subcase 1a of Lemma 3.1, the length of each dipath is

$$\sum_{l=0}^{j-1} x_l + ke + \sum_{l=j+k}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 2 - e'.$$

For $j \leq \lceil n/2 \rceil - 2$, the desired ω internally disjoint dipaths are similar to that in Subcase 2b of Lemma 3.1, the length of each dipath is at most

$$\sum_{l=0}^{j-1}(e' + x_l) + \sum_{l=j+1}^{n-1}(x_l - e) + d \leq n(d-1) - e(\lfloor n/2 \rfloor + 1) + 2 - e'.$$

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Lemma 3.5 Let $S = \{o, v\}$ be a subset of $V(C(d,n))$ with $o = (0,0,\cdots,0)$ and $v = (e,e,\cdots,e)$, $d,n \geq 4$. For $1 \leq \omega \leq n-1$, there exists ω internally disjoint dipaths of length at most $n(d-1) - f(n,d)$ from S to $x \in V(C(d,n)) - S$ if vertex x has some zero components.

Proof We consider the following cases:

Case 1. Vertex x has no components with value e.

Assume $x = (\overbrace{x_{n-1},\ldots,x_{i+j}}^{n-i-j}, \overbrace{x_{i+j-1},\ldots,x_i}^{j}, \overbrace{0,\ldots,0}^{i})$ for $e+1 \leq x_{n-1}, \cdots, x_{i+j} \leq d-1$ and $1 \leq x_{i+j-1}, \cdots, x_i \leq e-1$, $i \geq 1$.

Subcase 1a. $\lceil n/2 \rceil + 1 \leq i + j \leq n$. We can construct ω internally disjoint dipaths from o to x.

If $i = 1$, the length of each dipath is

$$\sum_{l=i}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 1 - d.$$

If $i \geq 2$, the length of each dipath is at most

$$\sum_{l=i}^{n-1} x_l + d \leq n(d-1) - \lceil n/2 \rceil e' + 2 - e.$$

Subcase 1b. $i + j = \lceil n/2 \rceil$. If $i \geq 2$, construct ω internally disjoint dipaths from v to x, the length of each dipath is

$$ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j}^{n-1} (x_l - e) \leq n(d-1) - e(\lfloor n/2 \rfloor + 1) + 2 - e.$$

If $i = 1$, it is similar to the case of $i = 1$ in Subcase 1a of Lemma 3.5. The length of each dipath is

$$\sum_{l=i}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 1 - e.$$

Subcase 1c. $i + j \leq \lceil n/2 \rceil - 1$. Construct ω internally disjoint dipaths from v to x, the length of each dipath is

$$ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j}^{n-1} (x_l - e) \leq n(d-1) - e(\lfloor n/2 \rfloor + 1) + 1 - e.$$

Case 2. Vertex x has some component with value e.

From the Case 2 of Lemma 3.2, the result follows. ∎

Finally, we can see that Lemma 3.4 and 3.5 yield the following theorem.

Theorem 3.6 If $d, n \geq 4$, then $\gamma_{l,\omega}(C(d,n)) = 2$ for $1 \leq \omega \leq n-1$ and $n(d-1) - f(n,d) \leq l \leq n(d-1) - 1$. ∎

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