SOME GENERALIZATIONS OF MULTIPLE LAGUERRE POLYNOMIALS VIA RODRIGUES FORMULA

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ABSTRACT. This paper aims to provide systematic investigation of the family of polynomials generated by the Rodrigues' formulas

$$K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p) = (-1)^{n_1+n_2} e^{px^k} [\prod_{j=1}^{2} x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{\alpha_j+n_j}] e^{-px^k},$$

and

$$M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k) = \frac{(-1)^{n_1+n_2}}{p_1^{n_1} p_2^{n_2}} x^{-\alpha_0} [\prod_{j=1}^{2} e^{p_j x^k} \frac{d^{n_j}}{dx^{n_j}} e^{-p_j x^k}] x^{n_1+n_2+\alpha_0},$$

which include the multiple Laguerre I and the multiple Laguerre II polynomials, respectively. The explicit forms, certain operational formulas involving these polynomials with some applications and linear generating functions for $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$ and $M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k)$ are obtained.

1. INTRODUCTION

Multiple orthogonal polynomials, which are the extension of the orthogonal polynomials have been an active research field during the last few decades. Multiple orthogonal polynomials are closely related to Hermite-Pade approximation of a system of Markov functions [12]. These polynomials have some general properties such as Rodrigues' formula and generating functions.

There are several attempts to generalize the known special functions via corresponding Rodrigues' formula [4],[5],[11],[14],[16] and new papers constantly coming out related with these generalizations [3],[6],[7],[8],[15].

Our starting point in this paper is to obtain some properties of multiple Laguerre I and multiple Laguerre II polynomials [2]. In obtaining these

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properties we consider the following general family of polynomials:

$$(1)\quad (-1)^{n_1+n_2}K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)=e^{px^k}[\prod_{j=1}^{2}x^{-\alpha_j}\frac{d^{n_j}}{dx^{n_j}}x^{\alpha_j+n_j}]e^{-px^k}$$

and

$$(2)\quad M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k)=\frac{(-1)^{n_1+n_2}}{p_1^{n_1}p_2^{n_2}}x^{-\alpha_0}[\prod_{j=1}^{2}e^{p_jx^k}\frac{d^{n_j}}{dx^{n_j}}e^{-p_jx^k}]x^{n_1+n_2+\alpha_0}$$

where $p, p_1, p_2 > 0$ with $p_1 \neq p_2$; $\alpha_0, \alpha_1, \alpha_2 > -1$ with $\alpha_1 \neq \alpha_2$ and k is a natural number. It should be noticed that, in the particular case $k = 1$ and $p = 1$, definition (1) gives the Rodrigues' formula of the multiple Laguerre I polynomials and the case $k = 1$ reduces (2) to the Rodrigues' formula of the multiple Laguerre II polynomials [2].

Except for the appropriate choices of the parameters, it is not easy to show that the polynomials defined by (1) and (2) satisfy multiple orthogonality properties in the sense of the classical definition [1]. On the other hand, they may satisfy a kind of multiple biorthogonality which is an open problem.

Recently, two classes of multiple orthogonal polynomials were discussed in [10], where J. Coussement and W. Van Assche obtained a linear differential equation for them by combining the lowering operator with the raising operator. Obtaining a differential equation for certain classes of polynomials including the polynomials defined by (1) and (2), is another open problem which is related with the above mentioned open problem.

We organize the paper as follows. In section 2, we obtain the explicit forms of the polynomials as

$$K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)=(-1)^{n_1+n_2}n_1!n_2!\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{p^i}{i!}(-1)^j\binom{i}{j}$$

$$\cdot\binom{\alpha_2+n_2+kj}{n_2}x^{ki}\sum_{l=0}^{n_1}\sum_{h=0}^{l}\frac{p^l}{l!}(-1)^h\binom{l}{h}\binom{\alpha_1+n_1+ki+kh}{n_1}x^{kl},$$

and

$$M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k)=\frac{(-1)^{n_1+n_2}n_1!n_2!}{p_1^{n_1}p_2^{n_2}}\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{p_2^i}{i!}(-1)^j\binom{i}{j}$$

$$\cdot\binom{\alpha_0+n_1+n_2+kj}{n_2}x^{ki}\sum_{l=0}^{n_1}\sum_{h=0}^{l}\frac{p_1^l}{l!}(-1)^h\binom{l}{h}\binom{\alpha_0+n_1+ki+kh}{n_1}x^{kl}$$

which shows that the polynomials are of exact degree $k(n_1 + n_2)$. In section 3, we obtain some operational formulas involving the polynomials

$K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$ and $M_{n_1,n_2}^{(\alpha_0,p_1,\cdot p_2)}(x,k)$ and give some applications to these results. In section 4, linear generating functions are obtained for the polynomials defined by (1) and (2).

2. Explicit Expression of the Polynomials

Chatterjea [5], defined the generalized Laguerre polynomials via Rodrigues' formula by

$$(3)\qquad T_{kn}^{(\alpha)}(x,p)=\frac{1}{n!}x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k});\quad p,k\in\mathbb{N},$$

and obtained its explicit formula as

$$(4)\qquad T_{kn}^{(\alpha)}(x,p)=\sum_{i=0}^{n}\frac{p^i}{i!}x^{ki}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{\alpha+n+kj}{n}.$$

In obtaining the explicit form of the polynomials $T_{kn}^{(\alpha)}(x,p)$, he used the following generalized rule for the derivative of $f(z)$ [13],

$$(5)\qquad D_z^s(f(z))=\sum_{k=0}^{s}\frac{(-1)^k}{k!}D_z^kf(z)\sum_{j=0}^{k}(-1)^j\binom{k}{j}z^{k-j}D_z^sz^j,$$

where $D_z^s(f(z))=\dfrac{d^s(f(z))}{dz^s}$.

Now, using (1), (3) and (4), we get

$$(-1)^{n_1+n_2}K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)=e^{px^k}[x^{-\alpha_1}\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}][x^{-\alpha_2}\frac{d^{n_2}}{dx^{n_2}}x^{\alpha_2+n_2}e^{-px^k}]$$

$$=n_2!e^{px^k}[x^{-\alpha_1}\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}e^{-px^k}T_{kn_2}^{(\alpha_2)}(x,p)]$$

$$=n_2!e^{px^k}x^{-\alpha_1}\sum_{i=0}^{n_2}\frac{p^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2}\frac{d^{n_1}}{dx^{n_1}}(x^{\alpha_1+n_1+ki}e^{-px^k})$$

$$=n_2!e^{px^k}x^{-\alpha_1}\sum_{i=0}^{n_2}\frac{p^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2}$$

$$\cdot\sum_{s=0}^{n_1}\binom{n_1}{s}D^{n_1-s}(x^{\alpha_1+n_1+ki})D^s(e^{-px^k}).$$

Taking into account (5), we obtain

$$(-1)^{n_1+n_2}K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p) = n_1!n_2!\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{p^i}{i!}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2}x^{ki}$$

$$\cdot\sum_{s=0}^{n_1}\binom{\alpha_1+n_1+ki}{n_1-s}\sum_{l=0}^{s}\frac{p^l}{l!}\sum_{h=0}^{l}(-1)^h\binom{l}{h}\binom{kh}{s}x^{kl}$$

$$= n_1!n_2!\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{p^i}{i!}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2}x^{ki}$$

$$\cdot\sum_{l=0}^{n_1}\frac{p^l}{l!}x^{kl}\sum_{h=0}^{l}(-1)^h\binom{l}{h}\sum_{s=l}^{n_1}\binom{\alpha_1+n_1+ki}{n_1-s}\binom{kh}{s}.$$

On the other hand, since (see [5])

$$\sum_{h=0}^{l}(-1)^h\binom{l}{h}\sum_{s=0}^{l-1}\binom{\alpha_1+n_1+ki}{n_1-s}\binom{kh}{s} = 0,$$

we can state the following theorem, which gives the explicit form of the polynomials defined by (1).

Theorem 2.1. *The family of polynomials $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$ has the explicit expression*

$$K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p) = (-1)^{n_1+n_2}n_1!n_2!\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{p^i}{i!}(-1)^j\binom{i}{j}$$

$$\cdot\binom{\alpha_2+n_2+kj}{n_2}x^{ki}\sum_{l=0}^{n_1}\sum_{h=0}^{l}\frac{p^l}{l!}(-1)^h\binom{l}{h}\binom{\alpha_1+n_1+ki+kh}{n_1}x^{kl}.$$

In a similar way, it can be obtained that

Theorem 2.2. *The family of polynomials $M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k)$ defined by (2) has the explicit expression*

$$M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k) = \frac{(-1)^{n_1+n_2}n_1!n_2!}{p_1^{n_1}p_2^{n_2}}\sum_{i=0}^{n_2}\sum_{j=0}^{i}\frac{p_2^i}{i!}(-1)^j\binom{i}{j}$$

$$\cdot\binom{\alpha_0+n_1+n_2+kj}{n_2}x^{ki}\sum_{l=0}^{n_1}\sum_{h=0}^{l}\frac{p_1^l}{l!}(-1)^h\binom{l}{h}\binom{\alpha_0+n_1+ki+kh}{n_1}x^{kl}.$$

Remark 2.1. *Theorem 2.1 and Theorem 2.2 state that the polynomials $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$ and $M_{n_1,n_2}^{(\alpha_0,p_1,.p_2)}(x,k)$ are of exact degrees $k(n_1+n_2)$.*

3. Operational Formulas

In this section we obtain some operational formulas for the polynomials defined by (1) and (2). We should note that some operational formulas have recently been given in [9].

In [5], Chatterjea obtained the operational formula

$$(6)\qquad x^{-\alpha}e^{px^k}D^n(x^{\alpha+n}e^{-px^k}Y)=\prod_{j=1}^{n}(xD-pkx^k+\alpha+j)Y$$

where Y is any sufficiently differentiable function of x. Now considering

$$\begin{aligned} & e^{px^k}[\prod_{j=1}^{2}x^{-\alpha_j}\frac{d^{n_j}}{dx^{n_j}}x^{\alpha_j+n_j}]e^{-px^k}Y \\ = \quad & e^{px^k}x^{-\alpha_1}\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}x^{-\alpha_2}\frac{d^{n_2}}{dx^{n_2}}x^{\alpha_2+n_2}e^{-px^k}Y\end{aligned}$$

and using (6), we get

$$(7)\qquad \begin{aligned} & e^{px^k}[x^{-\alpha_1}\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}x^{-\alpha_2}\frac{d^{n_2}}{dx^{n_2}}x^{\alpha_2+n_2}e^{-px^k}Y] \\ = & \prod_{i=1}^{n_1}(xD-pkx^k+\alpha_1+i)\prod_{j=1}^{n_2}(xD-pkx^k+\alpha_2+j)Y.\end{aligned}$$

Recalling Chatterjea's result [5]

$$\begin{aligned} D^n(x^{\alpha+n}e^{-px^k}Y) &= \sum_{r=0}^{n}\binom{n}{r}D^{n-r}(x^{\alpha+n}e^{-px^k})D^rY \\ &= n!x^{\alpha}e^{-px^k}\sum_{r=0}^{n}\frac{x^r}{r!}T_{k(n-r)}^{(\alpha+r)}(x,p)D^r(Y)\end{aligned}$$

where the polynomials $T_{kn}^{(\alpha)}(x,p)$ be defined by (3) and (4).

Considering (7), we can write

$$[\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}][x^{-\alpha_2}\frac{d^{n_2}}{dx^{n_2}}x^{\alpha_2+n_2}e^{-px^k}Y]$$

$$= n_2!\sum_{r=0}^{n_2}\frac{d^{n_1}}{dx^{n_1}}(\frac{x^{\alpha_1+n_1+r}}{r!}e^{-px^k}T_{k(n_2-r)}^{(\alpha_2+r)}(x,p)D^rY)$$

$$= n_2!\sum_{r=0}^{n_2}\sum_{i=0}^{n_2-r}\sum_{j=0}^{i}\frac{p^i}{i!r!}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2-r}$$

$$\cdot\frac{d^{n_1}}{dx^{n_1}}(x^{\alpha_1+n_1+ki+r}e^{-px^k}D^rY)$$

$$= n_2!n_1!x^{\alpha_1}e^{-px^k}\sum_{r=0}^{n_2}\sum_{i=0}^{n_2-r}\sum_{j=0}^{i}\frac{p^i}{i!r!}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2-r}x^{ki+r}$$

$$\cdot\sum_{s=0}^{n_1}\frac{x^s}{s!}T_{k(n_1-s)}^{(\alpha_1+ki+r+s)}(x,p)D^s(D^rY)$$

$$= n_2!n_1!x^{\alpha_1}e^{-px^k}\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\sum_{i=0}^{n_2-r}\sum_{j=0}^{i}\frac{p^i}{i!}(-1)^j\binom{i}{j}\binom{\alpha_2+n_2+kj}{n_2-r}x^{ki}$$

$$\cdot\sum_{l=0}^{n_1-s}\sum_{h=0}^{l}\frac{p^l}{l!}(-1)^h\binom{l}{h}\binom{\alpha_1+n_1+ki+kh+r}{n_1-s}x^{kl}D^s(D^rY).$$

By the explicit form of $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$, we obtain

$$[\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}][x^{-\alpha_2}\frac{d^{n_2}}{dx^{n_2}}x^{\alpha_2+n_2}e^{-px^k}Y] = n_2!n_1!x^{\alpha_1}e^{-px^k}$$

$$\cdot\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\frac{(-1)^{n_1-s+n_2-r}}{(n_1-s)!(n_2-r)!}K_{n_1-s,n_2-r}^{(\alpha_1+s+r,\alpha_2+r)}(x,k,p)D^s(D^rY).$$

Hence

$$\text{(8)}\qquad \frac{x^{-\alpha_1}e^{px^k}}{n_1!n_2!}[\frac{d^{n_1}}{dx^{n_1}}x^{\alpha_1+n_1}][x^{-\alpha_2}\frac{d^{n_2}}{dx^{n_2}}x^{\alpha_2+n_2}e^{-px^k}Y]$$

$$=\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\frac{(-1)^{n_1-s+n_2-r}}{(n_1-s)!(n_2-r)!}K_{n_1-s,n_2-r}^{(\alpha_1+s+r,\alpha_2+r)}(x,k,p)D^s(D^rY).$$

Then by (7) and (8), we get the following general operational formula

$$\text{(9)}\qquad \prod_{i=1}^{n_1}(xD-pkx^k+\alpha_1+i)\prod_{j=1}^{n_2}(xD-pkx^k+\alpha_2+j)Y$$

$$= n_1!n_2!\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\frac{(-1)^{n_1-s+n_2-r}}{(n_1-s)!(n_2-r)!}K_{n_1-s,n_2-r}^{(\alpha_1+s+r,\alpha_2+r)}(x,k,p)D^s(D^rY\ ).$$

By appropriately choosing the differentiable function Y one can obtain various operational formulas from (9). For instance, setting $Y=1$ in (9) we can state the following theorem.

Theorem 3.1. *For* $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$, *we have*

(10)

$$(-1)^{n_1+n_2}K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)=\prod_{i=1}^{n_1}(xD-pkx^k+\alpha_1+i)\prod_{j=1}^{n_2}(xD-pkx^k+\alpha_2+j)1.$$

In fact Theorem 3.1 can be obtained directly from the equation (6). Letting $k=1$ and $p=1$ in (9), we have the following result.

Corollary 3.2. *For the multiple Laguerre I polynomials, we have*

$$\prod_{i=1}^{n_1}(xD-x+\alpha_1+i)\prod_{j=1}^{n_2}(xD-x+\alpha_2+j)Y$$

$$=\quad n_1!n_2!\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\frac{(-1)^{n_1-s+n_2-r}}{(n_1-s)!(n_2-r)!}L_{n_1-s,n_2-r}^{(\alpha_1+s+r,\alpha_2+r)}(x)D^s(D^rY\ ).$$

Furthermore letting $Y=1$ in the above Corollary, we get

Corollary 3.3. *For the multiple Laguerre I polynomials, we have*

$$(-1)^{n_1+n_2}L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x)=\prod_{i=1}^{n_1}(xD-x+\alpha_1+i)\prod_{j=1}^{n_2}(xD-x+\alpha_2+j)1.$$

Following the same procedure which is used to obtain operational formulas for the polynomials $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$, we have the following analogous result of (9) for the polynomials $M_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x,k)$:

$$\prod_{i=1}^{n_1}(xD-p_1kx^k+\alpha_0+i)\prod_{j=1}^{n_2}(xD-p_2kx^k+n_1+\alpha_0+j)Y \tag{11}$$

$$=n_1!n_2!\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\frac{(-1)^{n_1-s+n_2-r}p_1^{n_1-s}p_2^{n_2-r}}{(n_1-s)!(n_2-r)!}$$

$$\cdot M_{n_1-s,n_2-r}^{(\alpha_0+r+s,p_1,p_2)}(x,k)D^s(D^rY),$$

where Y is any sufficiently differentiable function of x. Setting $Y=1$ in (11) we can state the following theorem.

Theorem 3.4. *For* $M_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x,k)$, *we have*

$$\prod_{i=1}^{n_1}(xD-p_1kx^k+\alpha_0+i)\prod_{j=1}^{n_2}(xD-p_2kx^k+n_1+\alpha_0+j)1 \tag{12}$$

$$=(-1)^{n_1+n_2}p_1^{n_1}p_2^{n_2}M_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x,k).$$

Now letting $k=1$ in (11), we conclude that

Corollary 3.5. *For the multiple Laguerre II polynomials, we have*

$$\begin{aligned}&\prod_{i=1}^{n_1}(xD-p_1x+\alpha_0+i)\prod_{j=1}^{n_2}(xD-p_2x+n_1+\alpha_0+j)Y\\ =\;& n_1!n_2!\sum_{r=0}^{n_2}\frac{x^r}{r!}\sum_{s=0}^{n_1}\frac{x^s}{s!}\frac{(-1)^{n_1-s+n_2-r}p_1^{n_1-s}p_2^{n_2-r}}{(n_1-s)!(n_2-r)!}\\ &\cdot L_{n_1-s,n_2-r}^{(\alpha_0+r+s,p_1,p_2)}(x)\; D^s(D^rY).\end{aligned}$$

Finally, taking $Y=1$ in the above corollary, we get

Corollary 3.6. *For the multiple Laguerre II polynomials, we have*

$$\begin{aligned}&(-1)^{n_1+n_2}p_1^{n_1}p_2^{n_2}L_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x)\\ =\;&\prod_{i=1}^{n_1}(xD-p_1x+\alpha_0+i)\prod_{j=1}^{n_2}(xD-p_2x+n_1+\alpha_0+j)1.\end{aligned}$$

4. Some Applications Of The Operational Formulas

In this section, by use of the operational formulas obtained in the above section, we are aimed to obtain some recurrence relations and useful formulas for the polynomials $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$ and $M_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x,k)$ defined by (1) and (2). We start with the following theorem:

Theorem 4.1. *For $K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)$, we have*

$$K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)=-(xD-pkx^k+\alpha_1+n_1)K_{n_1-1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p).$$

Proof. Writing (10) in the form

$$\begin{aligned}&(-1)^{n_1+n_2}K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)=(xD-pkx^k+\alpha_1+n_1)\\ &\cdot\prod_{i=1}^{n_1-1}(xD-pkx^k+\alpha_1+i)\prod_{j=1}^{n_2}(xD-pkx^k+\alpha_2+j)1,\end{aligned}$$

we get the desired result. □

The particular case $k=1$ and $p=1$ in the above theorem gives the following recurrence relation for the multiple Laguerre I polynomials:

Corollary 4.2. *For the multiple Laguerre I polynomials, we have*

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x)=-(xD-x+\alpha_1+n_1)L_{n_1-1,n_2}^{(\alpha_1,\alpha_2)}(x).$$

Another application of the formula (10) is the following:

Theorem 4.3. *The recurrence formula*

$$(-1)^{n_1+m_1}K^{(\alpha_1,\alpha_2)}_{n_1+m_1,n_2+m_2}(x,k,p)$$
$$=\prod_{i=1}^{m_1}(xD-pkx^k+\alpha_1+n_1+i)\prod_{l=1}^{n_1}(xD-pkx^k+\alpha_1+l)K^{(n_2+\alpha_2,\alpha_2)}_{m_2,n_2}(x,k,p)$$

holds for $K^{(\alpha_1,\alpha_2)}_{n_1,n_2}(x,k,p)$.

Proof. Replacing n_1 by n_1+m_1 and n_2 by n_2+m_2 in (10), we can write

$$(-1)^{n_1+m_1+n_2+m_2}K^{(\alpha_1,\alpha_2)}_{n_1+m_1,n_2+m_2}(x,k,p)$$
$$=\prod_{i=1}^{n_1+m_1}(xD-pkx^k+\alpha_1+i)\prod_{j=1}^{n_2+m_2}(xD-pkx^k+\alpha_2+j)$$
$$=\prod_{i=1}^{m_1}(xD-pkx^k+\alpha_1+n_1+i)\prod_{l=1}^{n_1}(xD-pkx^k+\alpha_1+l)$$
$$\cdot\prod_{j=1}^{m_2}(xD-pkx^k+\alpha_2+n_2+j)\prod_{h=1}^{n_2}(xD-pkx^k+\alpha_2+h)$$
$$=(-1)^{m_2+n_2}\prod_{i=1}^{m_1}(xD-pkx^k+\alpha_1+n_1+i)$$
$$\cdot\prod_{l=1}^{n_1}(xD-pkx^k+\alpha_1+l)K^{(n_2+\alpha_2,\alpha_2)}_{m_2,n_2}(x,k,p).$$

Whence the result. □

Corollary 4.4. *For the multiple Laguerre I polynomials, we have*

$$(-1)^{n_1+m_1}L^{(\alpha_1,\alpha_2)}_{n_1+m_1,n_2+m_2}(x)$$
$$=\prod_{i=1}^{m_1}(xD-x+\alpha_1+n_1+i)\prod_{l=1}^{n_1}(xD-x+\alpha_1+l)L^{(n_2+\alpha_2,\alpha_2)}_{m_2,n_2}(x).$$

On the other hand, considering (9) with $Y=K^{(n_2+\alpha_2,\alpha_2)}_{m_2,n_2}(x,k,p)$ in (10) we get

Corollary 4.5. *For* $K^{(\alpha_1,\alpha_2)}_{n_1,n_2}(x,k,p)$, *we have*

$$K^{(\alpha_1,\alpha_2)}_{n_1+m_1,n_2+m_2}(x,k,p)$$
$$=m_1!n_1!\sum_{r=0}^{n_1}\frac{x^r}{r!}\sum_{s=0}^{m_1}\frac{x^s}{s!}\frac{(-1)^{-(s+r)}}{(m_1-s)!(n_1-r)!}K^{(\alpha_1+n_1+s+r,\alpha_1+r)}_{m_1-s,n_1-r}(x,k,p)$$
$$\cdot D^{s+r}(K^{(\alpha_2+n_2,\alpha_2)}_{m_2,n_2}(x,k,p)).$$

Corollary 4.6. *For the multiple Laguerre I polynomials, we have*

$$L_{n_1+m_1,n_2+m_2}^{(\alpha_1,\alpha_2)}(x)$$
$$= m_1!n_1!\sum_{r=0}^{n_1}\frac{x^r}{r!}\sum_{s=0}^{m_1}\frac{x^s}{s!}\frac{(-1)^{-(s+r)}}{(m_1-s)!(n_1-r)!}L_{m_1-s,n_1-r}^{(\alpha_1+n_1+s+r,\alpha_1+r)}(x)$$
$$\cdot D^{s+r}(L_{m_2,n_2}^{(\alpha_2+n_2,\alpha_2)}(x)).$$

Before leaving this section, it should be noticed that similar formulas for the polynomials $M_{n_1,n_2}^{(\alpha_0,p_1,\cdot p_2)}(x,k)$ can be obtained in a similar manner.

5. Generating Functions

In this section we obtain generating functions of a class of polynomials defined by (1) and (2) by using Chatterjea's result [5]

$$(1-t)^{-\alpha-1}e^{px^k(1-(1-t)^{-k})} = \sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,p)t^n, \tag{13}$$

where the polynomials $T_{kn}^{(\alpha)}(x,p)$ are the class of polynomials defined by (3). We start with the following theorem:

Theorem 5.1. *Let the polynomials* $\left\{K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)\right\}$ *be defined by (1).*

Then we have

$$G_{k,p}^{(\alpha_1,\alpha_2)}(x,t_1,t_2) = \sum_{n_2=0}^{\infty}\sum_{n_1=0}^{\infty} K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!},$$

where

$$G_{k,p}^{(\alpha_1,\alpha_2)}(x,t_1,t_2) = (1+t_1)^{-\alpha_1-1}(1+t_2)^{-\alpha_2-1}e^{px^k(1-v(t_1)v(t_2))}$$

and $v(t) = (1+t)^{-k}$.

Proof. Considering (3) in (1), we can write

$$\sum_{n_1=0}^{\infty} K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)\frac{t_1^{n_1}}{n_1!}$$

$$= (-1)^{n_2}n_2!x^{-\alpha_1}e^{px^k}\sum_{n_1=0}^{\infty}(-1)^{n_1}\frac{d^{n_1}}{dx^{n_1}}(x^{\alpha_1+n_1}e^{-px^k}T_{kn_2}^{(\alpha_2)}(x,p))\frac{t_1^{n_1}}{n_1!}$$

$$= (-1)^{n_2}n_2!x^{-\alpha_1}e^{px^k}\frac{1}{2\pi i}\oint_{C_1}\frac{z^{\alpha_1}e^{-pz^k}T_{kn_2}^{(\alpha_2)}(z,p)}{(z-x)}\sum_{n_1=0}^{\infty}\frac{(-zt_1)^{n_1}}{(z-x)^{n_1}}dz$$

$$= \frac{(-1)^{n_2}n_2!x^{-\alpha_1}e^{px^k}}{1+t_1}\frac{1}{2\pi i}\oint_{C_2}\frac{z^{\alpha_1}e^{-pz^k}T_{kn_2}^{(\alpha_2)}(z,p)}{z-\frac{x}{1+t_1}}dz$$

$$= \frac{(-1)^{n_2}n_2!e^{px^k}}{(1+t_1)^{\alpha_1+1}}e^{-p(\frac{x}{1+t_1})^k}T_{kn_2}^{(\alpha_2)}(\frac{x}{1+t_1},p),$$

where C_1 is a circle in the complex $z-$plane, cut along the negative real axis, (centered at $z = x$) with radius $\varepsilon > 0$, which is described in the positive direction (counter-clockwise). where the closed contour C_2 in the complex $\eta-$plane, cut along the negative real axis, is a circle (centered at $\eta = \frac{x}{1+t_1}$) of sufficiently small radius. Thus

$$\sum_{n_2,n_1=0}^{\infty} K_{n_1,n_2}^{(\alpha_1,\alpha_2)}(x,k,p)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!} = \frac{e^{px^k}e^{-p(\frac{x}{1+t_1})^k}}{(1+t_1)^{\alpha_1+1}}\sum_{n_2=0}^{\infty}(-1)^{n_2}T_{kn_2}^{(\alpha_2)}(\frac{x}{1+t_1},p)t_2^{n_2}.$$

Using (13) in the above equality, the proof is completed. □

In a similar way, we can state the following theorem.

Theorem 5.2. *Let the polynomials* $\left\{M_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x,k)\right\}$ *be defined by (3).*

Then we have

$$H_k^{(\alpha_0,p_1,p_2)}(x,t_1,t_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} M_{n_1,n_2}^{(\alpha_0,p_1,p_2)}(x,k)\frac{t_1^{n_1}}{n_1!}\frac{t_2^{n_2}}{n_2!},$$

where

$$H_k^{(\alpha_0,p_1,p_2)}(x,t_1,t_2) = (1+\frac{t_1}{p_1})^{-\alpha_0-1}(1+\frac{t_2}{p_2})^{-\alpha_0-1-n_1} \cdot e^{p_1x^kw(\frac{t_1}{p_1})}e^{p_2x^kw(\frac{t_2}{p_2})(1-w(\frac{t_1}{p_1}))}$$

and $w(t) = 1-(1+t)^{-k}$.

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