Solid Bursts-From Hamming to RT-Spaces

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Abstract. In this paper, we generalize the notion of solid bursts from classical codes equipped with Hamming metric [14] to array codes endowed with RT-metric [13] and obtain some bounds on the parameters of RT-metric array codes for the correction and detection of solid burst array errors.

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1. Introduction

In a classical coding setting, codes are subsets/subspaces of ambient space F_q^n and are investigated with respect to the Hamming metric [12, 14]. In [13], m-metric or RT-metric array codes which are subsets/subspaces of linear space of all m by s matrices $\mathrm{Mat}_{m\times s}(F_q)$ with entries from a finite field F_q endowed with a non-Hamming metric were introduced and some bounds on code parameters were obtained.

Here is a model of an information transmission for which m-metric array coding is useful and the non-Hamming metric defined in [13] is the natural quality characteristic of a code. Suppose that a sender transmits messages, each being an s-tuple of m-tuples of q-ary symbols, transmitted over m parallel channels. There is an interfering noise in the channels which creates errors in the transmitted message. An important and practical situation is when errors are not scattered randomly in the code matrix but are in cluster from and are confined to a submatrix part of the code matrix. These errors arise, for example, due to lightning and thunder in deep space and satellite communications. With this motivation, the author has already introduced the notion of usual burst errors [7], CT-burst errorrs [8] and cyclic burst errors [11] in the space $\mathrm{Mat}_{m\times s}(F_q)$ equipped with m-metric as a generalization of classical burst of respective types. In this paper, we generalize and get another category of classical bursts viz. solid bursts to

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array codes and obtain some bounds on the parameters of m-metric array codes for the correction and detection of solid burst arry errors.

2. Definitions and Notations

Let F_q be a finite field of q elements. Let $\text{Mat}_{m\times s}(F_q)$ denote the linear space of all $m \times s$ matrices with entries from F_q. An m-metric array code is a subset of $\text{Mat}_{m\times s}(F_q)$ and a linear m-metric array code is an F_q-linear subspace of $\text{Mat}_{m\times s}(F_q)$. Note that the space $\text{Mat}_{m\times s}(F_q)$ is identifiable with the space F_q^{ms}. Every matrix in $\text{Mat}_{m\times s}(F_q)$ can be represented as a $1 \times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in F_q^{ms} can be represented as an $m \times s$ matrix in $\text{Mat}_{m\times s}(F_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates.

The weight and metric defined by Rosenbloom and Tsfasman [9] on the space $\text{Mat}_{m\times s}(F_q)$ are as follows :

Let $X \in \text{Mat}_{m\times 1}(F_q)$ with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{pmatrix},$$

then column weight (or weight) of X is given by

$$wt_c(X) = \begin{cases} m - \ \max \{ \ i \mid x_k = 0 \quad \text{for any } k \le i\} \ \text{if } X \ne 0 \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \ X = 0. \end{cases}$$

This definition of wt_c can be extended to $m \times s$ matrices in the space $\text{Mat}_{m\times s}(F_q)$ as

$$wt_c(A) = \sum_{j=1}^{s} wt_c(A_j)$$

where $A = [A_1, A_2, \cdots, A_s] \in \text{Mat}_{m\times s}(F_q)$ and A_j denotes the j^{th} column of A. Then wt_c satisfies $0 \le wt_c(A) \le n(= ms)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ if we set $d(A, A') = wt_c(A - A') \ \forall \ A, A' \in \text{Mat}_{m\times s}(F_q)$. We call this metric as column-metric. Note that for $m = 1$, it is just the usual Hamming metric.

There is an alternative equivalent way of defining the weight of an $m \times s$ matrix using the weight of its rows [4]:

Let $Y \in \text{Mat}_{1\times s}(F_q)$ with $Y = (y_1, y_2, \cdots, y_s)$. Define row weight (or weight) of Y as

$$wt_\rho(Y) = \begin{cases} \max\{\, i \mid y_i \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Extending the definitions of wt_ρ to the class of $m \times s$ matrices as

$$wt_\rho(A) = \sum_{i=1}^{m} wt_\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in \mathrm{Mat}_{m\times s}(F_q)$ and R_i denotes the i^{th} row of A. Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq n(= ms)\ \forall\ A \in \mathrm{Mat}_{m\times s}(F_q)$ and determines a metric on $\mathrm{Mat}_{m\times s}(F_q)$ known as row-metric.

It turns out that row weight of a vector is equal to the column weight of transpose of the vector with its component reversed and hence the two metrices viz. row-metric and column-metric give rise to equivalent codes and both the metrices have been known as m-metric or RT-metric.

In this paper, we take distance and weight in the sense of row-metric (or ρ-metric). Throughout this paper, $< x, y >$ will denote the minimum of x and y and $[x]$ as the greatest integer less than equal to x.

3. Solid Bursts in m-Metric Array Codes

We now define bursts in m-metric array codes:

Definition 3.1. A solid burst of order pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\mathrm{Mat}_{m\times s}(F_q)$ is an $m \times s$ matrix A having a $p \times r$ submatrix B with all entries in B nonzero and remaining entries in the $m \times s$ matrix A as zero.

Note. For $p = 1$, Definition 3.1 reduces to the definition of solid-burst for classical codes [14].

Example 3.2. Consider the linear space $\mathrm{Mat}_{3\times 3}(F_2)$. Then all the solid-bursts of order 2×2 are given by:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We now obtain a bound for the correction of solid-burst array errors in linear m-metric array codes.

Theorem 3.3. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ must satisfy*

$$q^{n-k} \geq 1 + S^{p\times r}_{m\times s}(F_q), \tag{1}$$

where $S^{p\times r}_{m\times s}(F_q)$ is the number of solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ in $Mat_{m\times s}(F_q)$ and is given by

$$S^{p\times r}_{m\times s}(F_q) = (m-p+1)(s-r+1)(q-1)^{pr}. \tag{2}$$

Proof. Consider a solid-burst $A \in \mathrm{Mat}_{m\times s}(F_q)$ of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$. Let B be the $p \times r$ nonzero submatrix of A such that all the nonzero entries in B are nonzero and entries in A outside B are zero. Since the number of starting positions for the submatrix B of order pr in the $m \times s$ matrix A is $(m-p+1)(s-r+1)$ and entries iin B can be filled in $(q-1)^{pr}$ ways, therefore, the total number of solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$S^{p\times r}_{m\times s}(F_q) = (m-p+1)(s-r+1)(q-1)^{pr}.$$

Now, since the linear m-metric array code $V \subseteq \mathrm{Mat}_{m\times s}(F_q)$ corrects all solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$, therefore, all the solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ including the null $m \times s$ matrix must belong to different cosets of the standard array. Since number of available cosets $= q^{n-k}$. Therefore, we must have

$$q^{n-k} \geq 1 + S^{p\times r}_{m\times s}(F_q)$$

where $S^{p\times r}_{m\times s}(F_q)$ is given by (2) and we get (1). □

Remark 3.4.

(i) Take $m = s = 3, p = r = 2$ and $q = 2$ in $S^{p\times r}_{m\times s}(F_q)$ computed in (2), we get $S^{2\times 2}_{3\times 3}(F_2) = 2 \times 2 = 4$ and these 4 solid-bursts of order 2×2 in $\mathrm{Mat}_{3\times 3}(F_2)$ are listed in Example 3.2.

(ii) Take $m = s = 3, p = 1, r = 2$ and $q = 2$ in $S^{p\times r}_{m\times s}(F_q)$ computed in (2). We get $S^{1\times 2}_{3\times 3}(F_2) = 3 \times 2 = 6$ and these 6 solid-bursts of order 1×2 in $\mathrm{Mat}_{3\times 3}(F_2)$ are listed below:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We now obtain a lower bound on the number of parity check digits required to correct all weighted solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ in $\mathrm{Mat}_{m\times s}(F_q)$ having ρ-weight w or less $(1 \leq w \leq ms)$. To obtain the desired bound, we first prove a lemma that enumerates the number of solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight w or less.

Lemma 3.5. *The number of solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ in $Mat_{m\times s}(F_q)$ having ρ-weight w or less $(1 \leq w \leq ms)$ is given by*

$$S^{p\times r}_{m\times s}(F_q, w) = \begin{cases} m \times \min(w, s) \times (q-1) \text{ if } p = r = 1, \\ m \times \min(w - r + 1, s - r + 1)(q-1)^r \\ \qquad\qquad\qquad \text{if } p = 1, r \geq 2 \\ (m - p + 1) \times L \times (q-1)^{pr} \quad \text{if } p > 1, r \geq 1 \end{cases} \tag{3}$$

where

$$L = \max\left\{0, \min\left\{[w/p] - r + 1, s - r + 1\right\}\right\}. \tag{4}$$

Proof. Consider a solid-burst $A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{bmatrix}$ where $A_i = (a_{i_1}, a_{i_2}, \cdots, a_{i_s})$, of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight w or less $(1 \leq w \leq ms)$. Let B be the $p \times r$ nonzero submatrix of A such that all the entries in B are nonzero and entries in A outside B are zero. There are three cases depednding upon the values of p and r.

Case 1. When $p = 1,\ r = 1$.

In this case, the number of starting positions for the 1×1 nonzero submatrix B in $m \times s$ matrix A is $m \times \min(w, s)$ and these $m \times \min(w, s)$ positions can be filled by $(q-1)$ nonzero elements from F_q. Therefore, the number of solid-bursts of order 1×1 having ρ-weight w or less in $\mathrm{Mat}_{m\times s}(F_q)$ is given by

$$S^{1\times 1}_{m\times s}(F_q, w) = m \times \min(w, s) \times (q-1).$$

Case 2. When $p = 1,\ r \geq 2$.

In this case, the number of starting positions for the $1 \times r$ nonzero submatrix B in $m \times s$ matrix A is $m \times \min(w-r+1, s-r+1)$ and entries in the $1 \times r$ submatrix B can be selected in $(q-1)^r$ ways. Therefore, the number of solid-bursts of order $1 \times r$ having ρ-weight w or less in $\text{Mat}_{m\times s}(F_q)$ is given by

$$S_{m\times s}^{1\times r}(F_q, w) = m \times \min(w - r + 1, s - r + 1) \times (q - 1)^r.$$

Case 3. When $p > 1,\ r \geq 1$.

In this case, the number of starting positions for the $p \times r$ nonzero submatrix B in $m \times s$ matrix A is $(m - p + 1) \times L$ when L is given by (4). and entries in B can be filled in $(q-1)^{pr}$ ways. Therefore, the number of solid-bursts of order $p \times r$ having ρ-weight w or less in $\text{Mat}_{m\times s}(F_q)$ is given by

$$S_{m\times s}^{p\times r} = (m - p + 1) \times L \times (q - 1)^{pr}$$

where L is given by (4). □

Example 3.6. Take $m = s = 3, p = r = 2, q = 2$ and $w = 3$. Then number of solid-bursts of order 2×2 having ρ-weight 3 or less in $\text{Mat}_{3\times 3}(F_2)$ is given by:

$$\begin{aligned} S_{3\times 3}^{2\times 2}(F_2, 3) &= 2 \times \max\{0, \min\{0, 2\}\} \times 1 \\ &= 2 \times 0 = 0. \end{aligned}$$

Thus, there is no solid burst of order 2×2 having ρ-weight 3 or less in $\text{Mat}_{3\times 3}(F_2)$.

Example 3.7. Take $m = s = 3, p = r = 2, q = 2$ and $w = 4$ in Lemma 3.5. Then $S_{3\times 3}^{2\times 2}(F_2, 4)$ is given by:

$$\begin{aligned} S_{3\times 3}^{2\times 2}(F_2, 4) &= 2 \times \max\{0, \min\{1, 2\}\} \times 1 \\ &= 2 \times \max\{0, 1\} \\ &= 2 \times 1 = 2. \end{aligned}$$

The 2 solid-bursts of order 2×2 having ρ-weight 4 or less in $\text{Mat}_{3\times 3}(F_2)$ is given by:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Now, we obtain the lower bound on the number of parity check digits for the correction of solid-bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight w or less $(1 \leq w \leq ms)$.

Theorem 3.8. *An (n,k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all solid-bursts of order pr $(1 \le p \le m, 1 \le r \le s)$ having ρ-weight w or less $(1 \le w \le ms)$ must satisfy*

$$q^{n-k} \ge 1 + S^{p\times r}_{m\times s}(F_q, w) \tag{5}$$

where $S^{p\times r}_{m\times s}(F_q, w)$ is given by (3) in Lemma 3.5.

Proof. The proof follows from the fact that the number of available cosets must be greater than or equal to the number of correctable error matrices including the null matrix. □

4. Construction Bounds for Solid-Bursts Error Detection and Correction in Linear m-Metric Array Codes

In this section, we obtain construction bounds for solid-burst error detection and correction. To obtain the desired bounds, we shall identify the space $\mathrm{Mat}_{m\times s}(F_q)$ with the space F_q^{ms} i.e. an $m \times s$ matrix over F_q is considered as an ms-tuple over F_q arranged into m groups of s elements each. Each group of s elements in an ms-tuple is called a block. Also, s is called the block length or block size and m is the number of blocks. Each block of an ms-tuple has a ρ-weight and sum of ρ-weights of all the m blocks of an ms-tuple is the ρ-weight of that ms-tuple. Also, columns of generator matrix G and parity check matrix H of a linear m-metric array code V are grouped into m blocks of s columns each. Therefore, generator matrix G and parity check matrix H of a linear m-metric array code V are represented as $G = [G_1, G_2, \cdots, G_m], H = [H_1, H_2, \cdots, H_m]$ where G_i and H_i are the i^{th} block $(1 \le i \le m)$ of generator and parity check matrix respectively of the code V and are given by

$$G_i = [G_{i1}, G_{i2}, \cdots, G_{is}],$$

and

$$H_i = [H_{i1}, H_{i2}, \cdots, H_{is}],$$

where each $G_{ij} (1 \le i \le m, 1 \le j \le s)$ is a $k \times 1$ column vector and each $H_{ij} (1 \le i \le m, 1 \le j \le s)$ is an $(ms - k) \times 1$ column vector.

Note. Throughout our discussion, by *strict linear combination* we mean a linear combination where all the scalars are nonzero.

Now, we obtain the construction (upper) bound for burst error detection in linear m-metric array codes.

Theorem 4.1. *Let q be prime or power of prime and m, s, p, r, k be positive integers satisfying $1 \le p \le m, 1 \le r \le s$ and $1 \le k \le ms$, then there exists* *an $[m \times s, k]$ linear m-metric array code over F_q i.e. a linear m-metric array code with m as the number of blocks and s as the block size, that has no solid-burst of order pr as a code array provided*

$$q^{ms-k} > 1 + \sum_{j=1}^{s} (q^{<j,r>})^{p-1} q^{<j-1,r-1>}. \tag{6}$$

Proof. The existence of such a code will be proved by constructing a suitable $(ms - k) \times ms$ parity check matrix H for the desired code. To detect any solid-burst of order pr, it is necessary and sufficient that no strict linear combination involving r consecutive columns in p consecutive blocks should be zero. Suppose that $i - 1 (1 \leq i \leq m)$ blocks $H_1, H_2, \cdots, H_{i-1}$ have been chosen suitably. Then j^{th} column $(1 \leq j \leq s)$ in the i^{th} block may be added provided it is not a strict linear combination of $l_j^{th}, (l_j + 1)^{th}, \cdots, j^{th}$ columns from the immediately preceding $< i - 1, p - 1 >$ blocks (where $l_j =< 1, j - r + 1 >$) together with strict linear combination of $l_j^{th}, (l_j + 1)^{th}, \cdots, (j - 1)^{th}$ columns in the i^{th} block. Therefore, column $H_{ij} (1 \leq j \leq s)$ in the i^{th} block can be added to H provided

$$\begin{aligned} H_{ij} \quad \neq \quad & \sum_{g=i-<i,p>+1}^{i-1} (\alpha_{g,l_j} H_{g,l_j} + \alpha_{g,l_j+1} H_{g,l_j+1} + \cdots + \alpha_{g,j} H_{g,j}) \\ & + \alpha_{i,l_j} H_{i,l_j} + \alpha_{i,l_j+1} H_{i,l_j+1} + \cdots + \alpha_{i,j-1} H_{i,j-1}. \end{aligned} \tag{7}$$

Note that summation in (7) will not run at all if the lower limit of the summation is greater than the upper limit and this will occur when $< i, p >= 1$ and in this case value of the summation is assumed to be zero.

Now, the number of strict linear combinations occuring in (7) is given by

$$((q - 1)^{<j,r>})^{<i-1,p-1>} (q - 1)^{<j-1,r-1>}. \tag{8}$$

Therefore, i^{th} block can be added to H provided the summation of number of strict linear combinations enumerated in (8) for $1 \leq j \leq s$ including the pattern of all zeros is less than the total number of $(ms - k)$-tuples. Therefore, i^{th} block H_i can be added to H provided that

$$q^{ms-k} > 1 + \sum_{j=1}^{s} ((q - 1)^{<j,r>})^{<i-1,p-1>} (q - 1)^{<j-1,r-1>}. \tag{9}$$

For the existence of an $[m \times s, k]$ linear m-metric array code, inequality (9) should hold for $i = m$ so that it is possible to add up to the m^{th} block to form an $(ms - k) \times ms$ parity check matrix and we get (6).

(Note that $1 \leq p \leq m$ gives $< m-1, p-1 >= p-1$). $\square$

Example 4.2. Take $m = 3, s = 2, p = r = 2, k = 2$ and $q = 3$.

Then

$$\begin{aligned} \text{R.H.S. of(6)} &= 1 + \sum_{j=1}^{2} (2^{<j,2>})^1 2^{<j-1,1>} \\ &= 1 + (2^{<1,2>}) 2^{<0,1>} + (2^{<2,2>}) 2^{<1,1>} \\ &= 1 + 2 \times 2^0 + 2^2 \times 2^1 \\ &= 11 \end{aligned}$$

Also, L.H.S. of (6) $= q^{ms-k} = 3^4 = 81$.

Therefore, L.H.S. of (6) $= 81 > 11 =$ R.H.S. of (3).

Thus, sufficient condition of Theorem 4.1. is satisfied for the chosen parameters and hence there exists a $[3 \times 2, 2]$ linear m-metric array code over F_2 detecting all solid-bursts of order 2×2. Consider the following $(3 \times 2 - 2) \times (3 \times 2) = 4 \times 6$ parity check matrix of a $[3 \times 2, 2]$ linear m-metric array code over F_2 constructed by the algorithm discussed in the proof of Theorem 4.1.

$$H = \begin{bmatrix} 1 & 0 & \vdots & 0 & 0 & \vdots & 2 & 2 \\ 0 & 1 & \vdots & 0 & 0 & \vdots & 0 & 2 \\ 0 & 0 & \vdots & 1 & 0 & \vdots & 2 & 0 \\ 0 & 0 & \vdots & 0 & 1 & \vdots & 0 & 0 \end{bmatrix}_{4\times 6}$$

The generator matrix of the code corresponding to the parity check matrix H is given by

$$G = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 0 & \vdots & 0 & 1 \end{bmatrix}_{2\times 6}$$

The 9 code arrays of the code $V \subseteq \text{Mat}_{3\times 2}(F_2)$ with G as generator matrix and H as parity check matrix are given by

$$v_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, v_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 2 & 0 \end{pmatrix}, v_5 = \begin{pmatrix} 2 & 2 \\ 0 & 0 \\ 0 & 2 \end{pmatrix}, v_6 = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 2 & 2 \end{pmatrix}, v_7 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 1 & 2 \end{pmatrix},$$

$$v_8 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 2 & 1 \end{pmatrix}.$$

We note that none of the code arrays is a solid-burst of order 2×2 over F_3. Therefore, construction bound (6) is verified.

Now, we obtain a construction upper bound for solid burst error correction.

Theorem 4.3. *Let q be prime or power of prime and m, s, p, r, k be positive integers satisfying $1 \le p \le [m/2], 1 \le r \le s$ and $1 \le k \le ms$, then a sufficient condition for the existence of an $[m \times s, k]$ linear m-metric array code over F_q that corrects all solid-bursts of order pr is given by*

$$q^{ms-k} > 1 + \left(S^{p\times r}_{(m-p)\times s)}(F_q)\right) \times \times \left(\sum_{j=1}^{s}((q-1)^{<j,r>})^{p-1}(q-1)^{<j-1,r-1>}\right) \tag{10}$$

where $S^{p\times r}_{(m-p)\times s}(F_q)$ is given by (3).

Proof. The existence of such a code will be proved as in previous theorem by constructing a suitable parity check matrix for the code. To correct all solid-bursts of order pr, it is necessary and sufficient that no code array consist of the sum of two solid-bursts of order pr. Thus, no strict linear combination involving two sets of r consecutive columns in p consecutive blocks should be zero. Suppose that $m-1$ blocks $H_1, H_2, \cdots, H_{m-1}$ of the parity check matrix H have been chosen suitably. Then j^{th} column $(1 \le j \le s)$ in the m^{th} block may be added, provided that it is not a strict linear combination of $l_j^{th}, (l_j+1)^{th}, \cdots, j^{th}$ columns from the immediately preceding $p-1$ blocks (where $l_j =< 1, j-r+1 >$) together with a strict linear combination of $l_j^{th}, (l_j+1)^{th}, \cdots, (j-1)^{th}$ columns in the m^{th} block and any set of r consecutive columns in p consecutive blocks among the first $(m-p)$ blocks. (Note that the condition $1 \le p \le [m/2]$ is used here because if $p \le [m/2]$, only then $(m-p)$ blocks can contain p consecutive blocks). In other words, column $H_{mj} (1 \le j \le s)$ in the m^{th} block can be added to H provided that

$$\begin{aligned} H_{mj} \;\; \neq \;\; & \sum_{g=m-p+1}^{m-1} (\alpha_{g,l_j} H_{g,l_j} + \alpha_{g,l_j+1} H_{g,l_j+1} + \cdots, \alpha_{g,j}, H_{g,j}) \\ & + \alpha_{m,l_j} H_{m,l_j} + \alpha_{m,l_j+1} H_{m,l_j+1} + \cdots, \alpha_{m,j-1}, H_{m,j-1} \quad (11) \\ & + \text{strict linear combination which form a solid-burst} \\ & \text{of order } pr \text{ among the first } (m-p) \text{ blocks} \end{aligned}$$

$$= G_j + P.$$

where

$$\begin{aligned} G_j &= \sum_{g=m-p+1}^{m-1} (\alpha_{g,l_j} H_{g,l_j} + \alpha_{g,l_j+1} H_{g,l_j+1} + \cdots + \alpha_{g,j} H_{g,j}) \\ &\quad + \alpha_{m,l_j} H_{m,l_j} + \alpha_{m,l_j+1} H_{m,l_j+1} + \cdots + \alpha_{m,j-1} H_{m,j-1}, \\ P &= \text{strict linear combinations which form a solid-burst} \\ &\quad \text{of order } pr \text{ among the first}(m-p)\text{blocks.} \end{aligned}$$

Now, there are $((q-1)^{<j,r>})^{p-1}(q-1)^{<j-1,r-1>}$ distinct linear combinations occuring in G_j. The number of linear combinations occuring in P which form a solid-burst of order pr in the space of $(m-p) \times s$ matrices is given by

$$S^{p\times r}_{(m-p)\times s}(F_q).$$

Thus, the number of strict linear combinations for a given $j(1 \leq j \leq s)$ occuring in the R.H.S. of (8) is given by

$$\left(((q-1)^{<j,r>})^{p-1}(q-1)^{<j-1,r-1>} \right) \times \; S^{p\times r}_{(m-p)\times s}(F_q). \qquad (12)$$

To add all the s columns in the m^{th} block, the number of available $(ms-k)$-tuples must be greater than the summation of the number enumerated in (12) for $j = 1$ to s including the pattern of all zeros. Therefore, m^{th} block H_m can be added to H provided

$$\begin{aligned} q^{ms-k} &> 1 + \sum_{j=1}^{s} \left(((q-1)^{<j,r>})^{p-1}(q-1)^{<j-1,r-1>} \right) \left(S^{p\times r}_{(m-p)\times s}(F_q) \right) \\ &= 1 + \left(S^{p\times r}_{(m-p)\times s}(F_q) \right) \left(\sum_{j=1}^{s} ((q-1)^{<j,r>})^{p-1}(q-1)^{<j-1,r-1>} \right). \end{aligned}$$

Thus we conclude that if (10) is satisfied, then it is possible to construct an $(ms-k) \times ms$ parity check matrix of an $[m \times s, k]$ linear m-metric array code which corrects all solid-bursts of order pr. □

Further, for a given $r(1 \leq r \leq s)$, let p_g^r be the largest value of p satisfying inequality (10). Then for $p = p_g^r + 1$, the opposite inequality is satisfied and the following theorem giving another upper bound on the number of parity checks holds:

Theorem 4.3. *There exists an $[m \times s, k]$ linear m-metric array code over F_q that corrects any single solid-burst of order $p_g^r \times r$ where $1 \leq r \leq s, 1 \leq$*

$p_g^r < [m/2]$, *for which the following inequality is satisfied:*

$$ms - k \leq log_q\Bigg(1 + \left(S_{(m-p_g^r-1)\times s}^{(p_g^r+1)\times r}(F_q)\right) \times \times\left(\sum_{j=1}^{s}(q-1)^{<j,r>})^{p_g^r}(q-1)^{<j-1,r-1>}\right)\Bigg). \qquad (13)$$

Example 4.4. Take $m = s = 3, p = 1, r = 2, q = 2$ and $k = 3$, Then

$$\begin{aligned}\text{R.H.S. of (10)} &= 1 + S_{3\times 3}^{1\times 2}(F_3)\Big)\left(\sum_{j=1}^{3}(2^{<j,2>})^0 2^{<j-1,1>}\right) \\ &= 1 + (6 \times 4 + (1+2+2) = 1 + 24 \times 5 = 1 + 120 \\ &= 121.\end{aligned}$$

Also, L.H.S. of (10) $= 3^{ms-k} = 3^6 = 729$.

Therefore, L.H.S. of (10) $= 729 > 121 =$R.H.S. of (10) there exists a $[3\times 3, 3]$ linear m-metric array code over F_3 that corrects any solid-bursts of order 1×2.

Consider the following $(3\times 3 - 3) \times (3 \times 3) = 6 \times 9$ parity check matrix of a $[3 \times 3, 3]$ linear m-metric array code over F_2 constructed by the synthesis procedure outlined in the proof of Theorem 4.2.

$$H = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & 0 & 0 & \vdots & 1 & 0 & 1 \\ 0 & 0 & 1 & \vdots & 0 & 0 & 0 & \vdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \vdots & 1 & 0 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 1 & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & 1 & \vdots & 1 & 0 & 0 \end{bmatrix}_{6\times 9}$$

We now claim that the code $V \subseteq \text{mat}_{3\times 3}(F_2)$ which is the null subspace of H corrects all solid-bursts of order 1×2. The claim is verified from Table 4.1 which shows that syndromes of all solid-burst errors of order 1×2 are all distinct.

Table 4.1

Solid-Burst Errors of order 1×2	Syndromes
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(110000)
$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(120000)
$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(210000)
$\begin{pmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(220000)
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(011000)
$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(012000)
$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(021000)
$\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(022000)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(000110)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(000120)
$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(000210)
$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	(000220)

Table contd.

Solid-Burst Errors of order 1×2	Syndromes
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	(000011)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	(00012)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	(000021)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	(000022)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	(111101)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$	(112201)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$	(221102)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$	(222202)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	(110100)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$	(220100)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$	(110200)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix}$	(220200)

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