# L(2,1)-labeling of a circular graph

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**Abstract**

In this paper the $\lambda$-number of the circular graph $C(km, m)$ is shown at most 9 where $m \geq 3$ and $k \geq 2$, and the $\lambda$-number of the circular graph $C(km+s, m)$ is shown at most 15 where $m \geq 3$, $k \geq 2$ and $1 \leq s \leq m-1$. In particular, the $\lambda$-numbers of $C(2m, m)$ and $C(n, 2)$ are determined, which are at most 8. All our results indicates that the Griggs and Yeh's conjecture holds for circular graphs. The conjecture says that for any graph $G$ with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$. Also, we determine $\lambda$-numbers of $C(n, 3)$, $C(n, 4)$ and $C(n, 5)$ if $n \equiv 0 (\mathrm{mod}\, 7)$.
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# 1 Introduction

An L(2,1)-labeling of a graph $G$ is a function $f$ from the vertex set of $G$ to a set of non-negative integers such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$, where $d(u, v)$ denotes the distance between two vertices $u$ and $v$ of $G$.

An L(2,1)-labeling of a graph $G$ that uses numbers in the set $\{0, 1, \ldots, k\}$ is called a $k$-labeling. The minimum $k$ such that $G$ has a $k$-labeling is called the $\lambda$-number of $G$ and is denoted by $\lambda(G)$. A $\lambda(G)$-labeling is called an optimal labeling of $G$.

Graph labelings are commonly used to model the channel assignment problem. The problem of labeling a graph with a condition at distance two

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was firstly investigated by Griggs and Yeh[5]. They shown that $\lambda(G) \leq \Delta^2 + 2\Delta$, where $\Delta$ denotes the maximum degree of $G$. Chang et al.[2] reduced the bound to $\Delta^2 + \Delta$, and Kral' et al.[8] reduced the bound to $\Delta^2 + \Delta - 1$. Recently, Goncalves [6] showed that $\lambda(G) \leq \Delta^2 + \Delta - 2$. For more details about labelings of graphs, readers can refer to [13]. In 1992, Griggs and Yeh[5] proposed the following conjecture.

**Conjecture** [5] For any graph $G$ with $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.

The conjecture is still open now.

The circular graph $C(n,m)$ is a graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{v_i v_{i+1}, v_i v_{i+m} | i = 0, 1, \ldots, n-1\}$, where $m$ and $n$ are positive integers satisfying $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$ and indices is read modulo $n$. If $n = 2m$, then $C(n,m)$ is a 3-regular graph. In other cases, $C(n,m)$ is a 4-regular graph.

People focus on verifying the Griggs and Yeh's conjecture and finding exact values for $\lambda$-numbers of particular graphs. In this paper we will show that the Griggs and Yeh's conjecture holds for circular graphs, and we will determine $\lambda$-numbers of some circular graphs. Circular graphs have closed relations with both Cartesian products of a path and a cycle and Generalized Petersen graphs. The L(2,1)-labeling of Cartesian product of any two graphs were considered in [12]. Jha[7] first considered the L(2,1)-labeling of the Cartesian product of a path and a cycle. The $\lambda$-numbers of the Cartesian products of a path and a cycle were determined by Klavžar and Vesel in [8] and by D.Kuo and J.H. Yan in [10] independently. $\lambda$-numbers of Generalized Petersen graphs have been discussed. Readers can refer to [1] and [3].

The structure of the paper is as follows. In Section 2 we will determine $\lambda$-number of $C(2m,m)$ and show that $\lambda(C(km,m)) \leq 9$ if $m \geq 3$ and $k \geq 3$. Also, We will show that $\lambda(C(km+s,m)) \leq 15$ if $m \geq 3$, $k \geq 2$ and $1 \leq s \leq m-1$. In Section 3 the $\lambda$-number of $C(n,2)$ will be determined, which is at least 8. In Section 4 we will determine $\lambda$-numbers of $C(n,3)$, $C(n,4)$ and $C(n,5)$ if $n \equiv 0 (\mathrm{mod}\, 7)$.

In the remainder of the section, we give some terms and theorems which will be used in the other sections.

For an L(2,1)-labeling of a graph $G$, if a triple (or a pair) of vertices are labeled the same number, then the triple (or the pair) is called a colored triple (or a colored pair). A path with $m$ vertices is denoted by $P_m$, and a cycle with $n$ vertices denoted by $C_n$. The Cartesian product of two graphs $G$ and $H$, denoted by $G \times H$, is defined as the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(x,y) : ux \in E(G) \text{ and } v = y, \text{ or } vy \in E(H) \text{ and } u = x\}$.

**Theorem 1.1**[5] If a graph contains three vertices with maximum degree $\Delta \geq 2$ and one of them is adjacent to the other two vertices, then its $\lambda$-number is no less than $\Delta + 2$.

**Theorem 1.2**[5] Let $G$ be a complete $k$-partite graph with $|V(G)| = \nu$. Then $\lambda(G) = \nu + k - 2$.

**Theorem 1.3**[5] $\lambda(C_n) = 4$.

**Theorem 1.4**[8] If $m \geq 4$ and $n \geq 3$, then

$$\lambda(P_m \times C_n) = \begin{cases} 6, & \text{if} \quad n \equiv 0 (\bmod 7), \\ 7, & \quad \text{otherwise.} \end{cases}$$

# 2 An upper bound for the $\lambda$-number of a circular graph

We consider an upper bound for the $\lambda$-number of the circular graph $C(2m, m)$ at first. If $m = 2$, then $C(2m, m)$ is the complete graph $K_4$ whose $\lambda$-number is 6. If $m = 3$, then $C(2m, m)$ is the complete bipartite graph $K_{3,3}$. By Theorem 1.2, $\lambda(C(6, 3)) = 6$. If $m = 4$, $C(2m, m)$ is the graph $V_8$. Since any two vertices of $V_8$ have the distance no more than two, $\lambda(V_8) \geq 7$. Define a labeling of $C(8, 4)$, say $f$, as follows: $f(v_i) \equiv 3i (\bmod 8)$ for $i = 0, 1, \ldots, 7$. Then $f$ is an L(2,1)-labeling of $C(8, 4)$, which means that $\lambda(C(8, 4)) = 7$. If $m \geq 5$, we have the following result.

**Theorem 2.1** If $m \geq 5$, then $\lambda(C(2m, m)) = 6$.

**Proof** A drawing of $C(2m, m)$ in the plane is shown in Figure-1. We will construct a labeling of $C(2m, m)$ using block combinations of matrices. The method is due to Schwars and Troxell[11].

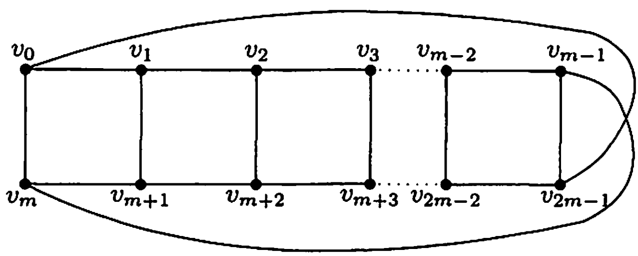


Figure 1 A drawing of $C(2m, m)$ in the plane

Now we suppose that $A_1 = \begin{pmatrix} 0 & 4 & 2 \\ 3 & 1 & 5 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 4 & 2 \\ 6 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 4 & 1 & 5 \\ 3 & 0 & 6 & 2 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 4 & 2 & 0 & 6 \\ 3 & 1 & 6 & 4 & 2 \end{pmatrix}$. If $m \equiv 0 \pmod 3$, then a labeling of $C(2m, m)$ is defined as $(A_2 A_1 \cdots A_1)$. If $m \equiv 1 \pmod 3$, then a labeling of $C(2m, m)$ is defined as $(A_1 \cdots A_1 B)$. Need to say that if $m = 7$, the above labeling of $C(2m, m)$ is $A_1 B$. If $m \equiv 2 \pmod 3$, then a labeling of $C(2m, m)$ is defined as $(A_1 \cdots A_1 C)$. In particular, the above labeling of $C(2m, m)$ is exactly $C$ if $m = 5$. Obviously, the above defined labeling of $C(2m, m)$ is an L(2,1)-labeling. So $\lambda(C(2m, m)) \leq 6$.

In order to prove the theorem, we need to show that $\lambda(C(2m, m)) \geq 6$. Since $C(2m, m)$ is a 3-regular graph, $\lambda(C(2m, m)) \geq 5$ by Theorem 1.1. Suppose that $\lambda(C(2m, m)) = 5$, and suppose that $f$ is an optimal labeling. Let $S = \{0, 1, 2, 3, 4, 5\}$. Without loss of generality, suppose that $f(v_0) = 0$. We have the following claims.

**Claim 1** $f(v_m) = 3$ or $5$.

Since $f(v_0) = 0$, we have that $f(v_m) \neq 0, 1$. If $f(v_m) = 2$, then $f(v_{m+1}) = 4$ or $5$. If $f(v_{m+1}) = 4$, then any number in $S$ can not label $v_1$. If $f(v_{m+1}) = 5$, then $f(v_1) = 3$ and $f(v_2) = 1$. In this case, any number in $S$ can not label $v_{m+2}$. So $f(v_m) \neq 2$.

If $f(v_m) = 4$, then $f(v_{m+1}) = 1$ or $2$. If $f(v_{m+1}) = 1$, then $f(v_1) = 3$ or $5$. If $f(v_1) = 3$, then $f(v_{m+2}) = 5$. In this case, any number in $S$ can not label $v_2$. Similarly, if $f(v_1) = 5$, then there is a contradiction. If $f(v_{m+1} = 2$, then $f(v_1) = 5$ and $f(v_{m+2}) = 0$. So $f(v_2) = 3$ and $f(v_3) = 1$. In this case, $f(v_{m+3}) = 4$ or $5$. If $f(v_{m+3}) = 5$, then $f(v_4) = 4$, $f(v_{m+4}) = 2$, and $f(v_{m+5}) = 0$. In this case, any number in $S$ can not label $v_5$. If $f(v_{m+3}) = 4$, then $f(v_{m+4}) = 2$ and $f(v_4) = 5$. If $m = 5$, there is a contradiction, since $v_5$ is adjacent to $v_4$. If $m \geq 6$, then $f(v_{m+5}) = 0$. In general, for $i = 1, 2, \ldots, m-1$, we have that $f(v_i) = 7 - 2j$ if $i \equiv j \pmod 3$ and $j = 1, 2, 3$, that $f(v_{m+i}) = 4 - 2j$ if $i \equiv j \pmod 3$ and $j = 1, 2$, and that $f(v_{m+i}) = 4$ if $i \equiv 3 \pmod 3$. So $f(v_{m-1}) \equiv 1 \pmod 2$. Since $v_m$ is adjacent to $v_{m-1}$, $f(v_{m-1}) = 1$. Then $f(v_{2m-1}) = 4$. Thus there is a contradiction, since the distance between $v_m$ and $v_{2m-1}$ is two.

**Claim 2** $f(v_m) \neq 5$.

If $f(v_m) = 5$, then $f(v_1) = 2, 3$ or $4$. If $f(v_1) = 2$, then any number in $S$ can not label $v_{m+1}$. If $f(v_1) = 3$, then $f(v_{m+1}) = 1$ and $f(v_{m+2}) = 4$. In this case, any number in $S$ can not label $v_2$. If $f(v_1) = 4$, then $f(v_{m+1}) = 1$ or $2$. If $f(v_{m+1}) = 1$, then $f(v_{m+2}) = 3$. In this case, any number in $S$ can not label $v_2$. Similarly, if $f(v_{m+1}) = 2$, there is a contradiction.

By Claim 1 and Claim 2, we have that $f(v_m) = 3$. So $f(v_{m+1}) = 1$ or $5$. If $f(v_{m+1}) = 1$, then $f(v_1) = 4$ or $5$. If $f(v_1) = 4$, then $f(v_{m+2}) = 5$

and $f(v_2) = 2$. Furthermore, $f(v_3) = 0$ and $f(v_{m+3}) = 3$. In general, for $i = 0, 1, \ldots, m-1$, we have that $f(v_i) = 6 - 2j$ if $i \equiv j \pmod 3$ and $j = 1, 2, 3$, that $f(v_{m+i}) = 9 - 2j$ if $i \equiv j \pmod 3$ and $j = 2, 3$, and that $f(v_{m+i}) = 1$ if $i \equiv 1 \pmod 3$. So $f(v_{m-1}) \equiv 0 \pmod 2$. Since $v_m$ is adjacent to $v_{m-1}$, $f(v_{m-1}) = 0$. So $f(v_{2m-1}) = 3$. Since the distance between $v_m$ and $v_{2m-1}$ is two, there is a contradiction. If $f(v_1) = 5$, then $f(v_{m+2}) = 4$, and $f(v_2) = 2$. So $f(v_3) = 0$. In this case, any number in $S$ can not label $v_{m+3}$.

If $f(v_{m+1}) = 5$, then $f(v_1) = 2$ and $f(v_2) = 4$. In this case, $f(v_{m+2}) = 0$ or 1. If $f(v_{m+2}) = 0$, then $f(v_3) = 1$, $f(v_{m+3}) = 3$, and $f(v_4) = 5$. In this case any number in $S$ can not label $v_{m+4}$. If $f(v_{m+2}) = 1$, then $f(v_3) = 0$ and $f(v_{m+3}) = 3$. In general, for $i = 0, 1, \ldots, m-1$, we have that $f(v_i) = 2j$ if $i \equiv j \pmod 3$ and $j = 1, 2, 3$, that $f(v_{m+i}) = 3 + 2j$ if $i \equiv j \pmod 3$ and $j = 0, 1$, and that $f(v_{m+i}) = 1$ if $i \equiv 2 \pmod 3$. Proceeding the similar argument to the above paragraph, there is a contradiction.

Therefore, $\lambda(C(2m, m)) \geq 6$, as desired. ∎

Now we consider the case that $k \geq 3$. A drawing of $C(km, m)$ in the plane is shown in Figure 2. We can see that the vertices of $C(km, m)$ can be presented by a $k$-by-$m$ matrix. Hence, labelings of the vertices of $C(km, m)$ can be presented by a $k$-by-$m$ matrix such that the entry on the $i$th row and the $j$th column is the label of the vertex.

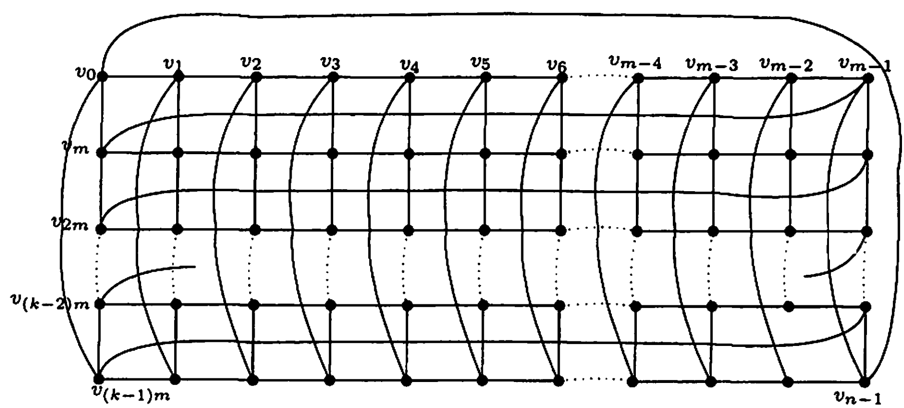


Figure 2 a drawing of $C(km, m)$ in the plane

**Theorem 2.2** If $m \geq 3$ and $k \geq 3$, then the $\lambda$-number of the circular graph $C(km, m)$ is at most 9, i.e., $\lambda(C(km, m)) \leq 9$.

**Proof** We define a labeling of $C(km, m)$ using block combinations of the matrices. We consider the following cases.

**Case 1.** $k \equiv 0 \pmod 3$. There are three cases to consider.

**Subcase 1.** $m \equiv 0 \pmod 3$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ A_1 & A_1 & \cdots & A_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_1 & A_1 & \cdots & A_1 \end{pmatrix}$, where $A_1 = \begin{pmatrix} 2 & 0 & 5 \\ 7 & 3 & 1 \\ 4 & 6 & 8 \end{pmatrix}$.

**Subcase 2.** $m \equiv 1 \pmod 3$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_2 & \cdots & A_2 & B_1 \\ A_2 & \cdots & A_2 & B_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_2 & \cdots & A_2 & B_1 \end{pmatrix}$, where $A_2 = \begin{pmatrix} 3 & 7 & 1 \\ 6 & 4 & 8 \\ 0 & 2 & 5 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 3 & 7 & 0 & 2 \\ 6 & 1 & 3 & 5 \\ 0 & 4 & 6 & 8 \end{pmatrix}$.

Note that if $m = 4$ then a labeling of $C(4k, 4)$ is defined as $({B_1}^T {B_1}^T \cdots {B_1}^T)^T$, where $X^T$ is the transpose of $X$. We shall not explain it in other cases.

**Subcase 3.** $m \equiv 2 \pmod 3$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_3 & \cdots & A_3 & C_1 \\ A_3 & \cdots & A_3 & C_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_3 & \cdots & A_3 & C_1 \end{pmatrix}$, where $A_3 = \begin{pmatrix} 0 & 6 & 3 \\ 7 & 2 & 5 \\ 4 & 8 & 1 \end{pmatrix}$ and $C_1 = \begin{pmatrix} 0 & 6 & 3 & 0 & 5 \\ 7 & 2 & 5 & 7 & 1 \\ 4 & 8 & 1 & 4 & 8 \end{pmatrix}$.

**Case 2.** $k \equiv 1 \pmod 3$. We consider the following cases.

**Subcase 1.** $m \equiv 0 \pmod 3$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_4 & A_4 & \cdots & A_4 \\ \cdots & \cdots & \cdots & \cdots \\ A_4 & A_4 & \cdots & A_4 \\ D_1 & D_1 & \cdots & D_1 \end{pmatrix}$, where $A_4 = \begin{pmatrix} 0 & 3 & 6 \\ 8 & 5 & 1 \\ 4 & 7 & 9 \end{pmatrix}$, and $D_1 = \begin{pmatrix} 0 & 3 & 6 \\ 8 & 5 & 1 \\ 3 & 0 & 7 \\ 5 & 8 & 2 \end{pmatrix}$ In particular, if $k = 4$ and $m = 3$, then a labeling is defined as $D_1$.

**Subcase 2.** $m \equiv 1 \pmod 3$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_1 & \cdots & A_1 & B_2 \\ \cdots & \cdots & \cdots & \cdots \\ A_1 & \cdots & A_1 & B_2 \\ D_2 & \cdots & D_2 & E \end{pmatrix}$, where $B_2 = \begin{pmatrix} 2 & 0 & 4 & 9 \\ 7 & 3 & 8 & 0 \\ 4 & 6 & 1 & 5 \end{pmatrix}$, $D_2 =$

$\begin{pmatrix} 2 & 0 & 5 \\ 7 & 3 & 9 \\ 0 & 6 & 2 \\ 4 & 9 & 7 \end{pmatrix}$ and $E = \begin{pmatrix} 2 & 0 & 4 & 9 \\ 7 & 5 & 8 & 2 \\ 0 & 9 & 3 & 7 \\ 4 & 6 & 1 & 5 \end{pmatrix}$

In particular, if $k = 4$ and $m = 4$, then a labeling is defined as $E$. We shall not explain it in other cases.

**Subcase 3.** $m \equiv 2 (\mathrm{mod}\, 3)$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_4 & \cdots & A_4 & C_2 \\ \cdots & \cdots & \cdots & \cdots \\ A_4 & \cdots & A_4 & C_2 \\ D_3 & \cdots & D_3 & F \end{pmatrix}$, where $C_2 = \begin{pmatrix} 0 & 3 & 6 & 0 & 2 \\ 8 & 5 & 1 & 3 & 9 \\ 4 & 7 & 9 & 5 & 7 \end{pmatrix}$

$D_3 = \begin{pmatrix} 0 & 3 & 6 \\ 8 & 5 & 1 \\ 3 & 0 & 7 \\ 5 & 8 & 2 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 3 & 6 & 0 & 2 \\ 8 & 5 & 1 & 4 & 9 \\ 3 & 0 & 8 & 6 & 1 \\ 5 & 7 & 4 & 9 & 7 \end{pmatrix}$.

**Case 3.** $k \equiv 2 (\mathrm{mod}\, 3)$. We consider the following cases.

**Subcase 1.** $m \equiv 0 (\mathrm{mod}\, 3)$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_1 & A_1 & \cdots & A_1 \\ \cdots & \cdots & \cdots & \cdots \\ A_1 & A_1 & \cdots & A_1 \\ G_1 & G_1 & \cdots & G_1 \end{pmatrix}$, where $G_1 = \begin{pmatrix} 2 & 0 & 5 \\ 7 & 4 & 1 \\ 3 & 9 & 6 \\ 8 & 5 & 0 \\ 4 & 7 & 9 \end{pmatrix}$.

**Subcase 2.** $m \equiv 1 (\mathrm{mod}\, 3)$. In this case a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A_1 & \cdots & A_1 & B_3 \\ \cdots & \cdots & \cdots & \cdots \\ A_1 & \cdots & A_1 & B_3 \\ G_2 & \cdots & G_2 & H_1 \end{pmatrix}$, where $B_3 = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 7 & 5 & 2 & 8 \\ 4 & 9 & 7 & 5 \end{pmatrix}$, $G_2 =$ $\begin{pmatrix} 2 & 0 & 5 \\ 7 & 4 & 1 \\ 3 & 9 & 6 \\ 8 & 2 & 0 \\ 4 & 6 & 9 \end{pmatrix}$ and $H_1 = \begin{pmatrix} 2 & 0 & 4 & 1 \\ 7 & 5 & 2 & 6 \\ 3 & 9 & 7 & 0 \\ 8 & 1 & 3 & 9 \\ 4 & 6 & 8 & 5 \end{pmatrix}$

**Subcase 3.** $m \equiv 2 (\mathrm{mod}\, 3)$. In this case a labeling of $C(km, m)$ is defined as

$\begin{pmatrix} A_5 & \cdots & A_5 & C_3 \\ \cdots & \cdots & \cdots & \cdots \\ A_5 & \cdots & A_5 & C_3 \\ G_3 & \cdots & G_3 & H_2 \end{pmatrix}$, where $C_3 = \begin{pmatrix} 1 & 3 & 5 & 9 & 3 \\ 7 & 9 & 2 & 7 & 0 \\ 4 & 6 & 8 & 4 & 6 \end{pmatrix}$,

$$A_5 = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{pmatrix}, G_3 = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 0 & 6 \\ 2 & 5 & 9 \\ 6 & 8 & 0 \end{pmatrix}, \text{ and } H_2 = \begin{pmatrix} 1 & 3 & 5 & 0 & 3 \\ 7 & 9 & 2 & 8 & 1 \\ 4 & 0 & 7 & 5 & 9 \\ 2 & 5 & 9 & 2 & 0 \\ 6 & 8 & 1 & 4 & 8 \end{pmatrix}.$$

Obviously, the above labeling is an L(2,1)-labeling. So $\lambda(C(km, m)) \leq 9$. ∎

**Theorem 2.3** Suppose $m \geq 3$, $k \geq 2$ and $1 \leq s \leq m-1$. Then $\lambda(C(km+s, m)) \leq 15$.

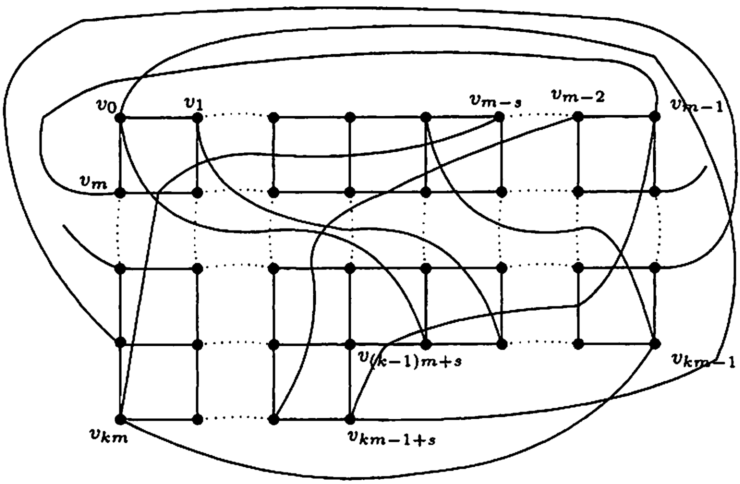


Figure 3 a drawing of $C(km+s, m)$ in the plane

**Proof** A drawing of $C(km+s, m)$ in the plane is shown in Figure 3. Let $T = \{v_{(k-1)m+s}, v_{(k-1)m+s+1}, \cdots, v_{km+s-1}\}$, let $H$ the subgraph of $C(km+s, m)$ induced by $V(C(km+s, m)) - T$, and let $H'$ the subgraph of $C(km+s, m)$ induced by $T$. Clearly, $H$ is a subgraph of $C(km, m)$, and $H'$ is isomorphic to the cycle $C_m$. Therefore, there exist a 9-L(2,1)-labeling $g$ of $H$ by Theorem 2.2 and a 4-L(2,1)-labeling $g'$ of $H'$ by Theorem 1.3. Define a labeling $f$ of $C(km+s, m)$ as

$$f(v) = \begin{cases} g(v) & \text{if } v \in V(H) \\ g'(v) + 11 & \text{if } v \in V(H') \end{cases}$$

Then $f$ is a 15-L(2,1)-labeling of $C(km+s, m)$. Therefore, $\lambda(C(km+s, m)) \leq 15$. ∎

# 3 The $\lambda$-number of $C(n, 2)$

In this section we determine the $\lambda$-number of $C(n, 2)$. We have known that $\lambda(C(4, 2)) = \lambda(C(6, 2)) = 6$ in Section 2. If $n = 5$, then $C(n, 2)$ is the complete graph $K_5$. So $\lambda(C(5, 2)) = 8$. Now we consider the case that $n \geq 7$.

**Theorem 3.1** If $n \equiv 0 (\bmod 7)$, then $\lambda(C(n, 2)) = 6$.

**Proof** Suppose $n = 7k$. Define a labeling $f$ of $C(n, 2)$ as follows: For $t = 0, 1, \ldots, k-1$, $f(v_{7t}) = 1$, $f(v_{7t+1}) = 6$, $f(v_{7t+2}) = 4$, $f(v_{7t+3}) = 2$, $f(v_{7t+4}) = 0$, $f(v_{7t+5}) = 5$, and $f(v_{7t+6}) = 3$. If $k = 1$, the labeling $f$ of $C(7, 2)$ is shown in Figure 4. It is easy to see that $f$ is an L(2,1)-labeling.

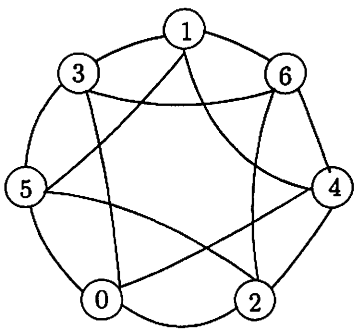


Figure 4 the labeling of $C(7, 2)$

We consider the case that $k \geq 2$. Suppose that $v_j$ is a vertex of $C(n, 2)$. The neighbors of $v_j$ are $v_{j+1}$, $v_{j+2}$,$v_{j-1}$ and $v_{j-2}$. No matter what $f(v_j)$ is, it can be checked that $|f(v_{j+p}) - f(v_j)| \geq 2$ if $p \in \{\pm 1, \pm 2\}$. The vertices of $C(n, 2)$ which have the distance 2 from $v_j$ are $v_{j+3}$,$v_{j+4}$, $v_{j-3}$ and $v_{j-4}$. Similarly, no matter what $f(v_j)$ is, it can be checked that $|f(v_{j+p}) - f(v_j)| \geq 1$, where $p \in \{\pm 3, \pm 4\}$. Hence, $f$ is an L(2,1)-labeling of $C(n, 2)$. Since $C(n, 2)$ is a 4-regular graph, $\lambda(C(n, 2)) \geq 6$ by Theorem 1.1. Therefore, $\lambda(C(n, 2)) = 6$. ∎

**Lemma 3.2** If $n \geq 8$ and $n \not\equiv 0 (\bmod 7)$, then $\lambda(C(n, 2)) \geq 7$.

**Proof** Since $C(n, 2)$ is a 4-regular graph, $\lambda(C(n, 2)) \geq 6$ by Theorem 1.1. Suppose that $\lambda(C(n, 2)) = 6$, and suppose that $f$ is an optimal 6-L(2,1)-labeling of $C(n, 2)$. Without loss of generality, suppose that $f(v_0) = 0$. By the symmetry of $v_1$ and $v_{n-1}$, it is sufficient to consider that $f(v_1) = 2, 3$, or 4. In this case $f(v_j) \neq 0, 1$ for $j \in \{1, 2, n-1, n-2\}$. Table 1 contains all possibilities of 6-L(2,1)-labelings of $v_1$, $v_2$, $v_{n-1}$ and $v_{n-2}$.

Let $S = \{0, 1, 2, 3, 4, 5, 6\}$. Since $v_3$ joins to $v_1$ and $v_2$, and the distance between $v_3$ and $v_0$ (or $v_{n-1}$) is 2, any number of $S$ can not label $v_3$ in cases (3), (4), (5) ,(10) and (11). Similarly, any number of $S$ can not label $v_{n-3}$ in case (9). In cases (7) and (8), $f(v_3) = 1$, but $v_4$ can not be labeled using any number of $S$. In case (6), $f(v_{n-3}) = 1$, but $v_{n-4}$ can not be labeled using any number of $S$. Hence, there is only one case (1) to be left. In case (1), $f(v_3) = 6$ and $f(v_{n-3}) = 1$. In general, we have the following claim.

| | $v_1$ | $v_2$ | $v_{n-1}$ | $v_{n-2}$ |
|---|---|---|---|---|
| (1) | 2 | 4 | 5 | 3 |
| (2) | 2 | 4 | 6 | 3 |
| (3) | 2 | 5 | 4 | 6 |
| (4) | 2 | 5 | 6 | 3 |
| (5) | 2 | 5 | 6 | 4 |
| (6) | 2 | 6 | 5 | 3 |
| (7) | 3 | 5 | 6 | 2 |
| (8) | 3 | 5 | 6 | 4 |
| (9) | 3 | 6 | 5 | 2 |
| (10) | 4 | 2 | 6 | 3 |
| (11) | 4 | 6 | 2 | 5 |

Table 1 all possibilities of L(2,1)-labelings of $v_1$,$v_2$, $v_{n-1}$ and $v_{n-2}$

**Claim 1** If $f(v_i) \equiv k(\text{mod } 7)$, then $f(v_{i+1}) \equiv k+2(\text{mod } 7)$, $f(v_{i+2}) \equiv k + 4(\text{mod } 7)$, $f(v_{i+3}) \equiv k + 6(\text{mod } 7)$, $f(v_{i-1}) \equiv k + 3(\text{mod } 7)$, $f(v_{i-2}) \equiv k + 5(\text{mod } 7)$ and $f(v_{i-3}) \equiv k + 1(\text{mod } 7)$.

Otherwise, let $g(v_j) = f(v_j) + (-k)$, where $j \in \{i, i+1, i+2, i+3, i-1, i-2, i-3\}$ and addition is read modulo 7. Then all possibilities of $g(v_j)$ are in Table 1 by the symmetry of $g(v_{j+1}))$ and $g(v_{j-1})$. Proceeding the similar argument to that in the paragraph before Table 1, one can show Claim 1.

Suppose that $n = 7t + s$, where $0 \leq s \leq 6$. Since $f(v_0) = 0$, we have the following claim by Claim 1.

**Claim 2** $f(v_j) = 0$ if $j = 7q$, where $1 \leq q \leq t$.

By Claim 2, $f(v_{7t}) = 0$. If $1 \leq s \leq 4$, then the distance between $v_{7t}$ and $v_0$ is at distance two. So there is a contradiction. If $s = 5$, then $f(v_{7t+1}) = 2$. Note that $f(v_{n-2}) = f(v_{7t+s-2}) = 3$ and $v_{n-2}$ is adjacent to $v_{7t+1}$, there is a contradiction. Similarly, if $s = 6$, there is a contradiction.

Therefore, $\lambda(C(n, 2)) \geq 7$. ∎

**Theorem 3.3** If $n \geq 8$, $n \not\equiv 0(\text{mod } 7)$ and $n \neq 9, 10, 11$ and 17, then $\lambda(C(n, 2)) = 7$.

**Proof** By Lemma 3.2, it is sufficient to show that $\lambda(C(n,2)) \leq 7$ if $n \geq 8$, $n \not\equiv 0(\bmod 7)$ and $n \neq 9, 10, 11$ and 17. Using the similar way of drawing to that of $C(km, m)$ and $C(km+s, m)$ shown in Figure 2 (or Figure 3), $C(n,2)$ has a drawing in the plane such that its vertices can be arranged two columns and $\lceil \frac{n}{2} \rceil$ rows. If $n \equiv 0(\bmod 2)$, then a labeling of $C(n,2)$ can be represented as the combination of matrices. If $n \not\equiv 0(\bmod 2)$, then a labeling of $C(n,2)$ is also represented as the combination of matrices, but the entry on the last row and the second column is absent, which is represented as *. We consider the following six cases.

**Case 1.** $n \equiv 0(\bmod 6)$. A labeling of $C(n,2)$ is defined as $\begin{pmatrix} A & A & \cdots & A \end{pmatrix}^T$, where $A = \begin{pmatrix} 0 & 3 & 7 \\ 6 & 1 & 4 \end{pmatrix}$.

**Case 2.** $n \equiv 1(\bmod 6)$. A labeling of $C(n,2)$ is defined as $\begin{pmatrix} A & \cdots & A & B \end{pmatrix}^T$, where $B = \begin{pmatrix} 0 & 3 & 7 & 2 \\ 6 & 1 & 4 & * \end{pmatrix}$.

**Case 3.** $n \equiv 2(\bmod 6)$. A labeling of $C(n,2)$ is defined as $\begin{pmatrix} A & \cdots & A & C \end{pmatrix}^T$, where $C = \begin{pmatrix} 0 & 3 & 5 & 2 \\ 6 & 1 & 7 & 4 \end{pmatrix}$.

**Case 4.** $n \equiv 3(\bmod 6)$. A labeling of $C(n,2)$ is defined as $\begin{pmatrix} A & \cdots & A & CB \end{pmatrix}^T$.

**Case 5.** $n \equiv 4(\bmod 6)$. A labeling of $C(n,2)$ is defined as $\begin{pmatrix} D & \cdots & D & E \end{pmatrix}^T$, where $D = \begin{pmatrix} 1 & 3 & 7 \\ 6 & 0 & 4 \end{pmatrix}$ and $E = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 6 & 0 & 2 & 4 \end{pmatrix}$.

**Case 6.** $n \equiv 5(\bmod 6)$. A labeling of $C(n,2)$ is defined as $\begin{pmatrix} F & \cdots & F & G \end{pmatrix}^T$, where $F = \begin{pmatrix} 7 & 1 & 4 \\ 3 & 6 & 0 \end{pmatrix}$ and $G =$ $\begin{pmatrix} 7 & 1 & 4 & 0 & 5 & 1 & 4 & 0 & 3 & 1 & 6 & 0 \\ 3 & 6 & 2 & 7 & 3 & 6 & 2 & 5 & 7 & 4 & 2 & * \end{pmatrix}$.

Obviously, the above labeling is an L(2,1)-labeling. Thus, the proposition holds. ∎

**Theorem 3.4** $\lambda(C(9,2)) = 8$, $\lambda(C(10,2)) = 8$, $\lambda(C(11,2)) = 8$ and $\lambda(C(17,2)) = 8$.

**Proof** L(2,1)-labelings of $C(9,2)$, $C(10,2)$, $C(11,2)$ and $C(17,2)$ are shown in the following matrices $A$, $B$, $C$ and $D$, respectively, where $A = \begin{pmatrix} 0 & 3 & 5 & 2 & 8 \\ 6 & 1 & 7 & 4 & * \end{pmatrix}^T$, $B = \begin{pmatrix} 0 & 2 & 8 & 4 & 6 \\ 4 & 6 & 0 & 2 & 8 \end{pmatrix}^T$,
$C = \begin{pmatrix} 0 & 3 & 5 & 7 & 1 & 4 \\ 6 & 8 & 0 & 3 & 8 & * \end{pmatrix}^T$, $D = \begin{pmatrix} 0 & 7 & 1 & 4 & 0 & 3 & 1 & 6 & 8 \\ 5 & 3 & 6 & 2 & 5 & 7 & 4 & 2 & * \end{pmatrix}^T$.
Hence, $\lambda(C(9,2)) \leq 8$, $\lambda(C(10,2)) \leq 8$, $\lambda(C(11,2)) \leq 8$ and $\lambda(C(17,2)) \leq$

8.

We now prove that $\lambda(C(k,2)) \geq 8$ for $k = 8, 10, 11, 17$.

(1) Since every pair of vertices of $C(9,2)$ has distance at most two, $\lambda(C(9,2)) \geq 8$. Hence, $\lambda(C(9,2)) = 8$.

(2) Assume that $\lambda(C(10,2)) \leq 7$. Suppose that $f$ is a 7-labeling of $C(10,2)$. We observe that there is only one vertex $v_{i+5}$ which has the distance more than two from $v_i$ in $C(10,2)$, and that any vertex in $V(C(10,2)) \setminus \{v_i, v_{i+5}\}$ is adjacent to $v_i$ or $v_{i+5}$. So there must be two colored pairs $(v_{i_1}, v_{i_1+5})$ and $(v_{i_2}, v_{i_2+5})$ in $C(10,2)$ under $f$. Thus, there are at most four numbers to label the rest six vertices. Hence, there must be two colored pairs $(v_{i_3}, v_{i_3+5})$ and $(v_{i_4}, v_{i_4+5})$. By our second observation, we have that $|f(v_{i_h}) - f(v_{i_l})| \geq 2$, where $h \neq j$. Therefore, the rest vertices can not be labeled, a contradiction. Thus, $\lambda(C(10,2)) = 8$.

(3) Assume that $\lambda(C(11,2)) \leq 7$. Suppose that $f$ is a 7-labeling of $C(11,2)$. We observe that there are exactly two vertices $v_{i+5}$ and $v_{i+6}$ which have distance more than two from $v_i$ in $C(11,2)$. So we have the following claims.

**Claim 1** If a colored pair labeled by $j_1$ under $f$, there is not the other colored pair labeled by $j_2$ under $f$ such that $|j_1 - j_2| = 1$.

**Claim 2** There is not a colored triple of vertices in $C(11,2)$ under $f$.

We also observe that there is only one vertex which has distance two from any colored pair of vertices under $f$. Hence, we have the following claim.

**Claim 3** If a colored pair labeled by $j_1$ under $f$, then there are not the other two vertices labeled by two distinct numbers $j_2$ and $j_3$, respectively, such that $|j_1 - j_2| = 1$ and $|j_1 - j_3| = 1$.

Let $w_2$ be the number of colored pairs. Then $3 \leq w_2 \leq 5$. If $w_2 = 5$, then $\lambda(C(11,2)) \geq 8$ by Claim 1. If $w_2 = 3$ or 4, there must be a colored pair labeled by $j$ and two vertices labeled $j-1, j+1$, respectively, which violates Claim 2. Hence, the assumption that $\lambda(C(11,2)) \leq 7$ does not hold. Thus, $\lambda(C(11,2)) = 8$.

(4) Assume that $\lambda(C(17,2)) \leq 7$. Suppose that $f$ is a 7-labeling of $C(17,2)$. We observe that there are not four vertices of $C(17,2)$ such that any two vertices have the distance more than two. So we the following claim.

**Claim 4** There are not four vertices labeled by the same number under $f$.

Suppose $(x_1, x_2, x_3)$ is a triple of vertices of $C(17,2)$ such that any two vertices have the distance more than two. Suppose $y_1$ and $y_2$ are two vertices in $V(C(17,2)) \setminus \{x_1, x_2, x_3\}$. We observe that there is one in $y_1$ and $y_2$ such that it is adjacent with some vertex in $x_1, x_2$ and $x_3$. Hence, we have the following claim.

**Claim 5** If a colored triple labeled by $j$ under $f$, then there can not be a colored pair labeled by $h$ under $f$ such that $|j-h| = 1$.

Let $w_3$ (or $w_2$) be the number of colored triples (or pairs). Then $0 \leq w_3 \leq 5$. If $w_3 = 0$, then $\lambda(C(17,2)) \geq 8$. If $w_3 = 5$, then $\lambda(C(17,2)) \geq 8$ by Claim 5. For $i = 1,2,3,4$, if $w_3 = i$, then there are at most $8-2i$ numbers to label colored pair of vertices under $f$. In other words, $w_2 \leq 8-2i$. In this case, $\lambda(C(17,2)) \geq 8$. Hence, $\lambda(C(17,2)) = 8$. ∎

**Remark:** Since the maximum degree of circular graphs is 4, all our above results indicates that the conjecture proposed by Griggs and Yeh [5] holds for circular graphs.

# 4 $\lambda$-numbers of some circular graphs

**Theorem 4.1** If $n \equiv 0 (\bmod 7)$, then $\lambda(C(n,3)) = 6$.

**Proof** A labeling $f$ of $C(n,3)$ is defined as $f(v_i) = 3i (\bmod 7)$ for $i = 0,1,\ldots,n-1$. It is easy to check that $f$ is a 6-L(2,1)-labeling of $C(n,3)$. Since $C(n,3)$ is a 4- regular graph if $n \equiv 0 (\bmod 7)$, $\lambda(C(n,3)) \geq 6$ by Theorem 1.1. Hence, $\lambda(C(n,3)) = 6$. ∎

Let us consider $\lambda$-number of $C(n,4)$ where $n \equiv 0 (\bmod 7)$. If a labeling of $v_j$ of $C(n,4)$ is defined as that of $C(n,3)$ in Theorem 4.1 for $j = 0,1,\ldots,n-1$, then it is easy to show the following result using the similar argument to that in the proof of Theorem 4.1.

**Theorem 4.2** If $n \equiv 0 (\bmod 7)$, then $\lambda(C(n,4)) = 6$.

Similarly, if $n \equiv 0 (\bmod 7)$ and if a labeling of $v_j$ of $C(n,5)$ is the same as that of $C(n,2)$ in Theorem 3.1, then we have the following theorem.

**Theorem 4.3** If $n \equiv 0 (\bmod 7)$, then $\lambda(C(n,5)) = 6$.

**Theorem 4.4** Let $m \geq 3$ and $k \geq 3$. Then

(1) If $k \equiv 0 (\bmod 3)$, then $\lambda(C(4k,4)) = 7$.

(2) If $k \equiv 0 (\bmod 4)$, then $\lambda(C(6k,6)) = 7$.

(3) If $k \equiv 0 (\bmod 3)$, $m \equiv 4 (\bmod 6)$ and $m \not\equiv 0 (\bmod 7)$, then $\lambda(C(km,m)) = 7$.

**Proof** If $m \geq 3$ and $k \geq 3$, $C(km,m)$ has a subgraph isomorphic to $P_m \times C_k$ which is obtained from $C(km,m)$ by deleting edges $v_0v_{n-1}$ and $v_{im-1}v_{im}$ for $i = 1,2,\ldots,k-1$. Hence, $\lambda(C(4k,4)) \geq 7$ and $\lambda(C(6k,6)) \geq 7$ by Theorem 1.4. Similarly, if $m \not\equiv 0 (\bmod 7)$, then $\lambda(C(km,m)) \geq 7$ where

$k \equiv 0 (\bmod 3)$ and $m \equiv 4 (\bmod 6)$. We need to show the inverses of the above mentioned inequalities.

(1) If $k \equiv 0 (\bmod 3)$, a labeling of $C(4k, 4)$ is defined as $\begin{pmatrix} A & A & \cdots & A \end{pmatrix}^T$, where $A = \begin{pmatrix} 3 & 6 & 0 \\ 7 & 1 & 4 \\ 0 & 3 & 6 \\ 4 & 7 & 1 \end{pmatrix}$. Obviously, the above labeling is an L(2,1)-labeling. So $\lambda(C(4k, 4)) \leq 7$. Then $\lambda(C(4k, 4)) = 7$.

(2) If $k \equiv 0 (\bmod 4)$, a labeling of $C(6k, 6)$ is defined as $\begin{pmatrix} B^T & B^T & \cdots & B^T \end{pmatrix}^T$, where $B = \begin{pmatrix} 3 & 7 & 5 & 2 & 0 & 4 \\ 6 & 1 & 3 & 7 & 5 & 2 \\ 0 & 4 & 6 & 1 & 3 & 7 \\ 5 & 2 & 0 & 4 & 6 & 1 \end{pmatrix}$. Clearly, the above labeling is an L(2,1)-labeling. So $\lambda(C(6k, 6)) \leq 7$. Then $\lambda(C(6k, 6)) = 7$.

(3) If $k \equiv 0 (\bmod 3)$ and $m \equiv 4 (\bmod 6)$,a labeling of $C(km, m)$ is defined as $\begin{pmatrix} A^T & C & \cdots & C \\ A^T & C & \cdots & C \\ \cdots & \cdots & \cdots & \cdots \\ A^T & C & \cdots & C \end{pmatrix}$, where $C = \begin{pmatrix} 6 & 1 & 3 & 7 & 0 & 4 \\ 0 & 4 & 6 & 1 & 3 & 7 \\ 3 & 7 & 0 & 4 & 6 & 1 \end{pmatrix}$. In particular, if $k = 3$ and $m = 10$, a labeling of $C(km, m)$ is defined as $(A^T\, C)$. Clearly, the above labeling is an L(2,1)-labeling. So $\lambda(C(km, m)) \leq 7$. Therefore, $\lambda(C(km, m)) = 7$. ∎

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