On path partition dimension of trees

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Abstract

For a connected graph G and any two vertices u and v in G, let $d(u,v)$ denote the distance between u and v and let $d(G)$ be the diameter of G. For a subset S of $V(G)$, the distance between v and S is $d(v,S) = \min\{d(v,x) \mid x \in S\}$. Let $\Pi = \{S_1, S_2, \dots, S_k\}$ be an ordered k-partition of $V(G)$. The representation of v with respect to Π is the k-vector $r(v \mid \Pi) = (d(v,S_1), d(v,S_2), \dots d(v,S_k))$. A partition Π is a resolving partition for G if the k-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension of G, and is denoted by $pd(G)$. A partition $\Pi = \{S_1, S_2, \dots, S_k\}$ is a resolving path k-partition for G if it is a resolving partition and each subgraph induced by S_i, $1 \le i \le k$, is a path. The minimum k for which there exists a path resolving k-partition of $V(G)$ is the path partition dimension of G, denoted by $ppd(G)$. In this paper path partition dimensions of trees and the existence of graphs with given path partition, partition and metric dimension, respectively are studied.

Keywords: distance, metric dimension, partition dimension, resolving partition, path partition dimension, resolving path partition.
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1 Introduction

As described in [8] and [3], dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. Perhaps the best known example of this process is graph coloring. In [4], the vertices of a connected graph are represented by other criterion, namely through partitions of vertex set and distances between each vertex and the subsets in the partition. Thus a new concept is introduced - resolving partition for a graph.

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices u and v in G, let $d(u,v)$ be the distance between u and v. The diameter of G, denoted by $d(G)$ is the greatest distance between any two vertices of G. For a subset S of $V(G)$ and a vertex v of G, the distance $d(v,S)$ between v and S is defined as $d(v,S) = \min\{d(v,x) \mid x \in S\}$.

For an ordered k-partition $\Pi = \{S_1, S_2, \dots, S_k\}$ and a vertex v of G, the *representation of v with respect to* Π is the k-vector

$$r(v \mid \Pi) = (d(v,S_1), d(v,S_2), \dots d(v,S_k)).$$

Partition Π is called a *resolving k-partition* for G if the k-vectors $r(v \mid \Pi)$, $v \in V(G)$ are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is the *partition dimension* of G and is denoted by $pd(G)$. A resolving partition of $V(G)$ containing $pd(G)$ classes is called a *minimum resolving partition.*

In [7] is considered a particular case of resolving partitions - connected resolving partitions.

Partition $\Pi = \{S_1, S_2, \dots, S_k\}$ is a *connected resolving k-partition* if it is a resolving partition and each subgraph induced by S_i, $1 \le i \le k$, is connected in G. The minimum k for which there is a connected resolving k partition of $V(G)$ is the *connected partition dimension* of G, denoted by $cpd(G)$.

Another type of resolving partitions, mentioned in [7] as topic for study, is resolving path partitions.

Partition $\Pi = \{S_1, S_2, \dots, S_k\}$ is a *resolving path k-partition* if it is a resolving partition and each subgraph induced by S_i, for $1 \le i \le k$, is a path. The minimum k for which there exists a resolving path k-partition of $V(G)$ is the *path partition dimension* of G, denoted by $ppd(G)$. A resolving path partition of $V(G)$ containing $ppd(G)$ classes is called a *minimum resolving path partition.*

Partition dimension of a graph is related to an older type of dimension of a graph, introduced by Slater in [10] and later in [9], and independently by Harary and Melter in [5] - metric dimension of a graph.

For an ordered set $W = \{w_1, w_2, \dots w_k\}$ of vertices of G and a vertex $v \in V(G)$, the *metric representation* of v with respect to W is the k-vector $r(v \mid W) = (d(v,w_1), d(v,w_2), \dots, d(v,w_k))$. If all vertices of G have distinct representations, then W is called a *resolving set* for G. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G. The number of vertices in a basis for G is the *metric dimension* of G and is denoted by $dim(G)$.

If $\Pi = \{S_1, S_2, \dots, S_k\}$ is a partition of $V(G)$ and $u_1, u_2, \dots, u_r$ are r distinct vertices, we say that $u_1, u_2, \dots, u_r$ are separated by classes S_{i_1},

..., S_{i_q} of partition Π if the q-vectors $(d(u_p, S_{i_1}), d(u_p, S_{i_2}), \ldots, d(u_p, S_{i_q}))$, $1 \leq p \leq r$ are distinct.

A partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ of $V(G)$ is an *induced-path partition* of G if each subgraph induced by S_i, $1 \leq i \leq k$, is a path. Hence, a resolving path partition is an induced-path partition which is also a resolving partition. The minimum cardinality of an induced-path partition of G is called the *induced-path number* of G and is denoted by $\rho(G)$. The concept of induced-path number was introduced by Chartrand et al. [2].

Next, we will use the term *path partition* for a tree T instead of induced-path partition, since any subpath of T is an induced path in T.

In [12] all graphs with path partition dimension 2, n or $n-1$ were characterized.

In this paper we will study path partition dimension of trees and the existence of graphs with given path partition dimension.

2 Path partition dimension of trees

First we remind some definitions and notation from [7].

Let G be a connected graph. A vertex of degree at least 3 of G is called *major vertex* of G. A vertex u of degree one of G is called a *terminal vertex of a major vertex v* of G if $d(u, v) < d(u, w)$, for every major vertex $w \neq v$ of G (v is the closest major vertex to u). The *terminal degree* of a major vertex v, denoted by $ter_G(v)$ or, if G is known, by $ter(v)$, is the number of terminal vertices of v. A major vertex v with $ter(v) > 0$ is said to be an *exterior major vertex* of G. We will call an exterior major vertex v with $ter(v) > 1$ a *branched major vertex* of G.

We denote by $\sigma(G)$ the sum of terminal degrees of the major vertices of G, by $\sigma_b(G)$ the sum of terminal degrees of the branched major vertices of G, by $ex(G)$ the number of exterior major vertices of G and by $ex_b(G)$ the number of branched major vertices of G.

For an exterior major vertex v of G a path Q to one of its terminal vertices u is called *terminal path* for vertex v.

For a tree T with $p = ex_b(T)$ branched major vertices we denote by $B(T) = \{v_1, v_2, \ldots, v_p\}$ the set of branched major vertices of T. For $1 \leq i \leq p$ we denote by $k_i = ter(v_i)$ and by u_{i1}, u_{i2}, ..., u_{ik_i} the terminal vertices of v_i, by P_{ij} the path from v_i to u_{ij}, for $1 \leq j \leq k_i$, with x_{ij} the vertex of P_{ij} adjacent with v_i and by Q_{ij} the subpath of P_{ij} from x_{ij} to u_{ij} (i.e., $Q_{ij} = P_{ij} - v_i$). For a branched major vertex v_i, $1 \leq i \leq p$, the paths Q_{ij}, $1 \leq j \leq k_i$ are the terminal paths of vertex v_i.

For example, the tree from figure 1 has 3 exterior major vertices: v_1, u and v_2. Only v_1 and v_2 are branched major vertices: $ter(v_1) = ter(v_2) = 2$. The terminal paths for v_1 are Q_{11} and Q_{12} and for v_2 are x_{21}, u_{21} and x_{22}.

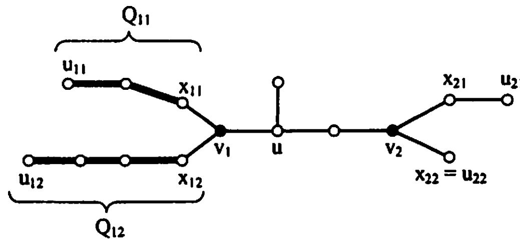

Figure 1: T

We denote by $T_{j_1 \ldots j_p}$, where $1 \leq j_i \leq k_i$, $\forall 1 \leq i \leq p$ the tree obtained from T by removing all its terminal paths excepting Q_{ij_i}, $1 \leq i \leq p$.

We will show that in order to find a minimum resolving path partition for tree T we need to build a certain minimum induced-path partition for $T_{j_1 \ldots j_p}$.

First, we remind that in [2] a formula for the induced-path number of a tree T is given.

Theorem 2.1 *([2]) Let T be a tree and let H be the forest induced by the vertices of T having degree 3 or more. Let H' be a spanning sub-forest of H of maximum size such that $deg_{H'}(v) \leq deg_T(v) - 2$ for every vertex v of H. Then*

$$\rho(T) = 1 + |E(H')| + \sum_{v \in V(H)} (deg_T(v) - 2 - deg_{H'}(v)).$$

Next we prove a recursive formula for $\rho(T)$, which also gives a recursive method to determine a minimum path partition for T, usefull in finding a minimum resolving path partition for a tree.

Theorem 2.2 *For a tree T with n vertices we have*

$$\rho(T) = \begin{cases} \sigma_b(T) - ex_b(T) + \rho(T'), & \text{if } T \neq P_n,\ n \geq 1 \\ 1, & \text{if } T = P_n,\ n \geq 1 \\ 0, & \text{if } n = 0, \end{cases}$$

where T' is the tree obtained from T by removing all its branched major vertices and their terminal paths.

Proof: Let T be a tree with $n \geq 1$ vertices, $T \neq P_n$.

It suffices to prove that any minimum path partition Π' of T can be transformed into a minimum path partition Π of T such that, for every branched major vertex v_i of T, $1 \leq i \leq p$, the following property $\mathcal{P}(\Pi)$ holds:

$\mathcal{P}(\Pi)$: ≪ Every vertex set $V(Q_{ij})$, $1 \le j \le k_i$ is a class of Π with two exceptions, say $V(Q_{i1})$ and $V(Q_{i2})$, and vertices from $V(Q_{i1}) \cup V(Q_{i2}) \cup \{v_i\}$ form a class of Π≫.

Indeed, let Π' be a minimum path partition that does not satisfy the required property for every branched major vertex. Let $I = \{i_1, \ldots, i_k\}$ be the set of indices of the branched major vertices that do not verify the property $\mathcal{P}(\Pi')$. Since Π' has minimum number of classes and a path Q_{ij} can be extended only through vertex v_i, it is easy to see that every branched major vertex v_i with $i \in I$ satisfies the following property:

≪All the sets $V(Q_{ij})$, $1 \le j \le k_i$ are classes of Π' with one exception, say $V(Q_{i1})$, and there exists a path L_i from v_i such that $V(L_i) \cap V(Q_{ij}) = \emptyset$ for every $1 \le j \le k_i$ and $V(Q_{i1}) \cup V(L_i)$ is a class of Π'≫.

We consider

$$\begin{aligned}\Pi'' = \ & \Pi' - \bigcup_{i \in I} \{V(Q_{i2}), V(Q_{i1}) \cup V(L_i)\} \cup \\ & \cup \bigcup_{i \in I} \{V(Q_{i1}) \cup V(Q_{i2}) \cup \{v_i\}, V(L_i) - \{v_i\}\}\end{aligned}$$

The number of branched major vertices that do not verify the property $\mathcal{P}(\Pi'')$ is smaller than the number of branched major vertices that do not verify the property $\mathcal{P}(\Pi')$. Thus, by applying the above transformation a finite number of times, we obtain a path partition Π such that every vertex satisfies $\mathcal{P}(\Pi)$. □

Remark 2.3 *For a tree T we have*

$$\sigma_b(T) - ex_b(T) = \sigma(T) - ex(T) = |term(T)| - ex(T)$$

where $term(T)$ is the set of terminal vertices of T.

Corollary 2.4 *Let T be tree and T' be the tree obtained from T by eliminating all its branched major vertices and their terminal paths. Then the path partition $\Pi_s(T)$ recursively defined as follows*

$$\Pi_s(T) = \begin{cases} \displaystyle\bigcup_{v_i \in B(T)} \left(\{V(Q_{i1}) \cup V(Q_{i2}) \cup \{v_i\}\} \cup \bigcup_{j=3}^{ter_T(v_i)} \{V(Q_{ij})\} \right) \cup \\ \qquad \cup \ \Pi_s(T'), \ \textit{if } T \textit{ is not a path} \\ \{V(T)\}, \ \textit{if } T \textit{ is a path} \end{cases}$$

is a minimum path partition.

Remark 2.5 *For a tree T the number $\rho(T_{j_1 \ldots j_p})$ does not depend of indices $j_1, \ldots, j_p$.*

If T is the path $x_1, \ldots, x_n$ then $ppd(T) = 2$, a minimum resolving path partition being $\{x_1\}$, $\{x_2, \ldots, x_n\}$.

Theorem 2.6 *For a tree T which is not isomorphic to a path we have*

$$ppd(T) = \sigma_b(T) - ex_b(T) + \rho(T_{j_1 \ldots j_p})$$

where $j_i \in \{1, \ldots, ter_T(v_i)\}$, for $1 \le i \le p$.

Proof: Let T be a tree different from a path.

Since a terminal path Q_{ij} can be extended only through vertex v_i, vertices from a set $V(Q_{ij})$, $1 \le i \le p$, $1 \le j \le k_i$ have distinct distances to any other fixed vertex of T. Moreover, with the above notation, any two distinct vertices x_{ij_1} and x_{ij_2}, adjacent to vertex v_i have equal distances to any fixed vertex from $V(T) - (V(Q_{ij_1}) \cup V(Q_{ij_2}))$.

Hence each set $V(Q_{ij})$, $1 \le i \le p$, $1 \le j \le k_i$ forms a separate class in any minimum resolving path partition, with at most one exception for every branched major vertex v_i, $1 \le i \le p$.

It follows that

$$ppd(T) \ge \sum_{v \in B(T)} (ter(v) - 1) + \rho(T_{\underbrace{1 \ldots 1}_{p}}) = \sigma_b(T) - ex_b(T) + \rho(T_{\underbrace{1 \ldots 1}_{p}}).$$

We consider the partition

$$\Pi = \{V(Q_{ij}) | 1 \le i \le p, 2 \le j \le k_i\} \cup \Pi_s(T_{\underbrace{1 \ldots 1}_{p}}) \tag{1}$$

where Π_s is given by Corollary 2.4.

We will prove that Π is a resolving partition. Since Π_s has $\rho(T_{\underbrace{1 \ldots 1}_{p}})$ classes, we shall deduce that

$$ppd(T) = \sigma_b(T) - ex_b(T) + \rho(T_{\underbrace{1 \ldots 1}_{p}}).$$

Indeed, vertices from a class $V(Q_{ij})$, $1 \le i \le p$, $2 \le j \le k_i$ are separated by any other class of Π.

We denote by $T^{(1)} = T_{\underbrace{1 \ldots 1}_{p}}$, $p^{(1)} = ex_b(T^{(1)})$ the number of branched major vertices of $T^{(1)}$, $v_1^{(1)}$, $v_2^{(1)}$, $\ldots$, $v_{p^{(1)}}^{(1)}$ the branched major vertices of $T^{(1)}$ and $Q_{ij}^{(1)}$ their terminal paths. We also denote by $T^{(2)}$ the tree obtained from $T^{(1)}$ by removing all its branched major vertices and their terminal paths.

If $T^{(1)}$ is a path, its vertices are separated by classes $V(Q_{12})$ and $V(Q_{22})$ if $ex_b(T) \geq 2$ or by $V(Q_{12})$ and $V(Q_{13})$ otherwise (since in this case $ter_T(v_1) \geq 3$).

If $T^{(1)}$ is not a path, by Corollary 2.4, we have

$$\Pi_s(T^{(1)}) = \bigcup_{v_i^{(1)} \in B(T^{(1)})} \left\{ \{V(Q_{i1}^{(1)}) \cup V(Q_{i2}^{(1)}) \cup \{v_i^{(1)}\}\} \cup \bigcup_{j=3}^{ter_{T^{(1)}}(v_i^{(1)})} \{V(Q_{ij}^{(1)})\} \right\} \cup \Pi_s(T^{(2)}).$$

Vertices from any class from $\Pi_s(T^{(1)}) - \Pi_s(T^{(2)})$ are separated by classes of Π. Indeed, vertices from $V(Q_{ij}^{(1)})$, $3 \leq j \leq ter_{T^{(1)}}(v_i^{(1)})$ are separated by class $\{V(Q_{i1}^{(1)}) \cup V(Q_{i2}^{(1)}) \cup \{v_i^{(1)}\}\}$, for any $1 \leq i \leq p^{(1)}$.

Since $Q_{i1}^{(1)}$ and $Q_{i2}^{(1)}$ are terminal paths in $T^{(1)}$, there exists a branched major vertex v_a of T such that $v_a \in V(Q_{i1}^{(1)})$ or $v_a \in V(Q_{i2}^{(1)})$. Moreover, since $v_i^{(1)}$ has degree at least 3, there exists a vertex $x \notin V(Q_{i1}^{(1)}) \cup V(Q_{i2}^{(1)})$ adjacent with $v_i^{(1)}$. Then vertices from $V(Q_{i1}^{(1)}) \cup V(Q_{i2}^{(1)}) \cup \{v_i^{(1)}\}$ are separated by the class that contains x and class $\{V(Q_{a2})\}$.

It remains to prove that vertices from classes of $\Pi_s(T^{(2)})$ have distinct representations with respect to Π.

Recursively, for $q \geq 2$, we have

$$\Pi_s(T^{(q)}) = \bigcup_{v_i^{(q)} \in B(T^{(q)})} \left\{ \{V(Q_{i1}^{(q)}) \cup V(Q_{i2}^{(q)}) \cup \{v_i^{(q)}\}\} \cup \bigcup_{j=3}^{ter_{T^{(q)}}(v_i^{(q)})} \{V(Q_{ij}^{(q)})\} \right\} \cup \Pi_s(T^{(q+1)}) \tag{2}$$

if $T^{(q)}$ is not a path, where $T^{(q+1)}$ is the tree obtained from $T^{(q)}$ by removing all its branched major vertices and their terminal paths and

$$\Pi_s(T^{(q)}) = \{V(T^{(q)})\}$$

if $T^{(q)}$ is a path.

Assume $T^{(q)}$ is not a path.

We denote by $p^{(q)} = ex_b(T^{(q)})$ the number of branched major vertices of $T^{(q)}$, by $v_1^{(q)}, v_2^{(q)}, \ldots, v_{p^{(q)}}^{(q)}$ the branched major vertices of $T^{(q)}$ and by $Q_{ij}^{(q)}$ their terminal paths, with extremities $x_{ij}^{(q)}$ and $u_{ij}^{(q)}$, $x_{ij}^{(q)}$ adjacent with $v_i^{(q)}$.

Vertices from any class from $\Pi_s(T^{(q)}) - \Pi_s(T^{(q+1)})$ are separated by classes of Π. Indeed, vertices from $V(Q_{ij}^{(q)})$, $3 \leq j \leq ter_{T^{(q)}}(v_i^{(q)})$ are separated by class $\{V(Q_{i1}^{(q)}) \cup V(Q_{i2}^{(q)}) \cup \{v_i^{(q)}\}\}$, for any $1 \leq i \leq p^{(q)}$. Moreover, since $Q_{i1}^{(q)}$ and $Q_{i2}^{(q)}$ are terminal paths in $T^{(q)}$, there exists a branched major vertex $v_a^{(q-1)}$ of $T^{(q-1)}$ such that $v_a^{(q-1)}$ is adjacent with $u_{i1}^{(q)}$ or with $u_{i2}^{(q)}$. Then vertices from $V(Q_{i1}^{(q)}) \cup V(Q_{i2}^{(q)}) \cup \{v_i^{(q)}\}$ are separated by class $\{V(Q_{a1}^{(q-1)}) \cup V(Q_{a2}^{(q-1)}) \cup \{v_a^{(q-1)}\}\}$.

If $T^{(q)}$ is a path we can similarly prove that vertices from $V(T^{(q)})$ are separated by classes of $\Pi_s(T^{(q-1)}) - \Pi_s(T^{(q)})$.

By induction, it follows that the vertices from classes of $\Pi_s(T^{(2)})$ have distinct representations with respect to Π, hence Π is a resolving partition. □

Corollary 2.7 *For a tree T which is not isomorphic to a path we have*

$$ppd(T) = dim(T) + \rho(T_{j_1 \dots j_p})$$

where $j_i \in \{1, \dots, ter_T(v_i)\}$, for $1 \leq i \leq p$.

Proof: By [1], for a tree T which is not isomorphic to a path we have

$$dim(T) = \sigma(T) - ex(T).$$

By Theorem 2.6, since $\sigma_b(T) - ex_b(T) = \sigma(T) - ex(T)$ it follows that

$$ppd(T) = dim(T) + \rho(T_{j_1 \dots j_p})$$

where $j_i \in \{1, \dots, ter_T(v_i)\}$, for $1 \leq i \leq p$. □

Next we describe a linear time algorithm for finding a minimum resolving path partition of a tree T with n vertices based on the proof of Theorem 2.6. Let us remark that in [6] a linear time algorithm for finding the induced-path number for graphs whose blocks are complete graphs, cycles or complete bipartite graphs is presented.

Denote by *deg* the vector of degrees and by *ter* the vector of terminal degrees.

We assume the tree is represented using adjacency lists.

The algorithm also uses the following lists:

LBranched - the list of all branched vertices of the current tree

LTerminal - the list of all terminal vertices of the current tree

$\mathcal{L}$ - a vector of lists, $\mathcal{L}[u]$ containing the terminal paths of vertex u in the current tree.

We assume that we have a function EXTEND(Q, v) having as parameters a path Q and a vertex v adjacent with one of the extremities of Q that extends the path Q through vertex v until a vertex v_0 such that $deg[v_0] \neq 2$. Vertex v_0 is not added to Q. The function returns the vertex v_0.

Also we consider two procedures - BUILD_TERMINAL_PATHS and EXTEND_TERMINAL_PATHS - as follows.

Procedure BUILD_TERMINAL_PATHS builds all terminal paths in T starting with terminal vertices from list *LTerminal* and store them in list $\mathcal{L}$, if T is not a path, or adds class $V(T)$ to the partition Π and the algorithm stops, if T is a path; the terminal degrees of vertices and the list of branched major vertices *LBranched* are updated according to the new terminal paths discovered:

```
procedure BUILD_TERMINAL_PATHS
    for every terminal vertex u in LTerminal do
        let Q be the path consisting of only vertex u
        let v be the vertex adjacent to u in T
        v := EXTEND(Q, v)
        if deg[v] ≥ 3 then
            add Q to 𝓛[v]
            ter[v] = ter[v] + 1
            if (ter[v] = 2) then add v to LBranched
        else
            add vertex v to Q
            add class V(Q) to partition Π and STOP.
    LTerminal = ∅
endprocedure
```

Procedure EXTEND_TERMINAL_PATHS(*LToExtend*) has as parameter a list o vertices *LToExtend* and extends the paths from $\mathcal{L}[v]$ (which were terminal paths at previous step of algorithm) to terminal paths in the current tree, for every vertex v in *LToExtend*; the terminal degrees of vertices and the list of branched major vertices *LBranched* are updated according to the new terminal paths discovered:

```
procedure EXTEND_TERMINAL_PATHS(LToExtend)
    let LBranchedNew be an empty list
    while LToExtend is not empty do
        extract a vertex v from LToExtend
        if 𝓛[v] ≠ ∅ then
            remove the path Q from 𝓛[v]
            ter[v] := ter[v] − 1
            v0 := EXTEND(Q, v)
            if deg[v0] ≥ 3 then
                add Q to 𝓛[v0]
                ter[v0] := ter[v0] + 1
                if (ter[v0] = 2) then
                    add v to LBranchedNew
```

else

add vertex v_0 to Q

add class $V(Q)$ to partition Π and STOP.

$LBranched := LBranchedNew$

endprocedure

Then the algorithms has the following steps:

1. Initialize

$ter[u] := 0$ for every $u \in V(T)$;

$\Pi = \emptyset$.

$LTerminal = \emptyset$; $LBranched = \emptyset$

$\mathcal{L}[u] = \emptyset$ for every $u \in V(T)$.

2. Calculate the degree $deg[u]$ for every $u \in V(T)$.

3. If T is a path then let x be a terminal vertex in T; the partition Π with classes $\{x\}$ and $V(T) - \{x\}$ is a minimum resolving path partition for T. STOP.

4. Add all terminal vertices of T to list $LTerminal$

5. BUILD_TERMINAL_PATHS

6. Build classes from terminal paths as in (1) as follows:

for each vertex v in $LBranched$ do

for each path Q in $\mathcal{L}[v]$ with one exception do

remove Q from $\mathcal{L}[v]$

$T := T - Q$

add class $V(Q)$ to partition Π

$ter[v] := 1$

7. Extend the remaining paths from $\mathcal{L}$ to terminal paths in the current tree and determine all major branched vertices for the current tree:

EXTEND_TERMINAL_PATHS($LBranched$)

8. While the current path is not a tree (hence still have branched major vertices) recursively add classes to Π as in (2) and update the terminal paths for the new tree:

while $LBranched$ is not empty

for every v in $LBranched$ do

extract two paths Q_1 and Q_2 from $\mathcal{L}[v]$

add class $V(Q_1) \cup V(Q_2) \cup \{v\}$ to partition Π

for every path Q in $\mathcal{L}[v]$ do

remove Q from $\mathcal{L}[v]$

$T := T - Q$

add class $V(Q)$ to partition Π

$T := T - (Q_1 \cup Q_2 \cup \{v\})$

let $LPreviousMajor$ be an empty list which will contain the non-terminal vertices from which major vertices were removed;

```
    for every v in LBranched do
        if there exists the vertex u from the current tree
        that was adjacent to v then
            if deg[u] = 1 then
                if u is not in LTerminal then
                    add u to LTerminal
            else
                if deg[u] > 1 then
                    if u is not in LPreviousMajor then
                        add u to LPreviousMajor
                else
                    add class {u} to partition Π and STOP.
    LBranched = ∅
    EXTEND_TERMINAL_PATHS(LPreviousMajor)
    BUILD_TERMINAL_PATHS
endwhile
```

Obviously the above algorithm has the complexity $O(n)$, since the operations of building or removing a path have the complexity equal to the length of path and the paths considered are edge-disjoint.

Examples

1. Let $S_n(p_1, \ldots, p_n)$ be the thorn star obtained from the star S_n with terminal vertices v_1, v_2, ..., v_n by attaching $p_i \geq 2$ new terminal vertices to each vertex v_i, $1 \leq i \leq n$.
We have

$$\begin{aligned} ppd(S_n(p_1, \ldots, p_n)) &= \sum_{i=1}^{n}(p_i - 1) + \rho(S_n(1, 1, \ldots, 1)) = \\ &= \sum_{i=1}^{n} p_i - n + n - 1 = \sum_{i=1}^{n} p_i - 1. \end{aligned}$$

2. Let $P_n(p_1, \ldots, p_n)$ be the caterpillar obtained from the path P_n with vertices v_1, v_2, ..., v_n by attaching $p_i \geq 2$ new terminal vertices to each vertex v_i, $1 \leq i \leq n$. We have

$$\begin{aligned} ppd(P_n(p_1, \ldots, p_n)) &= \sum_{i=1}^{n}(p_i - 1) + \rho(P_n(1, 1, \ldots, 1)) = \\ &= \sum_{i=1}^{n} p_i - n + \rho(P_n(1, 1, \ldots, 1)) \end{aligned}$$

and, by Theorem 2.2, if $n \geq 4$ we have

$$\rho(P_n(1, 1, \ldots, 1)) = 2 + \rho(P_{n-4}(1, 1, \ldots, 1)).$$

More,

$$\rho(P_3(1,1,1)) = 2 + \rho(P_0) = 2$$
$$\rho(P_2(1,1)) = \rho(P_3) = 1$$
$$\rho(P_1(1)) = \rho(P_2) = 1.$$

It follows that

$$\rho(P_n(1,1,\ldots,1)) = \begin{cases} 2k, & \text{if } n = 4k \\ 2k+1, & \text{if } n = 4k+1 \text{ or } n = 4k+2 \\ 2k+2, & \text{if } n = 4k+3 \end{cases}$$

hence

$$ppd(P_n(p_1,\ldots,p_n)) = \begin{cases} \sum_{i=1}^{n} p_i - 2k, & \text{if } n = 4k \text{ or } n = 4k+1 \\ \sum_{i=1}^{n} p_i - 2k - 1, & \text{if } n = 4k+2 \text{ or } n = 4k+3 \end{cases}$$

3 Existence of graphs with given path partition dimension

Theorem 3.1 *a) For any two integers a and b such that $3 \leq a \leq b$ there exists a connected graph G such that $pd(G) = a$ and $ppd(G) = b$.*

b) For any two integers a and b such that $3 \leq a \leq b$ there exists a connected graph G such that $dim(G) = a$ and $ppd(G) = b$.

Proof: Let a and b be two integers such that $3 \leq a \leq b$. We consider two cases.

Case 1 $a < b$.

a), b) We consider tree T from figure 2.

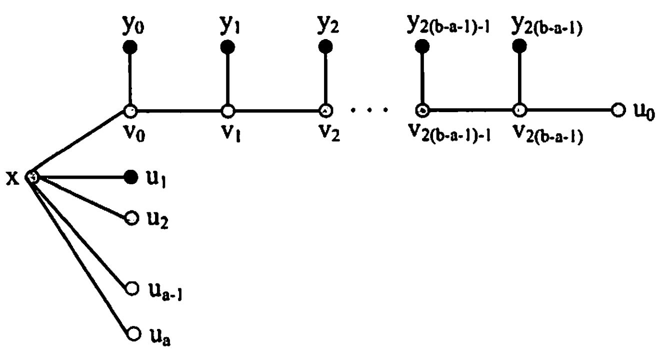

Figure 2

The external major vertices of T are x, v_0, v_1, ..., $v_{2(b-a-1)}$. For external major vertex x, its terminal vertices are u_j, $1 \leq j \leq a$, for $v_{2(b-a-1)}$ terminal vertices are $y_{2(b-a-1)}$ and u_0, and external major vertex v_i has one terminal vertex y_i, for each $0 \leq i \leq 2(b-a-1)-1$. It follows that $ter(x) = a$, $ter(v_i) = 1$, for any $0 \leq i \leq 2(b-a-1)-1$, $ter(v_{2(b-a-1)}) = 2$. By Theorem 2.6 we have

$$ppd(T) = ter(x) + ter(v_{2(b-a-1)}) - 2 + \rho(T') = a + \rho(T'),$$

where T' is the tree obtained from T by eliminating all its branched major vertices and their terminal paths, with one exception for each external terminal vertex.

By Theorem 2.2 we obtain $\rho(T') = b - a$, a minimum path partition of T' having classes $\{y_0, v_0, x, u_1\}$, $\{y_{2i-1}, v_{2i-1}, v_{2i}, y_{2i}\}$, $1 \leq i \leq b-a-1$.

It follows that $ppd(T) = b$.

In order to determine partition dimension of T, let Π be a resolving partition of T. Since vertices $u_1, \dots, u_a$ have the same distances to each of the other vertices of T, these vertices must belong to different classes of Π. It follows that $pd(T) \geq a$. Moreover, partition with classes $S_1 = \{u_1, y_0, y_1, \dots, y_{2(b-a-1)}\}$, $S_2 = \{u_2, x, v_0, v_1, \dots, v_{2(b-a-1)}, u_0\}$, $\{u_3\}$, ..., $\{u_{a+1}\}$ is a resolving partition, since $d(u_1, u_3) = 2$, $d(y_i, u_3) = i+3$, for $0 \leq i \leq 2(b-a-1)$, $d(x, u_3) = 1$, $d(u_2, u_3) = 2$, $d(v_i, u_3) = i+2$, for $0 \leq i \leq 2(b-a-1)$, $d(u_0, u_3) = 2(b-a-1)+3$ and $d(u_2, S_1) = 2$, $d(v_0, S_1) = 1$. Hence $pd(T) = a$.

By [1], we also have

$$dim(T) \quad = \sigma(T) - ext(T) = a.$$

Case 2 $a = b$.

a) Consider the graph $G = K_{a-1,a-1}$. We have [4],[12]

$$pd(G) = ppd(G) = a.$$

b) Consider the graph G with vertex set

$$V(G) = \{x_1, \dots, x_{a+1}, y_1, \dots, y_a, u_1, u_2, u_3, u_4, v_1, v_2\}$$

and edge set

$$\begin{aligned} E(G) = \quad & (\{x_i x_j | 1 \leq i < j \leq a+1\} - \{x_1 x_{a+1}\}) \cup \{x_i y_i | 1 \leq i \leq a\} \cup \\ & \cup \{u_i u_j | 1 \leq i < j \leq 4\} \cup \{u_1 v_1, u_2 v_2, u_4 x_1, u_3 x_{a+1}\}, \end{aligned}$$

as illustrated in figure 3.

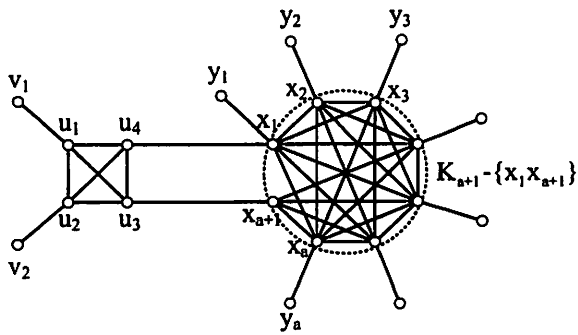

Figure 3

Since any two vertices y_i and y_j, $2 \le i \ne j \le a$ have equal distances to all the other vertices except x_i and x_j, it follows that a basis of G must contain at least one of vertices x_i, y_i for each $2 \le i \le a$ with one exception, and vertices y_i and y_j cannot belong to the same class in a resolving path partition of G.

Similarly, a basis of G must contain one of vertices v_1, u_1, v_2, u_2 and vertices v_1 and v_2 must belong to different classes in a resolving path partition of G.

It follows that

$$dim(G) \ge a - 2 + 1 = a - 1,$$

and, since from the way G was defined a path from a vertex y_i, $2 \le i \le a$ to any other terminal vertex y_1, v_1 or v_2 must contain one of the two vertices x_1 or x_{a+1}, we also have

$$ppd(G) \ge a - 1 + 1 = a.$$

But vertices x_1 and x_{a+1} have equal distances to vertices x_i, y_i, for $2 \le i \le a$ and to vertices v_1, u_1, v_2, u_2, hence

$$dim(G) \ge a.$$

It is easy to verify that the set $\{y_2, \ldots, y_{a-1}, v_1, x_1\}$ is a resolving set for G. It follows that

$$dim(G) = a.$$

Also, the partition with classes $\{y_1, x_1, u_4, u_1, v_1\}$, $\{y_a, x_a, x_{a+1}, u_3, u_2, v_2\}$, $\{y_i, x_i\}$, for $2 \le i \le a-1$ is a resolving path partition, hence

$$ppd(G) = a = dim(G).$$

□

Theorem 3.2 *For any two integers a and b such that $3 \le a \le b$ there exists a connected graph G such that $\rho(G) = a$ and $ppd(G) = b$.*

Proof: Let a and b be two integers such that $3 \le a \le b$. We consider two cases.

Case 1: $a < b$. Let $G = P_{a+1+3(b-a-1)} + \overline{K_{a-1}}$. Denote by $n = a+1+3(b-a-1)$, $x_1, \ldots, x_{a-1}$ vertices of K_{a-1} and by $y_1, \ldots, y_n$ vertices of P_n (figure 4).

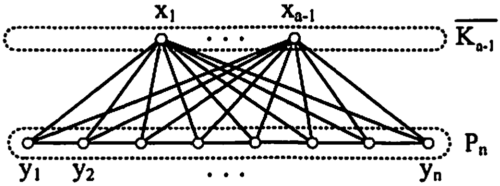

Figure 4

Then $\rho(G) = a$, a path partition of G with a classes being: $\{x_i, y_i\}$, for $1 \le i \le a-1$, $\{y_a, \ldots, y_n\}$.

We observe that if a class in a resolving path partition Π of G has three elements $\{y_i, y_{i+1}, y_{i+2}\}$, $1 \le i \le n-2$, vertices y_i, y_{i+1}, y_{i+2} have distinct representations with respect to Π only if $2 \le i \le n-3$ and y_{i-1} and y_{i+3} are not in the same class with any vertex x_j, $1 \le j \le a-1$. It follows that $ppd(G) \ge a-1+(b-a-1)+2 = b$. The path partition with classes $\{x_i, y_i\}$, $1 \le i \le a-1$, $\{y_a\}$, $\{y_{a+3k+1}, y_{a+3k+2}, y_{a+3k+3}\}$, $0 \le k \le b-a-2$, $\{y_n\}$ is a resolving partition, hence $ppd(G) = b$.

Case 2: $a = b$. Let G_a be the graph obtained from the star S_a by attaching a new terminal vertex x to one of the terminal vertices of the star and two new terminal vertices y and z to x (figure 5).

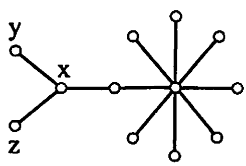

Figure 5

By Theorems 2.2 and 2.6 we obtain $ppd(G_a) = \rho(G_a) = a$. □

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