ENUMERATION OF NON-ISOMORPHIC SEMIGRAPHS IN Γ_4

K. KAYATHRI
Department of Mathematics
Thiagarajar College, Madurai-625 009
E-mail: *kayathrikanagavel@yahoo.com*
S. PETHANACHI SELVAM
Department of Mathematics
The Standard Fireworks Rajaratnam College for Women
Sivakasi - 626 123.
E-mail: *pethanachiselvam@yahoo.com*

Abstract

A semigraph G is *edge complete* if every pair of edges in G are adjacent. In this paper, we enumerate the non isomorphic semigraphs in one type of edge complete (p,3) semigraphs without isolated vertices.

Keywords: Semigraph, edge complete, m-vertex, end vertex.

1 Introduction

Sampathkumar [5, 6, 7, 8] introduced a new generalization of graphs called semigraphs. The edges of a graph G can be interpreted in the following two ways:

A. Each edge $\{u, v\}$ of G is a 2-element subset of the vertex set V of G.

B. Edges of G are 2-tuples (u, v) of vertices of G satisfying the following: (u, v) and (u', v') are equal if and only if (i) $u = u'$ and $v = v'$ or $u = v'$ and $v = u'$.

The Hypergraph theory [1] generalizes graphs using the approach A, whereas the semigraph theory generalizes graphs using the approach B.

Sampathkumar posed the problem of enumerating the edge complete semigraphs with $p \geq 6$. Edge complete $(p, 2)$ semigraphs are studied in [2] and the classification of edge complete $(p, 3)$ semigraphs is studied in

[3, 4]. In this paper we enumerate the non-isomorphic edge complete $(p, 3)$ semigraphs in one particular category.

2 Definitions

A *semigraph* G is an ordered pair (V, X) consisting of a non-empty set V of vertices and a set X of edges where X consists of n-tuples $(u_1, u_2, \ldots, u_n)$ of distinct elements belonging to the set V for various $n \geq 2$, with the following conditions:

(1)Any n-tuple $(u_1, u_2, \ldots, u_n) = (u_n, u_{n-1}, \ldots, u_1)$ and

(2)Any two such tuples have at most one element in common. A semigraph with p vertices and q edges is referred to as a (p, q) *semigraph.*

Let $E = (v_1, v_2, \ldots, v_n)$ be an edge of G. The end vertices of E are v_1 and v_n; and the *middle vertices* or *m-vertices* of E are $v_i, 2 \leq i \leq n-1$.

In diagrammatical representations of semigraphs, *thick dots* denote end vertices of an edge and *small circles* denote middle vertices of an edge.

If an m-vertex of an edge E_1 is an end vertex of another edge E_2, we draw a *small tangent* to the circle at the end of the edge E_2.

If a vertex v is an m-vertex of more than one edge of G, say $E_1, E_2, \ldots, E_t$, then v is represented as a *small regular polygon* with $2t$ corners $c_1, c_2, \ldots, c_{2t}$ with the convention that the jordon curve representing the edge E_i meets the polygon precisely at c_i and $c_{i+t}, i \in \{1, 2, \ldots, t\}$. (refer to Fig. (i)).

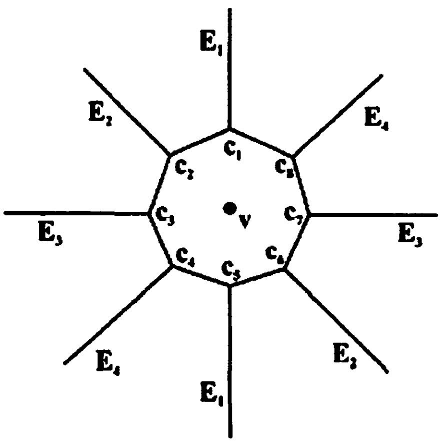


Fig. (i)

A vertex which is not a middle vertex of any edge is called a *strictly end vertex.*

A vertex which is not an end vertex of any edge is called a *strictly middle vertex.*

A vertex which does not lie in any edge is called an *isolated vertex*.

If $E = (v_1, v_2, \ldots, v_n)$ is an edge, then a *partial edge* of E, denoted by $E(v_i, v_j)$, is defined as $E(v_i, v_j) = (v_i, v_{i+1}, \ldots, v_j)$, where $1 \leq i \leq j \leq n$.

Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be two semigraphs. G_1 is isomorphic to G_2 if there exists a bijection $f : V_1 \to V_2$ such that $E = (v_1, v_2, \ldots, v_n)$ is an edge in G_1 iff $(f(v_1), f(v_2), \ldots, f(v_n))$ is an edge in G_2. In this case, we denote $(f(v_1), f(v_2), \ldots, f(v_n))$ as $f(E)$.

Two edges are *adjacent* if they have a common vertex. A semigraph G is *edge complete* if every pair of edges in G are adjacent.

3 Edge Complete $(p, 3)$ Semigraphs

Let Γ denote the class of all edge complete (p,3) semigraphs in which all edges have a common vertex. For every $G \in \Gamma$, let x denote the common vertex of all the three edges namely E_1, E_2, E_3. We shall categorize the semigraphs in Γ according to the position of x in E_1, E_2, E_3 respectively. Let e, m denote the positions 'end', 'middle' respectively.

The *label of a semigraph G in Γ*, denoted by $l(G)$, is an *ordered* triple containing the position of x in E_1, E_2, E_3 in order.

Let L' denote the set of all possible labelings of semigraphs in Γ. Then $|L'| = 2^3 = 8$.

We partition L' such that $L' = \bigcup_{j=1}^{4} L'_j$, where L'_j's are defined as follows:

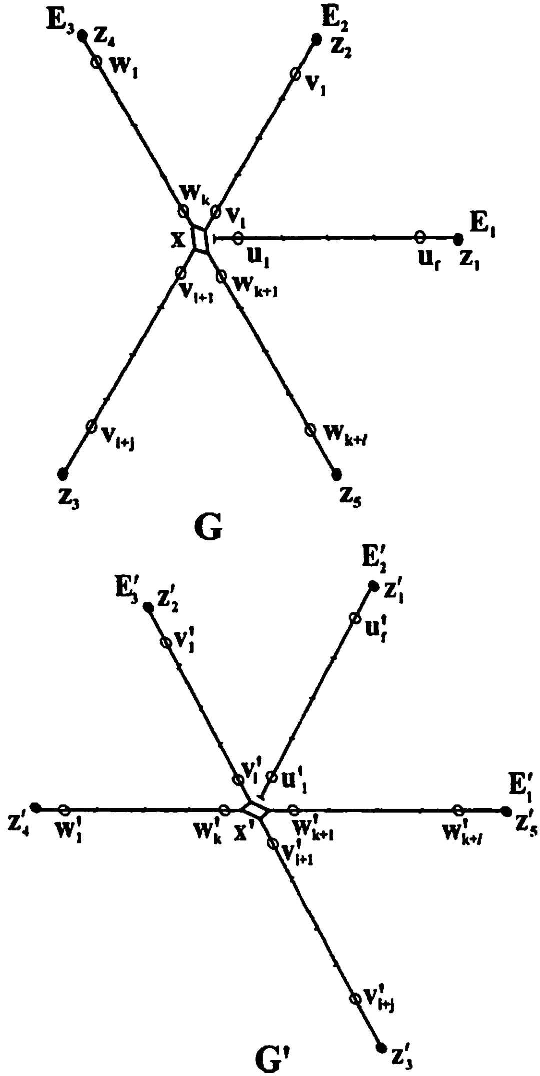
$L'_1 = \{(e, e, e)\}; L'_2 = \{(m, m, m)\};$
$L'_3 = \{(m, e, m), (m, m, e), (e, m, m)\}; L'_4 = \{(e, m, e), (e, e, m), (m, e, e)\}.$

3.1 Isomorphism of semigraphs in Γ

Theorem 3.1.1. *Any semigraph G in Γ with the labeling (η_1, η_2, η_3) is isomorphic to a semigraph in Γ with the labeling (η_3, η_1, η_2) and to a semigraph in Γ with the labeling (η_2, η_3, η_1).*

Proof. We prove the result for only one case, and the proof is similar in all the other cases. Let$(\eta_1, \eta_2, \eta_3) = (e, m, m)$. Then E_1, E_2 and E_3 are of the following form: $E_1 = (x, u_1, \ldots, u_f, z_1), E_2 = (z_2, v_1, \ldots, v_i, x, v_{i+1} \ldots, v_{i+j}, z_3)$, $E_3 = (z_4, w_1, \ldots, w_k, x, w_{k+1} \ldots, w_{k+l}, z_5)$. (Now $V(G) = \{x, z_1, z_2, z_3, z_4, z_5, u_1, \ldots, u_f, v_1, \ldots, v_{i+j}, w_1, \ldots, w_{k+l}\}$ and $X(G) = \{E_1, E_2, E_3\}$). Then consider the semigraph $G' = (V', X')$ in Γ with $V' = \{x', z'_1, z'_2, z'_3, cz'_4, z'_5, u'_1, \ldots, u'_f, v'_1, \ldots, v'_{i+j}, w'_1, \ldots, w'_{k+l}\}, X' = \{E'_1, E'_2, E'_3\})$, where $E'_1 = (z'_4, w'_1, \ldots, w'_k, x', w'_{k+1} \ldots, w'_{k+l}, z'_5), E'_2 = (x', u'_1, \ldots, u'_f, z'_1)$, $E'_3 = (z'_2, v'_1, \ldots, v'_i, x', v'_{i+1} \ldots, v'_{i+j}, z'_3)$. Now $l(G') = (m, e, m) = (\eta_3, \eta_1, \eta_2)$. Define a bijection $F : V \to V'$ by $F(v) = v'$, for all $v \in V$.

Then $F(E_1) = E_2'$. Similarly, $F(E_2) = E_3', F(E_3) = E_1'$ and so $G \cong G'$. Repeating the procedure for G', G is also isomorphic to a semigraph G'' with the labeling (η_2, η_3, η_1). (refer to Fig (ii)) $\square$



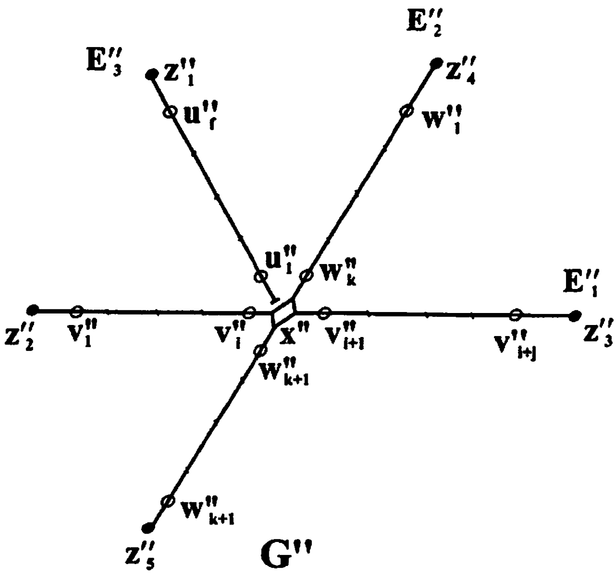


Fig. (ii)

3.2 Classification of semigraphs in Γ

Using the partition of L', we shall partition Γ into 4 subclasses such that semigraphs in different subclasses are non-isomorphic.

We define the following subfamilies of $\Gamma : \Gamma_i = \{G \in \Gamma | l(G) \in L_i'\}$, for $i = 1,2,3,4$. $\Gamma_1' = \Gamma_1; \Gamma_2' = \Gamma_2; \Gamma_3' = \{G \in \Gamma | l(G) = (e,m,m)\}; \Gamma_4' = \{G \in \Gamma | l(G) = (m,e,e)\}$. Note that $\Gamma_i' \subseteq \Gamma_i$ for $i = 1,2,3,4$.

Lemma 3.2.1. *For $i = 3,4$, the number of non-isomorphic semigraphs in Γ_i is the same as the number of non-isomorphic semigraphs in Γ_i'.*

Proof. We prove the result for $i = 4$. The proof is similar in the other case. Let $G \in \Gamma_4/\Gamma_4'$ and let $l(G) = (\eta_1, \eta_2, \eta_3)$. Now, $l(G) \in \{(e,e,m),(e,m,e)\}$. We shall consider both possibilities.

(i) Suppose $(\eta_1, \eta_2, \eta_3) = (e,e,m)$. By Theorem 3.1.1, there exists a semigraph G' with $l(G') = (\eta_3, \eta_1, \eta_2) = (m,e,e)$. Now, $G' \in \Gamma_4'$ and $G \cong G'$.

(ii) Suppose$(\eta_1, \eta_2, \eta_3) = (e,m,e)$. By Theorem 3.1.1, there exists a semigraph G' with $l(G') = (\eta_2, \eta_3, \eta_1) = (m,e,e)$. Now, $G' \in \Gamma_4'$ and $G \cong G'$.

Thus, in both cases, for every semigraph $G \in \Gamma_4/\Gamma_4'$, there exists a semigraph G' in Γ_4' which is isomorphic to G. This completes the proof. □

3.3 Non-isomorphic semigraphs in Γ_4

We enumerate the non-isomorphic semigraphs in Γ_4 without isolated vertices. As the number of non-isomorphic semigraphs in Γ_4 is the same as the number of non-isomorphic semigraphs in Γ'_4, we shall enumerate the non-isomorphic semigraphs in Γ'_4.

For every $G \in \Gamma'_4, l(G) = (m, e, e)$ and x is the common vertex which is an m-vertex of E_1 and end vertices of E_2 and E_3. Let the labels of the pair of end vertices of E_1, E_2, E_3 respectively be $(z_1, z_2), (z_4, x), (z_3, x)$. Then E_1, E_2, E_3 are of the following form: $E_1 = (z_1, \ldots, x, \ldots, z_2), E_2 = (z_4, \ldots, x), E_3 = (z_3, \ldots, x)$.

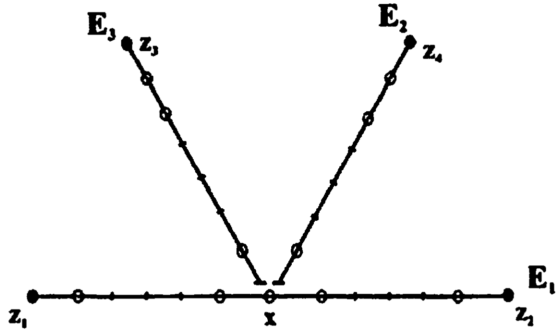


Fig. (iii)

Let k, i respectively denote the number of m-vertices in the edges E_2 and E_3; and let f, g respectively denote the number of m-vertices in the partial edges $E_1(z_1, x), E_1(x, z_2)$, where $f + g + k + i = p - 5$. (Note that $p \geq 5$).

Now any semigraph in Γ'_4 can be denoted by C_{fgki}.

For non-negative integers f, g, k, i with $f + g + k + i = p - 5$, up to isomorphism, there is just one semigraph C_{fgki} in Γ'_4 (of order $p = f + g + k + i + 5$).

Hence, C_{fgki} can be thought of as an unlabelled semigraph in Γ'_4.

Let $\mathcal{B}$ denote the family of all unlabelled semigraphs C_{fgki} in Γ'_4 of order p with $f+g+k+i = p-5$. Then the number of non-isomorphic semigraphs in Γ_4 is the same as the number of non-isomorphic semigraphs in $\mathcal{B}$.

3.3.1 Subfamilies of $\mathcal{B}$

For every $G(= C_{fgki}) \in \mathcal{B}$, let $G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}$ denote the following semigraphs in $\mathcal{B}$: $G^{(1)} = G = C_{fgki}, G^{(2)} = C_{fgik}, G^{(3)} = C_{gfki}, G^{(4)} = C_{gfik}$.

We define the following subfamilies of $\mathcal{B}$:

$\mathcal{B}_1 = \{G \in \mathcal{B} / G^{(1)} = G^{(2)}\}$,
$\mathcal{B}_2 = \{G \in \mathcal{B} / G^{(1)} = G^{(3)}\}$,
$\mathcal{B}_3 = \{G \in \mathcal{B} / G^{(1)} = G^{(2)} = G^{(3)} = G^{(4)}\}$.

3.3.2 Isomorphism of semigraphs in $\mathcal{B}$

Lemma 3.3.1. *Let $G \in \mathcal{B}$. Then*

(i) $G \in \mathcal{B}_1 \Leftrightarrow k = i$

(ii) $G \in \mathcal{B}_2 \Leftrightarrow f = g$

(iii) $G \in \mathcal{B}_3 \Leftrightarrow f = g$ *and* $k = i$.

Proof. Proof is obvious. □

Theorem 3.3.1. *For every $G \in \mathcal{B}$, we have*

(i) $G^{(1)} = G^{(2)} \Leftrightarrow G^{(3)} = G^{(4)}$

(ii) $G^{(1)} = G^{(3)} \Leftrightarrow G^{(2)} = G^{(4)}$

(iii) $G^{(1)} = G^{(4)} \Leftrightarrow G \in \mathcal{B}_3$

(iv) $G^{(2)} = G^{(3)} \Leftrightarrow G \in \mathcal{B}_3$.

Proof. (i) Let $G(= C_{fgki}) \in \mathcal{B}$.

$G^{(1)} = G^{(2)}$ implies that $k = i$. Then $G^{(3)} = C_{gfki} = C_{gfik} = G^{(4)}$. The other implication can be proved in the similar way.

Proof is similar for (ii)

(iii) $G^{(1)} = G^{(4)}$ implies that $f = g$ and $k = i$.

Now $G^{(2)} = C_{fgik} = C_{fgki} = G^{(4)}$. Similarly $G^{(3)} = C_{gfki} = C_{fgki} = G^{(1)}$. Hence $G \in \mathcal{B}_3$. The other implication is obvious from the definition of $\mathcal{B}_3$.

Proof is similar for (iv). □

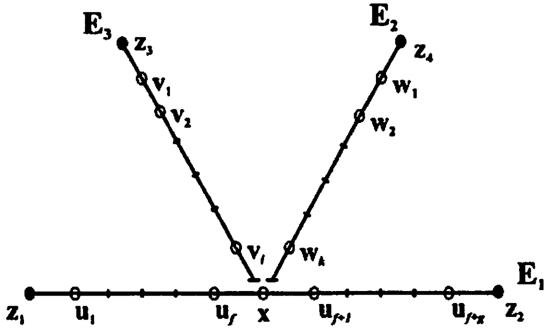
Note. $\mathcal{B}_3 \subseteq \mathcal{B}_i$, for $i = 1, 2$.

Theorem 3.3.2. *Let $G = C_{fgki}$ and $H = C_{f'g'k'i'}$ be two semigraphs in $\mathcal{B}$, where $(f, g, k, i) \neq (f', g', k', i')$. Then $G \cong H$ if and only if $H = G^{(t)}$ for some t, $t \in \{2, 3, 4\}$.*

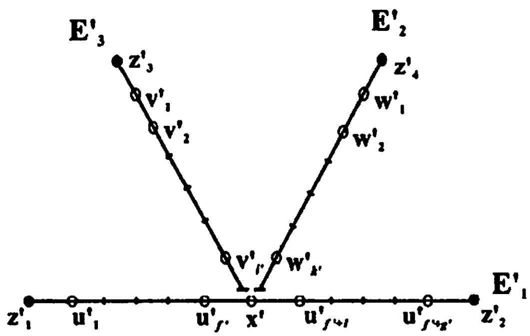
Proof. Though G and H are unlabelled semigraphs, we assign labels to vertices and edges in G and H for the purpose of referring to them.

Let $G = C_{fgki} = (V, X)$ and $H = C_{f'g'k'i'} = (V', X')$ with
$V = \{x, z_1, z_2, z_3, z_4, u_1, \ldots, u_{f+g}, v_1, \ldots, v_i, w_1, \ldots, w_k\}$,
$X = \{E_1, E_2, E_3\}$,
$E_1 = (z_1, u_1, \ldots, u_f, x, u_{f+1}, \ldots, u_{f+g}, z_2)$,

$E_2 = (z_4, w_1, \ldots, w_k, x)$,
$E_3 = (z_3, v_1, \ldots, v_i, x)$,
$V' = \{x', z_1', z_2', z_3', z_4', u_1', \ldots, u'_{f'+g'}, v_1', \ldots, v_{i'}', w_1', \ldots, w_{k'}'\}$
$X' = \{E_1', E_2', E_3'\}$,
$E_1' = (z_1', u_1', \ldots, u'_{f'}, x', u'_{f'+1}, \ldots, u'_{f'+g'}, z_2')$,
$E_2' = (z_4', w_1', \ldots, w_{k'}', x')$,
$E_3' = (z_3', v_1', \ldots, v_{i'}', x')$
and $f, g, k, i, f', g', k', i'$ are all non-negative integers with $f + g + k + i = f' + g' + k' + i' = p - 5$.



G=H_{fgki}



H=$H_{f'g'k'i'}$

Fig. (iv)

Suppose $G \cong H$. Then there exists a bijection $F : V \to V'$ such that $\{F(E_1), F(E_2), F(E_3)\} = \{E_1', E_2', E_3'\}$. Since x and x' respectively are the common vertices of E_1, E_2, E_3 and E'_1, E'_2, E'_3, we have $F(x) = x'$.

Moreover x is a middle vertex of E_1 and so $F(x)(= x')$ is also a middle vertex in $F(E_1)$ and hence $F(E_1) = E'_1$. Now we have one of the following two cases:

(i) $(F(z_1), F(u_1), \ldots, F(u_f), F(x), F(u_{f+1}), \ldots, F(u_{f+g}), F(z_2))$
$= (z'_1, u'_1, \ldots, u'_{f'}, x', u'_{f'+1}, \ldots, u'_{f'+g'}, z'_2)$

(ii) $(F(z_1), F(u_1), F(u_2), \ldots, F(u_f), F(x), F(u_{f+1}), \ldots, F(u_{f+g}), F(z_2))$
$= (z'_2, u'_{f'+g'}, \ldots, u'_{f'+1}, x', u'_{f'}, \ldots, u'_1 z'_1)$

If (i) is true then $f' = f$ and $g' = g$. Then $\{F(E_2), F(E_3)\} = \{E'_2, E'_3\}$ and so either $F(E_2) = E'_2$ and $F(E_3) = E'_3$ or $F(E_2) = E'_3$ and $F(E_3) = E'_2$. If $F(E_2) = E'_2$ and $F(E_3) = E'_3$ then $k = k'$ and $i = i'$. So $(f, g, k, i) = (f', g', k', i')$, a contradiction. If $F(E_2) = E'_3$ and $F(E_3) = E'_2$ then $i' = k$ and $k' = i$ and so $H = C_{fgik}$. Since H and G are unlabelled semigraphs in $\mathcal{B}$, it follows that $H = G^{(2)}$.

If (ii) is true then $f' = g$ and $g' = f$. Then $\{F(E_2), F(E_3)\} = \{E'_2, E'_3\}$. If $F(E_2) = E'_2$ and $F(E_3) = E'_3$ then $k' = k$ and $i' = i$. So $H = C_{gfki} = G^{(3)}$. If $F(E_2) = E'_3$ and $F(E_3) = E'_2$ then $i' = k$ and $k' = i$. So $H = C_{gfik} = G^{(4)}$.

Conversely suppose $H = G^{(2)}$. Then $f' = f, g' = g, k' = i, i' = k$.

We define a bijection $F : V \to V'$ by

$F(z_1) = z'_1$,
$F(z_2) = z'_2$,
$F(z_3) = z'_4$,
$F(z_4) = z'_3$,
$F(x) = x'$,
$F(u_\alpha) = u'_\alpha, 1 \le \alpha \le f + g$,
$F(w_\alpha) = v'_\alpha, 1 \le \alpha \le k$,
$F(v_\alpha) = w'_\alpha, 1 \le \alpha \le i$.
$F(E_1) = (F(z_1), F(u_1), \ldots, F(u_f), F(x), F(u_{f+1}), \ldots, F(u_{f+g}), F(z_2))$
$= (z'_1 u'_1, \ldots, u'_{f'}, x', u'_{f'+1}, \ldots, u'_{f'+g'}, z'_2) = E'_1$.
$F(E_2) = (F(z_4), F(w_1), \ldots, F(w_k), F(x)) = (z'_3, v'_1, \ldots, v'_{i'}, x') = E'_3$
$F(E_3) = (F(z_3), F(v_1), \ldots, F(v_i), F(x)) = (z'_4, w'_1, \ldots, w'_{k'}, x') = E'_2$
and so $G \cong H$.

Similarly when $H = G^{(t)}$ for $t = 3, 4$, we can show that $G \cong H$. This completes the proof. □

Corollary 3.3.3. *If G is a semigraph in $\mathcal{B}_3$ then G is not isomorphic to any other semigraph in $\mathcal{B}$.*

Proof. Since $G^{(1)} = G^{(2)} = G^{(3)} = G^{(4)}$ for every $G \in \mathcal{B}_3$, the result follows from Theorem 3.3.2. □

3.3.3 Non-isomorphism of semigraphs in $\mathcal{B}$

Lemma 3.3.2. $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_3$.

Proof. Let $G \in \mathcal{B}_1 \cap \mathcal{B}_2$. Then $G^{(1)} = G^{(2)}$ and $G^{(1)} = G^{(3)}$. Using Theorem 3.3.1(ii), $G^{(2)} = G^{(4)}$. Hence $G^{(1)} = G^{(4)}$ and so $G \in \mathcal{B}_3$. Thus $\mathcal{B}_1 \cap \mathcal{B}_2 \subseteq \mathcal{B}_3$. Obviously $\mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$ and so $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_3$. ☐

In all the following results, we set $G = C_{f_1 g_1 k_1 i_1}$ and $H = C_{f_2 g_2 k_2 i_2}$.

Lemma 3.3.3. *(i) If $G \in \mathcal{B}_1 \backslash \mathcal{B}_3$ then $G^{(3)}(= G^{(4)}) \in \mathcal{B}_1 \backslash \mathcal{B}_3$ and $G \neq G^{(3)}$,*

(ii) If $G \in \mathcal{B}_1 \backslash \mathcal{B}_3$ and $H \in \mathcal{B}_1 \backslash \mathcal{B}_3$ be two distinct semigraphs then $G \cong H$ if and only if $H = G^{(3)}$.

(iii) If $G \in \mathcal{B}_2 \backslash \mathcal{B}_3$ then $G^{(2)}(= G^{(4)}) \in \mathcal{B}_2 \backslash \mathcal{B}_3$ and $G \neq G^{(2)}$.

(iv) If $G \in \mathcal{B}_2 \backslash \mathcal{B}_3$ and $H \in \mathcal{B}_2 \backslash \mathcal{B}_3$ be two distinct semigraphs then $G \cong H$ if and only if $H = G^{(2)}$.

Proof. (i) Since $G \in \mathcal{B}_1 \backslash \mathcal{B}_3$ by Theorem 3.3.1(i) and (iii),

$$G^{(1)} = G^{(2)}, G^{(3)} = G^{(4)} \textit{ and } G^{(1)} \neq G^{(4)} \tag{1}$$

Also by Lemma 3.3.1(i) and (iii),

$$k_1 = i_1 \textit{ and } f_1 \neq g_1 \tag{2}$$

Let $H = G^{(3)}$. Then by (1) $H = G^{(4)}$ and $H \neq G$. Now $H = G^{(3)}$ implies that

$$f_2 = g_1, g_2 = f_1, k_2 = k_1, i_2 = i_1 \tag{3}$$

Using (2) and (3), $k_2 = i_2$ and $f_2 \neq g_2$. So $H(= G^{(3)}) \in \mathcal{B}_1 \backslash \mathcal{B}_3$.

(ii) Suppose $G \cong H$ and $G \neq H$.

Since $G \in \mathcal{B}_1 \backslash \mathcal{B}_3$, (1) and (2) holds. Since $G \cong H, H = G^{(t)}$ for some $t \in \{3, 4\}$. But $H \neq G$ and so by (1), $H = G^{(3)}$.

Conversely let $H = G^3$. Then by Theorem 3.3.2, $G \cong H$. Similarly we can prove (iii) and (iv). ☐

Lemma 3.3.4. *For $i = 1, 2$, the number of non-isomorphic semigraphs in $\mathcal{B}_i \backslash \mathcal{B}_3$ is $\frac{1}{2}|\mathcal{B}_i \backslash \mathcal{B}_3|$.*

Proof. Proof follows from Lemma 3.3.3. ☐

Lemma 3.3.5. *If $G \in \mathcal{B}_1 \backslash \mathcal{B}_3$ and $H \in \mathcal{B}_2 \backslash \mathcal{B}_3$ then G is not isomorphic to H.*

Proof. Assume that $G \cong H$. Then $G = H^{(t)}$, for some $t \in \{2,3,4\}$. By Lemma 3.3.2, $G \neq H$. Also $H^{(1)} = H^{(3)}$ and by Theorem 3.3.1 (ii), $H^{(2)} = H^{(4)}$. Hence $G = H^{(2)}$. Then

$$f_1 = f_2, g_1 = g_2, k_1 = i_2, i_1 = k_2 \tag{1}$$

Since $G \in \mathcal{B}_1 \backslash \mathcal{B}_3, G^{(1)} = G^{(2)}$ and so

$$k_1 = i_1 \tag{2}$$

By using (1) and (2), $f_1 = f_2, g_1 = g_2, i_1 = i_2, k_1 = k_2$. So $G = H$, which is a contradiction. □

Lemma 3.3.6. *If $G \in \mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$ and $H \in \bigcup_{i=1}^{2} \mathcal{B}_i$ then G is not isomorphic to H.*

Proof. On the contrary, suppose that $G \cong H$. By hypothesis , $G \neq H$.

Let $H \in \mathcal{B}_1$. Then

$$k_2 = i_2 \tag{1}$$

Also $H^{(1)} = H^{(2)}, H^{(3)} = H^{(4)}$. Since $G \neq H$ and $G \cong H$, we have $G = H^{(3)}$. Then

$$f_1 = g_2, g_1 = f_2, k_1 = k_2, i_1 = i_2 \tag{2}$$

(1) and (2) implies $k_1 = i_1$ and so $G \in \mathcal{B}_1$, a contradiction.

Let $H \in \mathcal{B}_2$. Then

$$f_2 = g_2 \tag{3}$$

Also $H^{(1)} = H^{(3)}, H^{(2)} = H^{(4)}$. Since $G \neq H$ and $G \cong H$, we have $G = H^{(2)}$. Then

$$f_1 = f_2, g_1 = g_2, k_1 = i_2, i_1 = k_2 \tag{4}$$

(3) and (4) implies $f_1 = g_1$ and so $G \in \mathcal{B}_2$, a contradiction. □

Lemma 3.3.7. *Let $G \in \mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$. If $H \neq G$ and $H \cong G$ then $H \in \mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$.*

Proof. Proof follows from Lemma 3.3.6. □

3.3.4 Enumeration

Theorem 3.3.4. *The number of distinct semigraphs in $\mathcal{B}$ is given by $T = \frac{1}{6}(p-4)(p-3)(p-2)$.*

Proof. For every $G \in \mathcal{B}, f+g+k+i = p-5$. Hence $T=$ The number of non-negative integer solutions of this equation. So

$$T = \binom{4+p-5-1}{p-5} = \binom{p-2}{3} = \frac{1}{6}(p-4)(p-3)(p-2). \qquad \square$$

Lemma 3.3.8. *For $i = 1, 2$, the number of distinct semigraphs in $\mathcal{B}_i$ is given by* $T_1 = \begin{cases} \left(\frac{p-3}{2}\right)^2 & \textit{if } p \textit{ is odd} \\ \left(\frac{p-4}{2}\right)\left(\frac{p-2}{2}\right) & \textit{if } p \textit{ is even.} \end{cases}$

Proof. We prove the result for $i = 1$ and the result is similar for $i = 2$. Let $G \in \mathcal{B}_1$. Then $k = i$. So $f + g + 2i = p - 5$. The number of non-negative integer solutions of this equation is the same as the number of non-negative integer solutions of the equation $x_1 + x_2 + x_3 = p - 5$ with the constraint that x_3 is even. This can be determined using the counting technique as follows:

Define $f_1(x) = f_2(x) = x^0 + x^1 + x^2 + \cdots = (1-x)^{-1}$

$f_3(x) = x^0 + x^2 + x^4 + \cdots = (1-x^2)^{-1}$ and

$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) = (1-x)^{-2} \cdot (1-x^2)^{-1}$.

The required solution is the coefficient of x^{p-5} in $f(x)$.

Now $f(x) = \sum_{r=0}^{\infty} \binom{r+1}{r} x^r \sum_{s=0}^{\infty} x^{2s}$.

The general term in the R.H.S is $(r+1)x^{r+2s}$. We need the coefficient of x^{p-5}. So $r + 2s = p - 5$.

Case i. p is odd.

Then r is even and $0 \le r \le p-5$. So $T_1 = \sum_{\substack{r=0 \\ r \text{ iseven}}}^{p-5} r+1 = \sum_{n=1}^{\frac{p-3}{2}} 2n-1$.

On simplification, we get $T_1 = \left(\frac{p-3}{2}\right)^2$.

Case ii. p is even.

Then r is odd and $0 \le r \le p-5$. So $T_1 = \sum_{\substack{r=0 \\ r \text{ is odd}}}^{p-5} r+1 = \sum_{n=1}^{\frac{p-4}{2}} 2n$. On simplification $T_1 = \left(\frac{p-4}{2}\right)\left(\frac{p-2}{4}\right)$ $\square$

Lemma 3.3.9. *The number of distinct semigraphs in $\mathcal{B}_3$ is given by*

$$T_2 = \begin{cases} \left(\frac{p-3}{2}\right) & \textit{if } p \textit{ is odd} \\ 0 & \textit{if } p \textit{ is even.} \end{cases}$$

Proof. Let $G \in \mathcal{B}_3$. Then $f = g$ and $k = i$. So $2f + 2k = p - 5$, ie $f + k = \frac{p-5}{2}$ and p is odd. The number of non-negative integer solutions is $\binom{2+\frac{p-5}{2}-1}{\frac{p-5}{2}} = \binom{\frac{p-5}{2}+1}{\frac{p-5}{2}} = \frac{p-3}{2}$ □

Theorem 3.3.5. *For $p \geq 5$, the number of non-isomorphic semigraphs in Γ_4 is given by* $S = \begin{cases} \frac{1}{24}p(p-2)(p-4) & \text{if } p \text{ is even} \\ \frac{1}{24}(p-1)(p-2)((p-3) & \text{if } p \text{ is odd.} \end{cases}$

Proof. The number of non-isomorphic semigraphs in Γ_4 is the same as the number of non-isomorphic semigraphs in $\mathcal{B}$. Hence S=Number of non-isomorphic semigraphs in $\mathcal{B}$.

Let $G \in \mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$. Then by using Theorem 3.3.1, $G^{(t)}$'s are all distinct for $t = 1, 2, 3, 4$. Also by Theorem 3.3.2, these four semigraphs are isomorphic.

Moreover by Lemma 3.3.7. $G^{(t)}$'s are all in $\mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$. Hence the number of non-isomorphic semigraphs in $\mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$ is $\frac{1}{4}|\mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i|$.

Now $|\mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i| = |\mathcal{B}| - |\bigcup_{i=1}^{2} \mathcal{B}_i| = |\mathcal{B}| - \left[\sum_{i=1}^{2} |\mathcal{B}_i| - |\mathcal{B}_1 \cap \mathcal{B}_2|\right] = T - [2T_1 - T_2]$, using Lemmas 3.3.2,3.3.8,3.3.9 and Theorem 3.3.4. Hence the number of non-isomorphic semigraphs in $\mathcal{B} \backslash \bigcup_{i=1}^{2} \mathcal{B}_i$ is $\frac{1}{4}[T - 2T_1 + T_2]$.

Using Lemmas 3.3.4,3.3.5 and Corollary 3.3.3, the number of non-isomorphic semigraphs in $\bigcup_{i=1}^{2} \mathcal{B}_i$ is equal to $\frac{1}{2}|\mathcal{B}_1 \backslash \mathcal{B}_3| + \frac{1}{2}|\mathcal{B}_2 \backslash \mathcal{B}_3| + |\mathcal{B}_3| = \frac{T_1 - T_2}{2} + \frac{T_1 - T_2}{2} + T_2 = T_1$.

By Lemma 3.3.6, the number of non-isomorphic semigraphs in $\mathcal{B}$ is given by

$S = \frac{1}{4}[T - 2T_1 + T_2] + T_1 = \frac{1}{4}[T + 2T_1 + T_2]$.

Case i. p is even.

$S = \frac{1}{4}\left[\frac{1}{6}(p-4)(p-3)(p-2) + 2\left(\frac{p-4}{2}\right)\left(\frac{p-2}{2}\right)\right]$.

On simplification, we get $S = \frac{1}{24}p(p-2)(p-4)$.

Case ii. p is odd.

$S = \frac{1}{4}\left[\frac{1}{6}(p-4)(p-3)(p-2) + 2\left(\frac{p-3}{2}\right)^2 + \frac{p-3}{2}\right]$.

Simplifying, we get $S = \frac{1}{24}(p-1)(p-2)(p-3)$. □

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