Acyclic Total Colorings of Planar Graphs

Xiang Yong Sun
School of Statistics and Mathematics, Shandong Economic University, Jinan, 250014, China

Jian Liang Wu*
School of Mathematics and Systems Science, Shandong University, Jinan, 250100, China

Abstract

In the paper, we give the definition of *acyclic total coloring* and *acyclic total chromatic number* of a graph. It is proved that the acyclic total chromatic number of a planar graph G with maximum $\Delta(G)$ and girth g is at most $\Delta(G)+2$ if $\Delta \geq 12$, or $\Delta \geq 6$ and $g \geq 4$, or $\Delta \geq 5$ and $g \geq 5$, or $g \geq 6$. Moreover, if G is a series-parallel graph with $\Delta \geq 3$ or a planar graph with $\Delta \geq 3$ and $g \geq 12$, then the acyclic total chromatic number of G is $\Delta(G) + 1$.

Keywords: acyclic total coloring, girth, planar graph, series-parallel graph

1 Introduction

In the paper, all graphs are finite, simple and undirected. We use $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ (simply for Δ) to denote the vertex set, the edge set, the minimum degree and the maximum degree of a graph G. If $uv \in E(G)$, then u is said to be the *neighbor* of v. We use $N(v)$ to denote the set of neighbors of a vertex v, and $d(v) = |N(v)|$ to denote the *degree* of v. A *k-vertex* is a vertex of degree k. Similarly, a *$\geq k$-vertex* is a vertex of degree at least k, and a *$\leq k$-vertex* is a vertex of degree at most k.

A *proper k-vertex-coloring* of a graph G is a mapping $\varphi : V(G) \to \{1, 2, \cdots k\}$ such that no two adjacent vertices receive the same color. A proper vertex coloring of a graph G is called *acyclic* if there is no 2-colored cycle in G. The *acyclic vertex chromatic number* $\chi_a(G)$ is the smallest

*This work was supported by National Natural Science Foundation of China(10971121, 10631070, 60673059).

integer k such that G has an acyclic vertex coloring. A *proper k-edge-coloring* of a graph G is a mapping $\psi : E(G) \to \{1, 2, \cdots k\}$ such that no two adjacent edges receive the same color. A proper edge coloring of a graph G is called *acyclic* if there is no 2-colored cycle in G. The *acyclic edge chromatic number* $\chi'_a(G)$ is the smallest integer k such that G has an acyclic edge coloring.

Grünbaum [1] posed the acyclic vertex coloring conjecture which asserted that every planar graph has an acyclic *5-coloring*. The conjecture was proved by Borodin [2]. For any graphs, some results are known recently. Fertin and Raspaud [3] showed that any graph of maximum degree 5 has acyclic chromatic number at most 9, and they gave a linear time algorithm that achieves this bound.

The acyclic edge coloring was introduced by Alon et al. in [4], and they proved that $\chi'_a(G) \leq 64\Delta(G)$ for all graphs. Molloy and Reed [5] showed that $\chi'_a(G) \leq 16\Delta(G)$ using the same method. In 2001, Alon, Sudakov and Zaks [6] proposed the acyclic edge coloring conjecture which stated that for any graph G, $\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2$. They proved in the same paper that this conjecture was true for almost all $\Delta(G)$-regular graphs G, and all $\Delta(G)$-regular graphs, whose girth (length of shortest cycle) is at least $c\Delta(G)\log\Delta(G)$ for some constant c. Alon and Zaks [7] proved that determining the acyclic edge chromatic number of an arbitrary graph is an NP-complete problem, even determining whether $\chi'_a(G) \leq 3$ for an arbitrary graph G. For planar graphs, Fiedorowicz, Haluszczak and Narayanan [8] proved that $\chi'_a(G) \leq \Delta(G) + 6$ if $g(G)$(length of shortest cycle of G)≥ 4 as well as G has an edge-partition into two forests. Hou, *etc* [9] showed that $\chi'_a(G) \leq max\{2\Delta(G) - 2, \Delta + 22\}$, $\chi'_a(G) \leq \Delta(G) + 2$ if $g(G) \geq 5$, and $\chi'_a(G) \leq \Delta(G) + 1$ if $g(G) \geq 7$. Furthermore, $\chi'_a(G) \leq \Delta(G) + 1$ if G is series-parallel graph.

A proper *total k-coloring* of a graph G is a mapping $\phi : E(G) \cup V(G) \to \{1, 2, \cdots k\}$ such that no two adjacent or incident elements receive the same color. The *total chromatic number* of G, $\chi''(G)$, is the smallest integer k such that G has a total *k-coloring*. For total colorings of graphs, Behzad [10] and Vizing [11] posed independently the famous total coloring conjecture which asserted that for any graph G, $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$. Clearly, the lower bound is trivial. The upper bound has not been proved for all values of Δ. Considering the colorings above, we have an idea of combining acyclic vertex (edge) coloring with total coloring. So we propose the following coloring.

Definition 1. An *acyclic total coloring* of a graph G is a proper total coloring such that for any cycle of G, there are at least 4 colors appeared on its vertices and edges. The *acyclic total chromatic number* of G, denoted by $\chi''_a(G)$, is the smallest integer k such that G has an acyclic total coloring using k colors.

In the paper, we mainly consider the acyclic total coloring of planar graphs. It is obtained that the acyclic total chromatic number of a planar graph G is at most $\Delta(G)+2$ if $\Delta \geq 12$, or $\Delta \geq 6$ and $g \geq 4$, or $\Delta \geq 5$ and $g \geq 5$, or $g \geq 6$. Moreover, if G is a series-planar graph or a planar graph with $g \geq 12$ and $\Delta \geq 3$, then its acyclic total chromatic number is $\Delta + 1$. On the basis of these results, we pose a conjecture on acyclic total coloring.

Conjecture 2. *For any graph G, $\Delta(G)+1 \leq \chi''_a(G) \leq \Delta(G)+2$.*

It is obvious that for any tree T, $\chi''_a(T) = \chi''(T)$. If C_n is a cycle of order n, then $\chi''_a(C_n) = 4$. For bipartite graphs, we have the following theorem.

Theorem 3. *Let G be a bipartite graph with two partite sets X and Y. Then $\chi''_a(G) \leq \Delta + 2$.*

Proof. Here, it suffices to give an acyclic total coloring of G as follows. Let $k = \Delta(G)+2$ and L be the color set $\{1, 2, \cdots, k\}$ for simplicity. First, we color every vertex of X with color 1 and color every vertex of Y with color 2. Then we color every edges of G using $3, 4, \cdots, k$ colors such that it is a proper edge coloring. Obviously, it is an acyclic total $(\Delta+2)$ coloring. □

2 Planar Graphs

In the section, we always assume that any graph G is planar and is embedded in the plane. We use $F(G)$ to denote the face set of G. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A $k(\geq k$, or $\leq k)$-*face* is a face of degree (at least, or at most) k. Let f is a 3-face. We use $\delta(f)$ to denote the least degree of vertices which are incident with f. A $(i, \leq j)$-edge $uv \in E(G)$ is the edge such that $d(u) = i$ and $d(v) \leq j$.

Theorem 4. *Let G be a planar graph with maximum degree Δ and girth g. Then $\chi''_a(G) \leq \Delta + 2$ if one of the following conditions holds.*

(i) $\Delta \geq 12$; (ii) $\Delta \geq 6$ *and* $g \geq 4$;
(iii) $\Delta \geq 5$ *and* $g \geq 5$; (iv) $g \geq 6$.

Proof. Let G be a minimal counterexample to the theorem. Then G is 2-connected. Let $t = \Delta(G)+2$ and let L be the color set $\{1, 2, \cdots, t\}$ for simplicity. First, we shall prove the following lemma.

Lemma 5. *G does not contain a k-vertex u with neighbors $u_1, u_2, \cdots, u_k$, $d(u_1) \le d(u_2) \cdots \le d(u_k)$, such that one of the following is true.*

(a) *If $\Delta \ge 3$, then $k \le 2$.*

(b) *If $\Delta \ge 5$, then $k = 3$ and $d(u_1) \le \Delta - 1$.*

(c) *If $\Delta \ge 6$, then $k = 4$, $d(u_1) \le \Delta - 3$ and $d(u_2) \le \Delta - 1$.*

(d) *If $\Delta \ge 8$, then $k = 5$, $d(u_1) \le \Delta - 4$, $d(u_2) \le \Delta - 2$ and $d(u_3) \le \Delta - 1$.*

Proof. Suppose, to be contrary, that such a k-vertex u does exists. By the minimality of G, $G' = G - uu_1$ has an acyclic total coloring ϕ with colors from L. For any $v \in V(G)$, let $\Phi(v) = \{\phi(v)\} \bigcup \{\phi(uv) | u \in N(v)\}$. In the following, we will extend ϕ to an acyclic total coloring φ of G with $\Delta(G)+2$ colors, which is a contradiction. First, we erase the color on vertex u, and let $\varphi(x) = \phi(x)$ for any $x \in (V(G') \bigcup E(G')) \bigcap (V(G) \bigcup E(G))$. For two adjacent edges wx and wy, if $\varphi(x) = \varphi(wy)$ and $\varphi(y) = \varphi(wx)$, then we call these two edges *a pair of paratactic edges*, denoted by $wx \| wy$.

Then, we will recolor some edges of G' such that there is no a pair of paratactic edges incident with u as follows. For (a), we do nothing since $d_{G'}(u) \le 1$. For (b), if $uu_2 \| uu_3$, then let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{\phi(u_3)\})$.

For (c), we have that $\Delta \ge 6$, $k = 4$, $d(u_1) \le \Delta - 3$ and $d(u_2) \le \Delta - 1$. Without loss of generality(WLOG), let $\phi(uu_i) = i, (i = 2, 3, 4)$. First, if $uu_3 \| uu_4$, let $\varphi(uu_3) \in L \backslash \Phi(u_3)$ to obtain that u_3, u_4, uu_3, uu_4 are colored at least three colors. If $\varphi(uu_3) = 2$, let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{2, 4\})$. Then, if $uu_2 \| uu_4$, let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{\varphi(uu_3)\})$ (since $d(u_2) \le \Delta - 1$, $|\Phi(u_2) \cup \{\varphi(uu_3)\}| \le \Delta + 1$). If $uu_2 \| uu_3$, it can be settled similarly.

For (d), we have $\Delta \ge 8$, $k = 5$, $d(u_1) \le \Delta - 4$, $d(u_2) \le \Delta - 2$, and $d(u_3) \le \Delta - 1$. Without loss of generality, let $\phi(uu_i) = i, (i = 2, 3, 4, 5)$. We consider the following cases.

Case 1. There are two pair of paratactic edges incident with u.

Subcase 1.1. $uu_2 \| uu_4$ and $uu_3 \| uu_5$ (the case that $uu_2 \| uu_5$ and $uu_3 \| uu_4$ can be settled similarly).

Let $\varphi(uu_3) \in L \backslash (\Phi(u_3) \cup \{4\})$ and $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{\varphi(uu_3), 5\})$.

Subcase 1.2. $uu_2 \| uu_3$ and $uu_4 \| uu_5$, that is, $\phi(u_2) = 3$, $\phi(u_3) = 2$ and $\phi(u_4) = 5$, $\phi(u_5) = 4$.

Let $\varphi(uu_4) \in L \backslash \Phi(u_4)$. If $\varphi(uu_4) \notin \{2, 3\}$, let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{\varphi(uu_4), 5\})$ since $d(u_2) \le \Delta - 2$. If $\varphi(uu_4) = 3$, then we can recolor edge uu_3 such that $\varphi(uu_3) \in L \backslash (\Phi(u_3) \cup \{5\})$. If $\varphi(uu_4) = 2$, let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{5\})$.

Case 2. There is only one pair of paratactic edges incident with u.

Subcase 2.1. $uu_2 \| uu_3$.

Let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{4, 5\})$.

Subcase 2.2. $uu_2 \| uu_4$ (the case that $uu_2 \| uu_5$ can be settled similarly).

Let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{3,5\})$ since $d(u_2) \leq \Delta - 2$.

Subcase 2.3. $uu_3 \| uu_4$ (the case that $uu_3 \| uu_5$ can be settled similarly).

Let $\varphi(uu_3) \in L \backslash (\Phi(u_3) \cup \{5\})$. Then $\varphi(uu_3) \notin \{3,4,5\}$. If $\varphi(uu_3) = 2$, let $\varphi(uu_2) \in L \backslash (\Phi(u_2) \cup \{4,5\})$. It is clearly that $\varphi(uu_2) \notin \{2,4,5\}$. If $\varphi(uu_2) = 3$ and $\phi(u_2) = 4$, or $\phi(u_2) = 5$ and $\phi(u_5) = \varphi(uu_2)$, then it return to Subcase 2.2.

Subcase 2.4. $uu_4 \| uu_5$.

Let $\varphi(uu_4) = c_1 \in L \backslash \Phi(u_4)$. If $c_1 \notin \{2,3\}$, then we have done. Suppose that $c_1 = 3$. Let $\varphi(uu_3) = f \in L \backslash (\Phi(u_3) \cup \{5\})$. It is clearly that $f \notin \{3,5\}$. If $f = 4$ and $\phi(u_3) = 5$, then it return to Subcase 2.3. If $\phi(u_2) = f$ and $\phi(u_3) = 2$, then it return to Subcase 2.1. If $f = 2$, let $\varphi(uu_2) = h \in L \backslash (\Phi(u_2) \cup \{3,5\})$. It is clearly that $h \notin \{2,3,5\}$. If $h = 4$ and $\phi(u_2) = 5$, then it return to Subcase 2.2. Suppose that $c_1 = 2$. Let $\varphi(uu_2) = p \in L \backslash (\Phi(u_2) \cup \{3,5\})$ by $d(u_2) \leq \Delta - 2$. It is clearly that $p \notin \{2,3,5\}$. If $p = 4$ and $\phi(u_2) = 5$, then it return to Subcase 2.2. If $\phi(u_3) = p$ and $\phi(u_2) = 3$, then it return to Subcase 2.1.

Thus, we recolor G' such that there is no a pair of paratactic edges incident with u. Now, let $\Gamma = \{\varphi(uu_2), \cdots, \varphi(uu_k)\}$. Finally, we begin to color uu_1 and recolor u as follows.

For (a) and (b), we first color uu_1 such that if $\phi(u_1) \in \Gamma$, WLOG, assume that $\varphi(uu_2) = \phi(u_1)$, let $\varphi(uu_1) \in L \backslash (\Phi(u_1) \cup \{\phi(u_2)\} \cup \Gamma)$. Otherwise, let $\varphi(uu_1) \in L \backslash (\Phi(u_1) \cup \Gamma)$. Later, let $\varphi(u) \in L \backslash \{\varphi(u_i), \varphi(uu_i) : 1 \leq i \leq k\}$.

For (c) and (d), we first recolor u such that $\varphi(u) \in L \backslash (\{\varphi(u_i), \varphi(uu_i) : 2 \leq i \leq k\} \cup \{\varphi(u_1)\})$. Then color uu_1 such that if $\phi(u_1) \in \Gamma$, WLOG, assume that $\varphi(uu_2) = \phi(u_1)$, let $\varphi(uu_1) \in L \backslash (\Phi(u_1) \cup \Gamma \cup \{\phi(u_2), \varphi(u)\})$. Otherwise, let $\varphi(uu_1) \in L \backslash (\Phi(u_1) \cup \Gamma \cup \{\varphi(u)\})$.

According to the above coloring, we obtain that for any i and $j (1 \leq i < j \leq k)$, $\{\varphi(u_i), \varphi(uu_i), \varphi(u_j), \varphi(uu_j), \varphi(u)\}| \geq 4$. Hence, φ is an acyclic total coloring of G with $\Delta(G) + 2$ colors. □

Now we are ready to prove (i) of the theorem. By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -4(|V| - |E| + |F|) = -8 < 0. \quad (1)$$

Now we define $w(x)$ to be the initial charge function to each $x \in V(G) \cup F(G)$. Let $w(v) = d(v) - 4$ for $v \in V(G)$ and $w(f) = d(f) - 4$ for $f \in F(G)$. In the following, we will reassign a new charge denoted by $w'(x)$ to each

$x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) < 0. \quad (2)$$

If we can show that $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, then we obtain a contradiction to (2), completing the proof.

A $(i, \leq j, \geq k)$ face uvw is a 3-face such that $d(u) = i \leq d(v) \leq j \leq k \leq d(w)$. Clearly $\delta(uvw) = i$. The other kind of 3-faces may be defined similarly. For simplicity, some special 3-faces are defined as follows. Let $f_a = (3, \Delta, \Delta)$, $f_b = (4, \leq \Delta - 3, \Delta)$, $f_c = (4, \geq \Delta - 2, \geq \Delta - 2)$, $f_d = (5, \leq \Delta - 4, \leq \Delta - 2)$, $f_e = (5, \leq \Delta - 4, \geq \Delta - 1)$, $f_f = (5, \geq \Delta - 3, \geq \Delta - 3)$ and $f_g = (\geq 6, \geq 6, \geq 6)$. By Lemma 5, any a 3-face must be a f_a-face, or f_b-face, or f_c-face, or f_d-face, or f_e-face, or f_f-face, or f_g-face.

For (i), the discharging rules are defined as follows.

R1-1: From each Δ-vertex to each of its adjacent 3-vertices, transfer $\frac{1}{3}$.

R1-2: From each vertex v with $4 \leq d(v) \leq (\Delta - 1)$ to each of its incident 3-faces, transfer $\frac{d(v)-4}{d(v)}$.

R1-3: From each Δ-vertex to each of its incident 3-faces f_a or f_c, transfer $\frac{1}{2}$; to each of its incident 3-faces f_e, transfer $\frac{3}{5}$; to each of its incident 3-faces f_f, transfer $\frac{2}{5}$; to each of its incident 3-faces f_g, transfer $\frac{1}{3}$.

R1-4: From each Δ-vertex to each of its incident $(4, 4, \Delta)$ 3-faces f_b, transfer 1, and to the other 3-faces f_b, transfer $\frac{4}{5}$.

R1-5: From each Δ-vertex v through its adjacent 5-vertex u to each 3-face f_d which is incident with the 5-vertex u, transfer $\frac{2}{45}$.

Let v be a vertex of G. We can get $d(v) \geq 3$ by Lemma 5(a). If $d(v) = 3$, then each neighbor of v is of maximum degree by Lemma 5(b). If 4-vertex v is adjacent to a vertex u with $d(u) \leq (\Delta - 3)$, then the other adjacent vertices of v must be Δ-vertices by Lemma 5(c). If 5-vertex v is adjacent to a vertex u with $d(u) \leq (\Delta - 4)$ and a vertex w with $d(w) \leq (\Delta - 2)$, then the other adjacent vertices of v must be Δ-vertices by Lemma 5(d).

If $d(v) = 3$, then v can receive $\frac{1}{3}$ from each adjacent vertex by R1-1. So $w'(v) \geq w(v) + \frac{1}{3} \times 3 = 0$. If $4 \leq d(v) \leq (\Delta - 1)$, then v can be incident with at most $d(v)$ 3-faces. So it follows by R1-2 that $w'(v) \geq w(v) - \frac{d(v)-4}{d(v)} \times d(v) = 0$.

If $d(v) = \Delta \geq 12$, then v can distribute value to 3-faces, 3-vertices by

R1-1 to R1-4 and 5-vertices by R1-5. Here, by means of an average ideal, we divide all the value, which v distribute to faces and vertices, by the amount of all faces which v is incident with. Now the value we consider is on an average face, called as *value on an average face*. In the following, we try to get the *value on an average face* in all cases.

When v is adjacent to one 3-vertex, v can distribute value to one 3-vertex and at most two 3-faces f_a by Lemma 5(b). Therefore, on an average face, v distribute value $(\frac{1}{3} + \frac{1}{2} \times 2)/2 = \frac{2}{3}$ by R1-1 and R1-3. When v is incident with $(4, 4, \Delta)$ faces, and at the same time, v must be incident with two $(4, \Delta, \Delta)$ faces by Lemma 5(c). So, on an average face, v distribute value $(1 + \frac{1}{2} \times 2)/3 = \frac{2}{3}$ by R1-3 and R1-4. When v is incident with one $(4, 5, \Delta)$ face, and at the same time, v may be incident with one $(4, 5, \Delta)$ faces and two $(4, \Delta, \Delta)$ faces by R1-3 and R1-4. v also can distribute to a 5-vertex $\frac{2}{45}$ by R1-5. So, on an average face, v distribute value $(\frac{1}{2} \times 2 + \frac{4}{5} \times 2 + \frac{2}{45})/4 = \frac{119}{180} < \frac{2}{3}$. When v is incident with $(4, 5, \Delta)$ faces, and at the same time, v may be incident with one $(5, 5, \Delta)$ face, one $(4, 5, \Delta)$ face and two $(4, \Delta, \Delta)$ faces by R1-3 and R1-4. v also can distribute to two 5-vertices $\frac{2}{45} \times 2$ by R1-5. So, on an average face, v distribute value $(\frac{3}{5} + \frac{4}{5} \times 2 + \frac{1}{2} \times 2 + \frac{2}{45} \times 2)/5 = \frac{148}{225} < \frac{2}{3}$. When v is incident with $(5, 5, \Delta)$ faces, and at the same time, v may be incident with two $(5, 5, \Delta)$ faces by R1-3. v also can distribute to four 5-vertices $\frac{2}{45} \times 4$ by R1-5. So, on an average face, v distribute value $(\frac{3}{5} \times 3 + \frac{2}{45} \times 4)/3 = \frac{89}{135} < \frac{2}{3}$. It is clearly in the other cases, v will distribute less than that of above cases. While the average of any two cases is at most $\frac{2}{3}$, we can see, on any an average face, v distribute at most $\frac{2}{3}$. So $w'(v) \geq w(v) - \frac{2}{3} \times d(v) = \frac{1}{3} \times d(v) - 4 \geq 0$.

Let f be a face of G. Suppose that $d(f) = 3$. If $\delta(f) = 3$, f must be a 3-face f_a by Lemma 5(b). So f can receive $\frac{1}{2} \times 2$ from its incident Δ-vertices by R1-3. If $\delta(f) = 4$, f could be a 3-face f_b or f_c by Lemma 5(c). So f can receive 1 by R1-4 while f is $(4, 4, \Delta)$ face, or receive at least $(\frac{1}{5} + \frac{4}{5})$ by R1-2 and R1-4 while f is the other 3-face f_b, and f can receive at least $2 \times \min\{\frac{6}{10}, \frac{1}{2}\}$ from its incident $(\geq (\Delta - 2))$-vertices by R1-2 and R1-3 while f is 3-face f_c. If $\delta(f) = 5$, f could be a 3-face f_d, f_e, or f_f by Lemma 5(d). So f can receive $\frac{1}{5}$,$\frac{1}{3}$, $\frac{3}{7}$ from each of its incident 5, 6, 7-vertex by R1-2 respectively. And a 5-vertex, which is incident with a 3-face f_d, must be adjacent to three Δ-vertices by Lemma 5(d). So f_d could receive $(\frac{2}{45} \times 3 + \frac{1}{5}) = \frac{1}{3}$ from the 5-vertex and three Δ-vertices by R1-5 and R1-2. Hence f_d can get at least $\frac{1}{3} \times 3 = 1$. If f is 3-face f_e, f can receive at least $\frac{1}{5} + \frac{1}{5} + \min\{\frac{3}{5}, \frac{7}{11}\} = 1$ by R1-3 and R1-2. If f is 3-face f_f, f can receive at least $\frac{1}{5} + 2 \times \min\{\frac{5}{9}, \frac{2}{5}\} = 1$ by R1-2 and R1-3. If f is 3-face f_g, f can receive at least $\frac{1}{3} \times 3 = 1$. Hence f can get at least 1 in any cases. If $d(f) \geq 4$, we can easy to get that $w'(f) = w(f) \geq 0$.

Hence we have $w'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction

If we can show that $w'''(x) \geq 0$ for each $x \in V(G)$, then we obtain a contradiction to (6), completing the proof.

For (iii), the discharging rule is defined as follows.

R3-1: From each Δ-vertex v to each of its adjacent 3-vertices, transfer $\frac{1}{3}$.

Let v be a vertex of G. Then $d(v) \geq 3$. If $d(v) = 3$, then $w'''(v) = w(v) + 3 \times \frac{1}{3} = 0$. If $4 \leq d(v) \leq \Delta - 1$, then $w'''(v) = w(v) \geq 0$. If $d(v) = \Delta \geq 5$, then $w'''(v) \geq w(v) - \frac{1}{3} \times d(v) = \frac{8}{3} \times d(v) - 10 \geq \frac{10}{3} > 0$. Hence we have $w'''(x) \geq 0$ for each $x \in V(G)$, a contradiction with (6). This contradiction proves (iii).

Now we are ready to prove (iv). By Euler's formula $|V| - |E| + |F| = 2$, we have $\sum_{v \in V(G)}(2d(v) - 6) + \sum_{f \in F(G)}(d(f) - 6) = -6(|V| - |E| + |F|) = -12$. So

$$\sum_{v \in V(G)} (2d(v) - 6) \leq \sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12 < 0.$$

It implies that $\delta(G) \leq 2$. If $\Delta \geq 3$, then it contradicts with Lemma 5(a). Otherwise, $\Delta = 2$, it is obvious that $\chi''_a(G) \leq \Delta(G) + 2$, a contradiction, too. Hence (iv) holds. □

Lemma 6. *Let G be a planar graph with $\delta(G) \geq 2$ and $g(G) \geq 12$. Then*

(1) *G contains a 2-vertex with neighbors are 2-vertices, or*

(2) *G contains a path $uvwxyz$ with $d(x) = 3$ and $d(v) = d(w) = d(y) =$* 2.

Proof. Let G be a counterexample to the lemma. Let $w(v) = 5d(v) - 12$ if $v \in V(G)$ and $w(f) = d(f) - 12$ if $f \in F(G)$. Then we have

$$\sum_{v \in V(G)} (5d(v) - 12) \leq \sum_{v \in V(G)} (5d(v) - 12) + \sum_{f \in F(G)} (d(f) - 12) = -24 < 0.$$

Now the discharging rules are defined as follows.

R4-1: From each 3-vertex u to each of its adjacent 2-vertex v, transfer 1 if u is adjacent to at least two 2-vertices; 2 otherwise.

R4-2: From each vertex v with $d(v) \geq 4$ to each of its adjacent 2-vertex, transfer 2.

We shall get a contradiction by proving that $w^{**}(x) \geq 0$ for each $x \in V(G) \cup F(G)$. Let v be a vertex of G. Suppose that $d(v) = 2$. If two neighbors of v are of degree at least 3, then $w^{**}(v) \geq w(v) + 2 \times 1 = 0$. Otherwise, $w^{**}(v) \geq w(v) + 2 = 0$. If $d(v) = 3$, then $w^{**}(v) \geq w(v) - \max\{3 \times 1, 2\} = 0$. If $d(v) \geq 4$, then $w^{**}(v) \geq w(v) - d(v) \times 2 = d(v) \times 3 - 12 \geq 0$. Hence, $w^{**}(f) \geq 0$. □

Theorem 7. *Let G be a planar graph with girth $g \geq 12$. Then $\chi''_a(G) \leq \max\{4, \Delta+1\}$.*

Proof. It is clearly that $\chi''_a(G) \leq 4$ if $\Delta \leq 2$. So we assume that $\Delta \geq 3$. Let G be a minimal counterexample to the theorem. Then G is 2-connected and has minimum degree at least 2. Let $k = \max\{4, \Delta+1\}$ and let L be the color set $\{1, 2, \cdots, k\}$ for simplicity. By lemma 6, we consider the following two cases.

Case 1. Suppose G contains a 2-vertex u with neighbors are 2-vertices v and w.

Let $N(v)\backslash\{u\} = \{x\}$ and $N(w)\backslash\{u\} = \{y\}$. Let $G' = G - \{u\}$. By the minimality of G, G' has an acyclic total coloring ϕ with colors from L. In the following, we will extend ϕ to an acyclic total coloring φ of G with $\Delta(G)+1$ colors, which is a contradiction. First, let $\varphi(z) = \phi(z)$ for any $z \in (V(G') \bigcup E(G'))\backslash\{v, w\}$. Then, if $|\{\phi(x), \phi(xv), \phi(wy), \phi(y)\}| \leq 3$, let $\varphi(v) = \varphi(w) \in L\backslash\{\phi(x), \phi(xv), \phi(wy), \phi(y)\}$, $\varphi(uv) \in L\backslash\{\varphi(v), \phi(xv)\}$, $\varphi(uw) \in L\backslash\{\varphi(uv), \varphi(w), \phi(wy)\}$ and $\varphi(u) \in L\backslash\{\varphi(v), \varphi(uv), \varphi(uw)\}$. Otherwise, let $\varphi(v) = \varphi(uw) = \phi(y)$, $\varphi(uv) = \phi(wy)$, $\varphi(w) = \phi(x)$, $\varphi(u) = \phi(vx)$. Since $|\{\phi(uv), \phi(u), \phi(uw), \phi(w)\}| = 4$, φ is an acyclic total coloring of G.

Case 2. G contains a path $uvwxyz$ with $d(x) = 3$ and $d(v) = d(w) = d(y) = 2$.

Let $N(x)\backslash\{w, y\} = \{x_1\}$. By the minimality of G, $G^* = G - vw$ has an acyclic total coloring ϕ with colors from L. Suppose $\Delta(G) \geq 4$. If $\{\phi(u), \phi(uv), \phi(v)\} = \{\phi(w), \phi(wx), \phi(x)\}$, let $\phi(vw) \in L\backslash\{\phi(w), \phi(wx), \phi(x)\}$. Otherwise, let $\phi(vw) \in L\backslash\{\phi(uv), \phi(v), \phi(w), \phi(wx)\}$. Thus ϕ is extended to an acyclic total coloring, which is a contradiction. So we have $\Delta(G) = 3$. In the following, we also extend ϕ to an acyclic total coloring φ of G with 4 colors, which is a contradiction. First, let $\varphi(z) = \phi(z)$ for any $z \in (V(G') \bigcup E(G'))\backslash\{v, w\}$.

Subcase 2.1. $\phi(uv) \neq \phi(x)$ (the case that $\phi(wx) \neq \phi(u)$ can be settled similarly).

If $\phi(uv) = \phi(wx)$, let $\varphi(v) \in L\backslash\{\phi(u), \phi(uv), \phi(x)\}$, $\varphi(w) \in L\backslash\{\phi(x), \phi(wx), \varphi(v)\}$ and $\varphi(vw) \in L\backslash\{\varphi(v), \varphi(w), \phi(uv)\}$. Otherwise, if $\phi(u) \neq \phi(wx)$, let $\varphi(v) = \phi(wx)$, $\varphi(vw) = \phi(x)$ and $\varphi(w) \in L\backslash\{\phi(uv), \phi(wx), \phi(x)\}$. Otherwise, let $\varphi(w) = \phi(uv)$, $\varphi(vw) = \phi(x)$ and $\varphi(v) \in L\backslash\{\phi(uv), \phi(wx), \phi(x)\}$.

Subcase 2.2. $\phi(uv) = \phi(x)$ and $\phi(wx) = \phi(u)$.

WLOG, assume that $\phi(x) = 1$, $\phi(wx) = 2$, $\phi(xx_1) = 3$ and $\phi(xy) = 4$. Suppose that $\phi(wx) = 2 \notin \{\phi(y), \phi(yz)\}$. Let $\varphi(wx) = 4$ and $\varphi(xy) = 2$, and at the same time, if $\phi(z) \neq 4$, let $\varphi(y) = 4$. Suppose that $\phi(yz) = 2$. Then $\phi(y) = 3$. If $\phi(x_1) = 2$, let $\varphi(x) = 4$ and $\varphi(xy) = 1$. Otherwise, we have $\phi(x_1) = 4$, and let $\varphi(xy) = 1$, $\varphi(x) = 2$, and $\varphi(wx) = 4$. Suppose that $\phi(y) = 2$. If $\phi(z) \neq 4$, let $\varphi(wx) = \varphi(y) = 4$ and $\varphi(xy) = 2$. Otherwise, if $\phi(yz) = 1$, let $\varphi(y) = 3$, $\varphi(wx) = 4$ and $\varphi(xy) = 2$. Otherwise, we have $\phi(yz) = 3$. Thus, if $\phi(x_1) = 2$, let $\varphi(x) = 4$ and $\varphi(xy) = 1$. Otherwise, let $\varphi(wx) = \varphi(y) = 1$ and $\varphi(x) = 2$. For all above discussions, we have $\varphi(uv) \neq \varphi(x)$ or $\varphi(wx) \neq \varphi(u)$. It deduces to Subcase 2.1. □

3 Series-parallel Graphs

A graph is a series-parallel graph (in short SP) if it contains no subgraphs homeomorphic to K_4. Duffin [12]showed that a connected SP graph can be obtained from a k_2 by repeatedly applying the following operation: inserting a vertex into an edge (series) or duplicating an edge by a path of length 2 (parallel). By the operation or by the definition of SP graph, it is easy to see that the connectivity of any SP graph is at most 2. Wu [13] obtained a structural property on SP graphs as follows.

Lemma 8. [13] *Let G be a 2-connected series-parallel graph of order at least 4. Then*

(1) G has two adjacent 2-vertices u and v, or

(2) G has a 3-cycle $uwvu$ such that $d(u) = 2$ and $d(v) = 3$, or

(3) G has a 4-cycle $uxvyu$ such that $d(u) = 2$ and $d(v) = 2$, or

(4) G has a 4-vertex w, $N(w)=\{u, v, x, y\}$, such that $d(u) = d(v) = 2$, $N(u) = \{x, w\}$ and $N(v) = \{w, y\}$.

Lemma 9. [13] *Let G be a 2-connected series-parallel graph having $\Delta(G) = 3$. Then G has a cycle such that there are just two 3-vertices on it.*

Now we shall prove the following theorem.

Theorem 10. *Let G be a series-parallel graph with maximum degree Δ. Then $\chi''_a(G) = \Delta + 1$ if $\Delta \geq 3$.*

Proof. We shall prove the theorem by induction on $|V(G)|$. We assume that G is 2-connected. Let $k = \Delta(G) + 1$ and L be the color set $\{1, 2, \cdots, k\}$ for simplicity.

Case 1. $\Delta \geq 4$.

By lemma 8, we consider the following cases.

Subcase 1.1. G has two adjacent 2-vertices u and v.

Let $N(u)\backslash\{v\} = \{x\}$ and $N(v)\backslash\{u\} = \{y\}$. Then $G' = G - u + xv$ is also a 2-connected series-parallel graph. By the induction hypothesis of G, G' has an acyclic total coloring ϕ with colors from L. Let $\phi(xu) = \phi(xv)$,

$\phi(u) \in L\backslash\{\phi(xu), \phi(vy), \phi(x), \phi(v)\}$ and $\phi(uv) \in L\backslash\{\phi(xu), \phi(vy), \phi(u), \phi(v)\}$. Since $|\{\phi(u), \phi(v), \phi(ux), \phi(vy)\}| = 4$, ϕ is extended to an acyclic total coloring of G with k colors.

Subcase 1.2. G has a 3-cycle $uwvu$ such that $d(u) = 2$ and $d(v) = 3$.

Let $\{y\} = N(v)\backslash\{w, u\}$. Then $G' = G - u$ is also a 2-connected series-parallel graph. By the induction hypothesis of G, G' has an acyclic total coloring ϕ with colors from L. First, let $\phi(uw) \in L\backslash\Phi(w)$. Then, if $\phi(uw) \in \Phi(v)$, let $\phi(uv) \in L\backslash(\Phi(v) \cup \{\phi(wu), \phi(w)\}$. Otherwise, let $\phi(uv) \in L\backslash(\Phi(v) \cup \{\phi(wu)\}$. Finally, let $\phi(u) \in L\backslash\{\phi(wu), \phi(uv), \phi(w), \phi(v)\}$. So ϕ is extended to an acyclic total coloring of G with k colors.

Subcase 1.3. G has a 4-cycle $uxvyu$ such that $d(u) = 2$ and $d(v) = 2$.

By the induction hypothesis of G, $G' = G\backslash\{u, v\}$ has an acyclic total coloring ϕ with colors from L. We choose two colors α_1, α_2 from $L\backslash\Phi(x)$ and two colors β_1, β_2 from $L\backslash\Phi(y)$. If $|\{\alpha_1, \alpha_2, \beta_1, \beta_2\}| \leq 3$, WLOG, assume that $\alpha_1 = \beta_1$, let $\phi(ux) = \phi(vy) = \alpha_1$, $\phi(uy) = \beta_2$ and $\phi(vx) = \alpha_2$. Otherwise, that is $|\{\alpha_1, \alpha_2, \beta_1, \beta_2\}| = 4$, assume that $\alpha_1 \neq \phi(y)$ and $\beta_1 \neq \phi(x)$, and let $\phi(ux) = \alpha_1$, $\phi(vx) = \alpha_2$, $\phi(uy) = \beta_2$ and $\phi(vy) = \beta_1$. Finally, let $\phi(u) \in L\backslash\{\phi(ux), \phi(uy), \phi(x), \phi(y)\}$ and $\phi(v) \in L\backslash\{\phi(vx), \phi(vy), \phi(x), \phi(y)\}$. Since $|\{\phi(u), \phi(ux), \phi(uy), \phi(x), \phi(y)\}| \geq 4$ and $|\{\phi(v), \phi(vx), \phi(vy), \phi(x), \phi(y)\}| \geq 4$, we obtain an acyclic total coloring of G with $\Delta(G) + 1$ colors.

Subcase 1.4. G has a 4-vertex w, $N(w)=\{u, v, x, y\}$, such that $d(u) = d(v) = 2$, $N(u) = \{x, w\}$ and $N(v) = \{w, y\}$.

By the induction hypothesis of G, $G' = G\backslash\{u, v\}$ has an acyclic total coloring ϕ with colors from L. Without loss of generality, let $\phi(wx) = 1$, $\phi(wy) = 2$ and $\phi(w) = 3$. First, let $\phi(ux) \in L\backslash\Phi(x)$ and $\phi(vy) \in L\backslash\Phi(y)$. Suppose that $\phi(ux) = \phi(vy)$, WLOG, assume that $\phi(ux) = 4$(the case that $\phi(ux) = 3$ can be settled similarly). Then we recolor edge wx, ux, w such that $\phi(wx) = 4$, $\phi(ux) = 1$ and $\phi(w) \in L\backslash\{4, 2, \phi(x), \phi(y)\}$. Since $\phi(ux) = \phi(vy) = 4$, $\phi(x) \neq 4$ and $\phi(y) \neq 4$. It follows that $|\{\phi(y), \phi(wx), \phi(wy), \phi(w)\}| = 4$ and ϕ is also an acyclic total coloring of G'. So we can assume that $\phi(ux) \neq \phi(vy)$.

If $\{\phi(ux), \phi(vy)\} \cap \{1, 2, 3\} = \emptyset$, let $\phi(uw) = \phi(vy)$, $\phi(wv) = \phi(ux)$, $\phi(u) = 1$ and $\phi(v) = 2$. Otherwise, WLOG, assume that $\phi(ux) \in \{2, 3\}$. First, if $\phi(vy) \neq 3$, let $\phi(wv) \in L\backslash\{1, 2, 3, \phi(vy)\}$. Otherwise, let $\phi(wv) \in L\backslash\{1, 2, 3, \phi(y)\}$. Then, let $\phi(v) = 2$. In the following, let $\phi(uw) \in L\backslash\{1, 2, 3, \phi(wv)\}$. If $\phi(uw) \neq \phi(x)$ or $\phi(ux) \neq 3$, let $\phi(u) = 1$. Otherwise, let $\phi(u) \in L\backslash\{1, 2, 3, \phi(wu)\}$. It is easy to check that we extend ϕ to an acyclic total coloring of G with $\Delta(G) + 1$ colors.

Case 2. $\Delta = 3$.

An acyclic total 4-coloring ϕ of G is called *neat* if for any two adjacent

2-vertices w_1 and w_2, $N(w_1) = \{w_1', w_2\}$ and $N(w_2) = \{w_2', w_1\}$, we have $\phi(w_1w_1') \neq \phi(w_2w_2')$ and $|\{\phi(w_1), \phi(w_1w_1'), \phi(w_2), \phi(w_2w_2')\}| = 3$. In the following, we will construct a neat acyclic total 4-coloring ϕ of a 2-connected series-parallel graph G with maximum degree 3.

Let n_i be the number of i-vertices of G, where $i = 2, 3$. Suppose that $n_3 = 2$. Let u, v be the two 3-vertices. Then G consists of three disjoint paths connecting u and v. If $|V(G)| \leq 8$, the result is trivial. Suppose that $|V(G)| > 8$. Then there must be a u-v path $uw_1w_2 \cdots w_kv$ such that $k \geq 4$, or $k = 3$ and $uv \notin E(G)$. If $k \geq 4$, then $G^* = G - \{w_1, w_2, w_3\} + uw_4$ is also a 2-connected series-parallel graph of maximum degree 3, and by the induction hypothesis, G^* has a neat acyclic total 4-coloring ϕ. Without loss of generality, assume that $\phi(uw_4) = 1$, $\phi(u) = 2$ and $\phi(w_4) = 3$. First, let $\phi(uw_1) = \phi(w_3w_4) = \phi(w_2) = 1$, $\phi(w_1) = \phi(w_3) = 4$. Then, if $k \geq 4$ and $\phi(w_4w_5) = 2$, let $\phi(w_1w_2) = 2$ and $\phi(w_2w_3) = 3$, otherwise let $\phi(w_1w_2) = 3$ and $\phi(w_2w_3) = 2$. Thus ϕ is extended to a neat acyclic total 4-coloring of G. If $k = 3$ and $uv \notin E(G)$, let $G^* = G - \{w_1, w_2, w_3\} + uv$ and its coloring is the same as above.

Suppose that $n_3 \geq 4$. By Lemma 9, let $u_1u_2 \cdots u_nu_1$ $(n \geq 3)$ be a longest cycle of G such that $d(u_1) = d(u_k) = 3$ for some $k(2 < n/2+1 \leq k \leq n)$. Suppose that $5 \leq k < n$ or $k \geq 6$. Then $G^* = G - \{u_2, u_3, u_4\} + u_1u_5$ is also a 2-connected series-parallel graph of maximum degree 3. So by the induction hypothesis, G^* has a neat total 4-coloring ϕ. Without loss of generality, assume that $\phi(u_1u_5) = 1$, $\phi(u_1) = 2$ and $\phi(u_5) = 3$. First, let $\phi(u_1u_2) = \phi(u_4u_5) = \phi(u_3) = 1$, $\phi(u_2) = \phi(u_4) = 4$. Then, if some edge incident with u_5 is colored with color 2, let $\phi(u_2u_3) = 2$ and $\phi(u_3u_4) = 3$, otherwise let $\phi(u_2u_3) = 3$ and $\phi(u_3u_4) = 2$. Thus ϕ is extended to a neat acyclic total 4-coloring of G.

Suppose that $k = n \leq 5$. Let $G^{**} = G - \{u_2, \cdots, u_{n-1}\}$. Then G^{**} is also a 2-connected series-parallel graph of maximum degree 3. By the induction hypothesis, G^{**} has a neat acyclic total 4-coloring ϕ. Since $d_{G^{**}}(u_1) = d_{G^{**}}(u_n) = 2$, we can color $u_1u_2, u_{n-1}u_n$ such that $\phi(u_1u_2) \in L\backslash\Phi(u_1)$ and $\phi(u_{n-1}u_n) \in L\backslash\Phi(u_n)$. Without loss of generality, assume that $\phi(u_1u_2) = \phi(u_n) = 1$, $\phi(u_1) = 2$ and $\phi(u_{n-1}u_n) = 3$. If $k = 3$, let $\phi(u_2) = 4$. If $k = 4$, let $\phi(u_2) = 3$, $\phi(u_2u_3) = 2$ and $\phi(u_3) = 4$. If $k = 5$, let $\phi(u_3) = 1$, $\phi(u_2u_3) = \phi(u_4) = 2$, $\phi(u_2) = 3$ and $\phi(u_3u_4) = 4$. So ϕ is extended to a neat acyclic total 4-coloring of G.

If $k < n$ and $k < 5$, then $n = k + 1 = 4$, or $n = k + 1 = 5$, or $n = k + 2 = 6$. These can be settled similarly, we omit here. Hence we complete the proof. □

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