SOME IDENTITIES OF SYMMETRY FOR CARLITZ q-BERNOULLI POLYNOMIALS INVARIANT UNDER S_4

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ABSTRACT. In this paper, we investigate some new identities of symmetry for the Carlitz q-Bernoulli polynomials invariant under S_4 which are derived from p-adic q-integrals on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$.

The p-adic norm is normalized as $|p|_p = \frac{1}{p}$. Let us assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-number of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$(1.1)\quad I_q(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)\, q^x, \quad (\text{see } [8]).$$

Thus, by (1.1), we get

$$(1.2)\quad qI_q(f_1) - I_q(f) = (q-1) f(0) + \frac{q-1}{\log q} f'(0), \quad (\text{see } [6\text{–}14]).$$

L. Carlitz defined the q-Bernoulli numbers as follows:

$$(1.3)\quad \beta_{0,q} = 1, \quad q(q\beta+1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \quad (\text{see } [3]).$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

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The q-Bernoulli polynomials were defined by L. Carlitz to be

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \beta_{l,q} \tag{1.4}$$
$$= \left(q^x \beta_q + [x]_q\right)^n, \quad \text{(see [1-17])}.$$

From (1.3), we have

$$\int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0). \tag{1.5}$$

When $x = 0$, $\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x)$, $(n \geq 0)$.

Indeed, by (1.2), we get

$$q \int_{\mathbb{Z}_p} [x+1]_q^n \, d\mu_q(x) - \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x) = \begin{cases} q-1, & \text{if } n=0, \\ 1, & \text{if } n=1, \\ 0, & \text{if } n>1. \end{cases} \tag{1.6}$$

Thus, from (1.6), we have

$$q\beta_{n,q}(1) - \beta_{n,q} = \delta_{1,n}, \quad \beta_{0,q} = 1.$$

By (1.5), we easily get

$$\beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \frac{l+1}{[l+1]_q}$$
$$= \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \beta_{l,q}.$$

In this paper, we investigate some properties of symmetry for the Carlitz q-Bernoulli polynomials invariant under S_4 arising from p-adic q-integrals on $\mathbb{Z}_p$. In addition, we give some new identities of symmetry for the Carlitz q-Bernoulli polynomials which are derived from our symmetric properties related to p-adic q-integrals on $\mathbb{Z}_p$.

2. Symmetric identities for the Carlitz q-Bernoulli polynomials invariant under S_4

In this section, we assume that $w_1, w_2, w_3, w_4 \in \mathbb{N}$.

From (1.1), we have

$$\int_{\mathbb{Z}_p} e^{[w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 i + w_4 w_1 w_3 j + w_4 w_1 w_2 k]_q t} d\mu_{q^{w_1 w_2 w_3}}(y) \tag{2.1}$$
$$= \lim_{N \to \infty} \frac{1}{[p^N]_{q^{w_1 w_2 w_3}}}$$

$$\times \sum_{y=0}^{p^N-1} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} q^{w_1w_2w_3y}$$

$$= \lim_{N\to\infty} \frac{1}{[w_4p^N]_{q^{w_1w_2w_3}}}$$

$$\times \sum_{y=0}^{p^N-1} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} q^{w_1w_2w_3y}$$

$$= \lim_{N\to\infty} \frac{1}{[w_4p^N]_{q^{w_1w_2w_3}}} \sum_{l=0}^{w_4-1}\sum_{y=0}^{p^N-1} q^{w_1w_2w_3(l+w_4y)}$$

$$\times e^{[w_1w_2w_3(l+w_4y)+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t}.$$

Thus, by (2.1), we get

(2.2)

$$\frac{1}{[w_1w_2w_3]_q}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1} q^{w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k}$$

$$\times \int_{\mathbb{Z}_p} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t} d\mu_{q^{w_1w_2w_3}}(y)$$

$$= \lim_{N\to\infty} \frac{1}{[w_1w_2w_3w_4p^N]_q}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}\sum_{l=0}^{w_4-1}\sum_{y=0}^{P^N-1} q^{[A]}$$

$$\times q^{w_1w_2w_3w_4y} e^{[w_1w_2w_3(l+w_4y)+w_1w_2w_3w_4x+w_4w_2w_3i+w_4w_1w_3j+w_4w_1w_2k]_q t},$$

where $A = w_1w_2w_3l + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k$

As this expression is invariant under any permutation $\sigma \in S_4$, we have the following theorem.

Theorem 2.1. *For $w_1, w_2, w_3, w_4 \in \mathbb{N}$, the following expressions*

$$\frac{1}{[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}]_q}\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}\sum_{k=0}^{w_{\sigma(3)}-1} q^B \int_{\mathbb{Z}_p} e^{[C]_q t} d\mu_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}(y),$$

where $B = w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k$ and $C = w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}y + w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}x + w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i + w_{\sigma(4)}w_{\sigma(1)} \times w_{\sigma(3)}j + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k$ are the same for any $\sigma \in S_4$.

Now, we observe that

$$\text{(2.3)} \quad [w_1w_2w_3y + w_1w_2w_3w_4x + w_4w_2w_3i + w_4w_1w_3j + w_4w_1w_2k]_q$$

$$= [w_1w_2w_3]_q \left[y + w_4x + \frac{w_4}{w_1}i + \frac{w_4}{w_2}j + \frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}.$$

Thus, by (2.3), we get

$$(2.4)\quad \int_{\mathbb{Z}_p} e^{[w_1w_2w_3y+w_1w_2w_3w_4x+w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k]_q t} d\mu_{q^{w_1w_2w_3}}(y)$$

$$=\sum_{n=0}^{\infty}[w_1w_2w_3]_q^n$$

$$\times \int_{\mathbb{Z}_p}\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}^n d\mu_{q^{w_1w_2w_3}}(y)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}[w_1w_2w_3]_q^n\,\beta_{n,q^{w_1w_2w_3}}\left(w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right)\frac{t^n}{n!}$$

From (2.4), we note that

$$(2.5)\quad \int_{\mathbb{Z}_p}[w_1w_2w_3y+w_1w_2w_3w_4x+w_2w_3w_4i$$

$$+w_1w_3w_4j+w_1w_2w_4k]_q^n\, d\mu_{q^{w_1w_2w_3}}(y)$$

$$=[w_1w_2w_3]_q^n\,\beta_{n,q^{w_1w_2w_3}}\left(w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right),\quad (n\geq 0).$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following theorem.

Theorem 2.2. *For $n\geq 0$, $w_1,w_2,w_3,w_4\in\mathbb{N}$, the following expressions*

$$\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q^{n-1}\sum_{i=0}^{w_{\sigma(1)}-1}\sum_{j=0}^{w_{\sigma(2)}-1}$$

$$\times\sum_{k=0}^{w_{\sigma(3)}-1}q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}i+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}j+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}k}$$

$$\times\beta_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x+\frac{w_{\sigma(4)}}{w_{\sigma(1)}}i+\frac{w_{\sigma(4)}}{w_{\sigma(2)}}j+\frac{w_{\sigma(4)}}{w_{\sigma(3)}}k\right)$$

are the same for any $\sigma\in S_4$.

From the definition of q-number, we note that

$$(2.6)\quad \left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}$$

$$=\frac{1-q^{w_1w_2w_3y+w_1w_2w_3w_4x+w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k}}{1-q^{w_1w_2w_3}}$$

$$(2.7)\quad =\frac{[w_4]_q}{[w_1w_2w_3]_q}[w_2w_3i+w_1w_3j+w_1w_2k]_{q^{w_4}}$$

$$+q^{w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k}[y+w_4x]_{q^{w_1w_2w_3}}$$

Thus, by (2.6), we get

$$(2.8)\quad \int_{\mathbb{Z}_p}\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}^n d\mu_{q^{w_1w_2w_3}}(y)$$

$$=\sum_{l=0}^{n}\binom{n}{l}\left(\frac{[w]_q}{[w_1w_2w_3]_q}\right)^{n-l}[w_2w_3i+w_1w_3j+w_1w_2k]_{q^{w_4}}^{n-l}$$

$$\times q^{l(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)}\int_{\mathbb{Z}_p}[y+w_4x]_{q^{w_1w_2w_3}}^l d\mu_{q^{w_1w_2w_3}}(y)$$

$$=\sum_{l=0}^{n}\binom{n}{l}\left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-l}[w_2w_3i+w_1w_3j+w_1w_2k]_{q^{w_4}}^{n-l}$$

$$\times q^{l(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)}\beta_{l,q^{w_1w_2w_3}}(w_4x).$$

From (2.8), we can derive the following equation:

$$[w_1w_2w_3]_q^{n-1}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}q^{w_2w_3w_4i+w_4w_1w_3j+w_4w_1w_2k}$$

$$\times\int_{\mathbb{Z}_p}\left[y+w_4x+\frac{w_4}{w_1}i+\frac{w_4}{w_2}j+\frac{w_4}{w_3}k\right]_{q^{w_1w_2w_3}}^n d\mu_{q^{w_1w_2w_3}}(y)$$

$$=\sum_{l=0}^{n}\binom{n}{l}[w_1w_2w_3]_q^{l-1}[w_4]_q^{n-l}\beta_{l,q^{w_1w_2w_3}}(w_4x)\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}$$

$$\times\sum_{k=0}^{w_3-1}q^{(l+1)(w_2w_3w_4i+w_1w_3w_4j+w_1w_2w_4k)}$$

$$\times[w_2w_3i+w_1w_3j+w_1w_2k]_{q^{w_4}}^{n-l}$$

$$=\sum_{l=0}^{n}\binom{n}{l}[w_1w_2w_3]_q^{l-1}[w_4]_q^{n-l}\beta_{l,q^{w_1w_2w_3}}(w_4x)T_{n,q^{w_4}}(w_1,w_2,w_3\mid l),$$

where

$$T_{n,q}(w_1,w_2,w_3\mid l)$$

$$=\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{k=0}^{w_3-1}q^{(l+1)(w_2w_3i+w_1w_3j+w_1w_2k)}[w_2w_3i+w_1w_3j+w_1w_2k]_q^{n-l}.$$

As this expression is an invariant under S_4, we have the following theorem.

Theorem 2.3. *Let $w_1, w_2, w_3, w_4 \in \mathbb{N}$. For $n \geq 0$, the following expressions*

$$\sum_{l=0}^{n} \binom{n}{l} \left[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}\right]_q^{l-1} \left[w_{\sigma(4)}\right]_q^{n-l} \times \beta_{l,q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}} \left(w_{\sigma(4)} x\right) T_{n,q^{w_{\sigma(4)}}} \left(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)} \mid l\right)$$

are all the same for $\sigma \in S_4$.

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