Cubic, edge critical, Hamilton laceable bigraphs

Gustavus J. Simmons
gsimmons30@comcast.net

Abstract:
Edge minimal Hamilton laceable[1] bigraphs on 2m vertices have at least $2\lfloor(m+3)/6\rfloor$ vertices of degree 2. If a bigraph is edge minimal with respect to Hamilton laceability, it is by definition edge critical, meaning the deletion of any edge will cause it to no longer be Hamilton laceable. The converse need not be true. The m-crossed prisms [8] on 4m vertices are edge critical for m > 2 but not edge minimal since they are cubic. A simple modification of m-crossed prisms forms a family of "sausage" bigraphs on 4m + 2 vertices that are also cubic and edge critical. Both these families share the unusual property that they have exponentially many Hamilton paths between every pair of vertices in different parts. Even so, since the bigraphs are edge critical, deleting an arbitrary edge results in at least one pair having none.

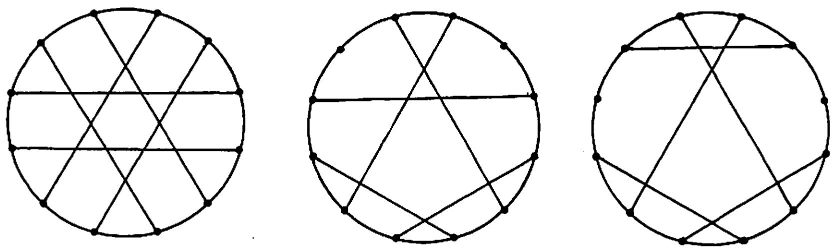
1 Introduction:

The Hamilton laceablity of some families of cubic bigraphs has been investigated [1] but without the stringent requirement of edge criticality. Since being Hamilton laceable is analogous for bipartite graphs to being Hamilton connected for general graphs, it is natural to ask whether families of cubic graphs, such as the Cayley graphs on dihedral groups, that are Hamilton connected when they are not bipartite are Hamilton laceable when they are. They are [2]. In the other direction, edge minimality – which implies edge criticality – has been investigated for the related, stronger, property of hyper-Hamilton laceability[2] [3]. But since the graph property is stronger, the minimality results are weaker. None of the results obtained in either of these approaches appear applicable to the problem of cubic bigraphs being edge critical, so no attempt has been made to reference all papers in either area. The ones cited lead to other related work.

1. A Hamilton path is only possible in a bipartite graph if the parts have the same cardinality (equitable) or differ by one (nearly equitable). A bipartite graph G is Hamilton-laceable if G is equitable and there exists a Hamilton path between every vertex in the one part and every vertex in the other or if G is nearly equitable and there exists a Hamilton path between every pair of vertices in the larger part.
2. A Hamilton laceable bigraph G is hyper-Hamilton laceable if G is equitable and for any vertex v, G-v is Hamilton laceable or G is nearly equitable and for any vertex v in the larger part, G-v is Hamilton laceable.

ARS COMBINATORIA 123(2015), pp. 291-302

A fundamental question in the study of Hamilton laceable bigraphs is: What is the least number of edges a bigraph must have in order to be Hamilton laceable? For nearly equitable bigraphs on 2m + 1 vertices the answer, $E_{2m+1} = 3m$, has been known since the notion of Hamilton laceablity was first introduced [3, 4]. For equitable bigraphs the best results known [5] are the bounds $3m - \lfloor m/3 \rfloor \leq E_{2m} \leq 3m - \lfloor (m+3)/6 \rfloor$. An exhaustive computer testing of small equitable bigraphs for Hamilton laceability revealed the Franklin graph on 12 vertices, Figure 1a, to be Hamilton laceable and edge critical. But it is not edge minimal, since it has 18 edges while there exist a pair of non-isomorphic Hamilton laceable bigraphs on 12 vertices with only 17 edges, Figures 1b and 1c, having 106 and 125 Hamilton paths respectively.



a - Franklin graph　　　b　　　c

Figure 1

An obvious generalization of the representation of the Franklin graph in 1a leads to an infinite family, the polygonal bigraphs P_{4m}, of cubic bigraphs on 4m vertices which are edge critical but not edge minimal for m > 2 [7]. This raises the natural question of whether there exists a comparable family of bigraphs on

4m + 2 vertices that are also cubic and edge critical.

The constructive characterization of P_{4m} is: Extend the sides of a regular polygon on 2m vertices to define the 4m(m-1)/2 finite points of intersection. Circumscribe a centrally symmetric circle large enough that all of the points of intersection are in its interior. The 4m points of intersection of the extended sides with the circle are the vertices of the polygonal bigraph, P_{4m}. The edges are the 4m arcs of the circle between the vertices and the 2m diagonals defined by the extended sides of the defining polygon. P_8 is the edge skeleton of the 3-cube, P_{12} the Franklin graph etc. The P_{4m} are isomorphic to the m-crossed prism graphs [8], shown schematically in Figure 2. Given that the P_{4m} are edge critical, but not edge minimal, an obvious place to look for bigraphs on 4m + 2 vertices with similar properties would be the polygonal bigraphs, P_{4m+2}, generated by regular polygons on 2m + 1 vertices using the same construction used to generate the P_{4m}. However, it has been shown that no P_{4m+2} is edge critical [6].

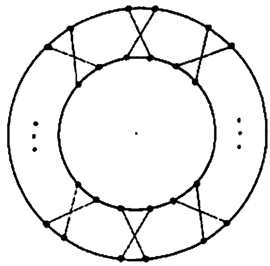


Figure 2

The solution found was to modify the m-crossed prism graph. Since each twisted quadrilateral in Figure 2 contributes four vertices and six edges, the problem was to find a way to add two vertices and three edges while preserving Hamilton laceability and edge criticality. To do this, replace the two edges linking a pair of adjacent twisted quadrilaterals with the H figure on vertices A and B as shown in Figure 3. The resulting "sausage" bigraphs, S_{4m+2}, are cubic, Hamilton laceable and edge critical. The balance of this paper is devoted to analyzing these bigraphs.

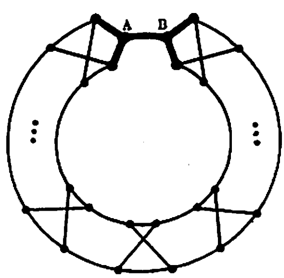


Figure 3

2 Essential properties of S_{4m+2}:

The positive result, that S_{4m+2} is Hamilton laceable, is easy to prove. The negative result, that S_{4m+2} is not Hamilton laceable if any edge is deleted, is more difficult.

It would be possible to directly prove S_{4m+2} is Hamilton laceable by exhibiting a construction for a Hamilton path between all sets of endpoints, but this would require treating a large number of sub-cases. A and B are obviously unique endpoints, as are the four vertices adjacent to one of them. Less obvious is the need to group other endpoints according to their proximity to one of A or B, so the number of sub-cases that would have to be treated becomes quite large. Instead we use a simple technique to extend Hamilton paths in S_{4m+2} to Hamilton paths in $S_{4(m+1)+2}$ which avoids the need to consider any sub-cases.

Theorem 1:

S_{4m+2} is Hamilton laceable.

Proof:

A Hamilton path must span the four vertices in each twisted quadrilateral. Figure 4 shows the only six ways this is possible if the quadrilateral doesn't host an endpoint, Figure 5 the five ways if it hosts only one and Figure 6 the eight ways if it hosts both.

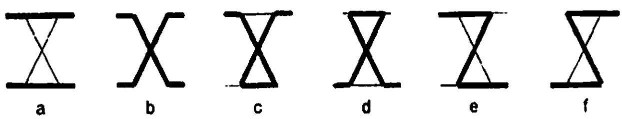


Figure 4

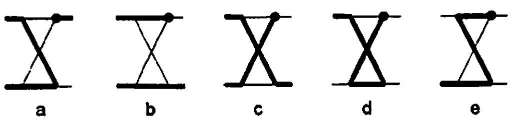


Figure 5

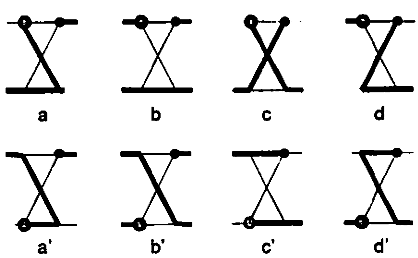


Figure 6

There are obvious symmetries and reflections which could be used to reduce the number of figures but there is no reason to do so since only two properties of the spanning sets of paths will be used. First, all of the sets, with the exception of 5d and 5e, are connected to the twisted quadrilaterals on either side with at least one edge. Second, there is exactly one set of spanning sub-paths that connect with the same entrant and exit edge(s) on both sides; 4a, 4c and 4d.

Associate a binary m-tuple with each Hamilton path in S_{4m+2} where a 1 indicates the associated quadrilateral hosts at least one endpoint, and a 0 that it hosts none. To extend a Hamilton path in S_{4m+2} splice a twisted quadrilateral into any one of the m-1 pairs of edges connecting adjacent quadrilaterals. Since the endpoints are in existing quadrilaterals, if at least one of the connecting edges is in the Hamilton path in S_{4m+2}, one of the sub-paths in either 4a, 4c or 4d will span the new twisted quadrilateral to produce a Hamilton path in $S_{4(m+1)+2}$. This is equivalent to inserting a 0 between two symbols in the binary m-tuple corresponding to the Hamilton path in S_{4m+2}. The only way this extension technique could fail is if the adjacent symbols in the binary m-tuple are both 1 and the two sub-paths are either 5d or 5e back to back so that no edges join the host quadrilaterals which would result in the four vertices in the new twisted quadrilateral being isolated. Again, it would be possible to treat these exceptional cases directly to show a Hamilton path would still exist in $S_{4(m+1)+2}$, but that would be contrary to the stated objective of avoiding having to consider any sub-cases.

For $m \geq 6$ a binary m-tuple with at most two 1's must have at least one pair of adjacent 0's and hence could have been formed by inserting a 0 into an (m-1)-tuple. The 5-tuple 01010 shows this need not be the case for $m < 6$. An exhaustive backtracking calculation of the Hamilton paths in S_{10}, S_{14}, S_{18} and S_{22} shows them to have 168, 592, 1384 and 5184 Hamilton paths respectively and to all be Hamilton laceable. Therefore, S_{4m+2} is Hamilton laceable for all m.

Theorem 2

S_{4m+2} is edge critical.

Proof

The method of proof will be the same in all cases; given an edge, identify an associated pair of endpoints, x and y, and show there cannot be a Hamilton path between them if the edge is deleted. In general there will be many such vertex pairs associated with each edge. The closest pair will be chosen to make the proof arguments be as local as possible.

There are four classes of edges to be considered.

1. edge A-B
2. an edge incident on only one of A or B
3. an edge on a quadrilateral
4. an edge not on a quadrilateral

Case 1

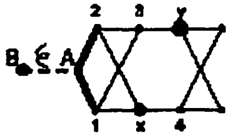


Figure 7

In order for A to be in the Hamilton path edges 1-A-2 must be in the path. If endpoint x is connected to 1 or 2 the path is forced to close prematurely (unless S_{4m+2} is $K_{3,3}$); x-1-A-2-3-y or x-2-A-1-3-y. If the path begins with x-4, y-3 is forced. Neither of the paths y-3-1-A-2 nor y-3-2-A-1 can be continued.

Therefore there cannot be a Hamilton path between x and y if edge A-B is deleted and m > 1.

Case 2

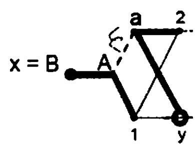
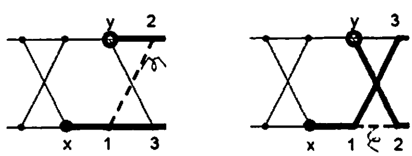


Figure 8

Choose edge a-A as representative of this class. Edges x-A-1 and y-a-2 are forced to include A and a respectively in the Hamilton path. The only possible continuation from vertex 1 is 1-2 which closes the path prematurely (unless S_{4m+2} is $K_{3,3}$).

Therefore, for all m > 1 the bigraph resulting from deleting any edge incident on either A or B is not Hamilton laceable.

Case 3



a b

Figure 9

Figure 9 makes it appear there are two cases, but they are the result of a simple half twist of S_{4m+2}: keep vertices y and 1 fixed, and interchange vertices 2 and 3, and the balance of S_{4m+2}. This operation will figure later in counting Hamilton paths, but all that matters here is that there is essentially only one construction involved. Since m > 1 there is another Q on one side or the other of the Q in which the edge is located. Choose endpoints x and y as shown in Figure 8a. In order for vertices 1 and 2 to be in a Hamilton path, the ends must start with x-1-3 and y-2, but these partial paths are directed into the same end of S_{4m+2} with the only connection back to the isolated vertices on the other side being edge A-B. Therefore there is no way for the two partial paths on x and y to join and to include the isolated vertices.

Case 4

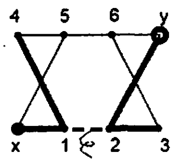


Figure 10

The only way 1 can be in a Hamilton path is for the path to start x-1-4. Similarly y-2-3 is forced. But vertices 5 and 6 are then either isolated or in a non-spanning path on eight vertices. Either way, the subgraph resulting from deleting an edge not on a quadrilateral is not Hamilton laceable.

3 Enumeration of Hamilton paths in S_{4m+2}:

Enumerating Hamilton paths normally involves an exponentially difficult backtracking construction of maximal length paths on each endpoint vertex. The unique structure of S_{4m+2} however permits the direct computation of Hamilton paths with no need for backtracking. The key observation is that each set of spanning paths in Figure 4 has a unique companion related by a half twist of the host quadrilateral; a transformation which leaves the twisted quadrilateral unchanged but permutes the spanning path(s). 4a and 4b are such a pair irrespective of the direction the Hamilton path is being traversed while 4c or 4d is paired with one of 4e or 4f, depending on the direction of the traverse.

Lemma:

A run of k twisted quadrilaterals, none of which host an endpoint, has 2^{k-1}:

i. spanning paths between each pair of endpoints on the ends of the run

or

ii. spanning pairs of paths that either enter and exit on the same cycles or else interchange cycles.

Proof:

By induction since the statements hold for a single twisted quadrilateral.

Given the results in the lemma, enumerating Hamilton paths in S_{4m+2} involves little more than identifying the runs of twisted quadrilaterals associated with endpoint pairs in Figure 11.

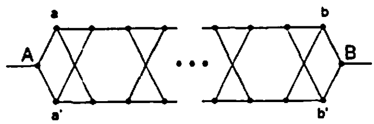


Figure 11

Theorem 3:

S_{4m+2} has a total of $2^{m+1}(2m^2 + 6m + 1)$ Hamilton paths, partitioned so uniformly over the $(2m + 1)^2$ pairs of vertices from different parts that every pair has exponentially many Hamilton paths between them.

Proof:

The number of Hamilton paths between the various classes of endpoint pairs will be computed first and the weighted sum formed to calculate the total.

There are four obviously distinct classes of adjacent vertices; A and B, A and a or a' or B and b or b', endpoints of an edge in a quadrilateral and endpoints of an edge not in a quadrilateral. There is no reason to expect that all adjacent pairs of vertices should have the same number of Hamilton paths between them, but surprisingly they do. There are exactly 2^{m+1} Hamilton paths between every pair of adjacent vertices.

When both endpoints are in quadrilaterals, due to parity the host quadrilaterals are either both between the endpoints (distal), or neither is (proximate). As might be expected these classifications have a significant impact on the number of Hamilton paths between the endpoints. In the case of proximate endpoints it makes a difference whether there is a quadrilateral between the host quadrilaterals or not. If there is none, the null-proximate case, all of the non-host quadrilaterals appear in runs and hence contribute a factor of 2 to the 2^m Hamilton paths. In the proximate case, one of the sets of paths through the quadrilaterals between the host quadrilaterals is forced and hence cannot contribute a factor of 2, so the number of Hamilton paths is only 2^{m-1} in this case.

Therefore there are five classes of endpoint pairs to be considered; adjacent, A or B and a non-adjacent vertex, null-proximate pairs, proximate pairs and distal pairs.

Case 1. adjacent endpoints

i. The endpoints are A and B

All m of the twisted quadrilaterals are in the run. By the lemma there are 2^{m-1} spanning paths between any pair of endpoints on the ends of the run. There are four ways A and B can connect to endpoints on the run, so there are a total of 2^{m+1} Hamilton paths between A and B.

ii. One endpoint is from the set (A, B), the other from the set (a, a', b, b')

Choose A and a as representative of this set of endpoints. There are two possibilities: either edge A-B is in the Hamilton path or it isn't. If it is, either edge b-B or else b'-B must be in the path. But by the lemma there are 2^{m-1} spanning paths between a and b and between a and b'. Therefore there are 2^m Hamilton paths between A and a that use edge A-B. If edge A-B is not in the Hamilton path, edge a'-A must be. The lemma says there are 2^m spanning paths though the run with vertex B appended, i.e. there are 2^{m+1} Hamilton paths between the endpoints for each of these four sets of endpoint pairs.

iii. The endpoints are on the same edge in a quadrilateral

There are four ways the host quadrilateral can be connected to the run of k quadrilaterals on one side and the run of m - k - 1 on the other, Figure 12.

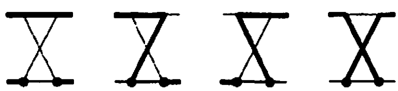


Figure 12

12a doesn't use edge A-B, the other three do. Each of these define 2^{m-1} Hamilton paths since the lemma constructions are multiplicative. Therefore there are 2^{m+1} Hamilton paths between the endpoints of any edge in a quadrilateral.

iv. The endpoints are on an edge not in a quadrilateral

In this case there are two ways the host quadrilaterals can be connected to the run of k quadrilaterals on one side and the run of m - k on the other, Figure 13.

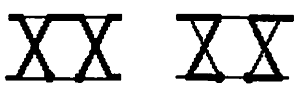


Figure 13

13a doesn't use edge A-B and 13 b does. In 13a the lemma says there will be 2^k spanning paths between A and the endpoint on the side A is on and 2^{m-k} between B and the other endpoint. Since the path multiplicities are multiplicative this says there are 2^m Hamilton paths that use edge A-B, By the same argument used earlier for the case in which edge A-B wasn't used, there are also 2^m Hamilton paths that do not use edge A-B for a total of 2^{m+1} in this case as well.

Therefore there are 2^{m+1} Hamilton paths between every pair of adjacent vertices in S_{4m+2}.

Case 2 One endpoint is from the set (A, B), say A, and the other a non-adjacent vertex, x.

By parity, x is on the side of the host quadrilateral closest to A. This means the path originating on x cannot span the vertices in the host quadrilateral since if it did it would exit on the side away from A which would then only be reachable by edge A-B. But x is not adjacent to A so any vertices between A and the quadrilateral containing x would be isolated. The only alternative is the subgraph shown in Figure 14 in which all m of the quadrilaterals have two choices for a spanning set of paths. Therefore there are 2^m Hamilton paths between endpoint pairs in this case.

Case 3 The endpoints are a null-proximate pair.

By definition there are no quadrilaterals between the pair of host quadrilaterals which divide the m quadrilaterals into a run of k on one side and m - k on the other. The lemma says that the one run will contribute 2^k spanning paths and the other 2^{m-k}, and as already remarked these are multiplicative so there are 2^m Hamilton paths between the endpoints in this case.

Case 4 The endpoints are a proximate pair

In this case there are k, $k \geq 1$, quadrilaterals between the pair of host quadrilaterals. The endpoints define a pair of vertices on the ends of the run, and the lemma says there will be only 2^{k-1} spanning paths between them. By the same arguments used in the previous case there will be 2^{m-k} spanning paths through the other quadrilaterals, or 2^{m-1} Hamilton paths between the endpoints.

Case 5 The endpoints are a distal pair

Unlike the previous cases in which there was a single starter for Hamilton paths, in this case there are three, Figure 14.

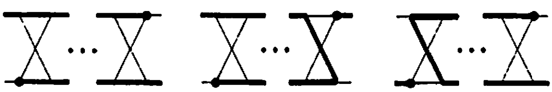


Figure 14

The analysis is complicated by the fact that the remaining m - 2 twisted quadrilaterals can be partitioned into three runs; k_1 to the left of the run containing the endpoints, k_2 to the right and $m - k_1 - k_2 - 2$ between the host quadrilaterals. For each choice of distal endpoints in the host quadrilaterals there are two sets of spanning paths, corresponding to the half twist of the region between the quadrilaterals that leaves the endpoints fixed. Therefore the total number of Hamilton paths for each starter is the contribution of 2^{m-2} paths from the three runs times 2 for the contribution from the host pair. Since there are three inequivalent starters, this says there are $3 \times 2^{m-1}$ Hamilton paths between each pair of distal endpoints.

Using the definitions of the five classes of endpoint pairs and referring to Figure 11 it is easy to see there are $3(2m+1)$ adjacent pairs, $2m(m-1)$ distal pairs, $2(m-1)$ null-proximate pairs, $2(m-1)(m-2)$ proximate pairs and $4(m-1)$ A or B to non-adjacent vertex pairs. Forming the weighted sum yields the result that S_{4m+2} has $2^{m+1}(2m^2+6m+1)$ Hamilton paths in total.

Table 1 collects the enumeration results proven here for Hamilton paths in S_{4m+2} (sausage bigraphs) and in [7] for m-crossed prisms (nee polygonal bigraphs P_{4m}).

Pair placement	m-crossed prism $\|V\| = 4m$	Sausage bigraph $\|V\| = 4m+2$
adjacent	2^m	2^{m-1}
null-proximate	$3 \times 2^{m-1}$	2^m
proximate	$3 \times 2^{m-2}$	2^{m-1}
distal	$3 \times 2^{m-2}$	$3 \times 2^{m-1}$
A or B to non-adjacent	-------	2^m
Total	$2^m(3m^2+3m)$	$2^{m+1}(2m^2+6m+1)$

Table 1

The significant thing to note in the table is that all entries double with each increase in m.

4 Conclusion:

There exist cubic, edge critical, Hamilton laceable bigraphs on every even cardinality set of ten or more vertices. The constructions used to prove this all have exponentially many Hamilton paths distributed so uniformly that every pair of vertices in different parts have exponentially many Hamilton paths connecting them. Even so, since the bigraphs are edge critical, deleting an arbitrary edge results in at least one pair having none.

5 Open question:

The quantity 3m, where $m = \lfloor |V|/4 \rfloor$, appears over and over in deriving bounds for the minimal number of edges a bigraph must have to be Hamilton laceable, or here in constructing cubic, edge critical, bigraphs. It could be this is simply an artifact of the constructions used to realize the bounds or to achieve criticality, but it could also be that 3m is a natural bound related in some deeper way to the bigraph property of Hamilton laceability. A deciding question is: Do there exist edge critical Hamilton laceable equitable bigraphs with more than 3m edges?

While a cubic graph on 2m vertices has 3m edges, the open question is not whether a critical Hamilton laceable equitable bigraph must be cubic since it is easy to construct such bigraphs with vertices of degree four, offset by an equal number vertices of degree two; Figure 15. The open question is whether there exists such a bigraph on 2m vertices with more than 3m edges.

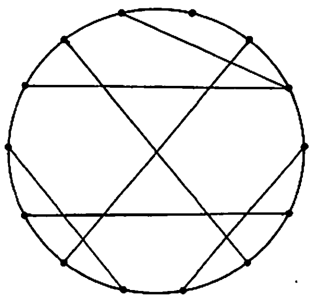


Figure 15

References:

1. B. Alspach, C.C. Chen, K. McAvaney On a class of Hamiltonian laceable 3-regular graphs, Journal of Discrete Mathematics, 151 (1996) 19-38
2. B. Alspach, C. Q. Zhang, Hamilton cycles in cubic Cayley graphs on dihedral groups, Ars Combinatoria 28 (1989) 101-108.
3. M. Lewinter, W. Widulski, Minimal hyper-Hamilton laceable graphs, Mathematical and Computer Modelling, 17 (1993) 125-127
4. G. J. Simmons, Minimal Hamilton-laceable graphs, Congressus Numerantium, 29 (1980), 893-900
5. G. J. Simmons Bounding the edge deficit in equitable Hamilton-laceable bigraphs, submitted Feb 2014, Journal of Graph Theory
6. G. J. Simmons, Enumeration of Hamilton paths in ladders, Congressus Numerantium 220 (2014), 109-124
7. G. J. Simmons, A surprising regularity in the number of Hamilton paths in polygonal bigraphs, Ars Combinatoria 115 (2014), 335-341
8. E. W. Weisstein, Crossed prism graph, Math World – A Wolfram Web Resource http://mathworld.wolfram.com/CrossedPrismGraph.html