Remarks on the complexity of signed k-independence on graphs

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Abstract

This paper is motivated by the concept of *the signed k-independence problem* and dedicated to the complexity of the problem on graphs. We show that the problem is linear-time solvable for any strongly chordal graph with a strong elimination ordering and polynomial-time solvable for distance-hereditary graphs. For any fixed positive integer $k \geq 1$, we show that the signed k-independence problem on chordal graphs and bipartite planar graphs is NP-complete. Furthermore, we show that even when restricted to chordal graphs or bipartite planar graphs, the signed k-independence problem, parameterized by a positive integer k and weight κ, is not *fixed parameter tractable.*

Keywords: Signed k-independence, Fixed parameter tractable, Strongly chordal graph

1. Introduction

Let $G = (V, E)$ be a finite, undirected, simple graph. For any vertex $v \in V$, the open neighborhood of v in G is $N_G(v) = \{u \in V | (u, v) \in E\}$ and the closed neighborhood of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $deg_G(v) = |N_G(v)|$. We also use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. Let Y be a subset of real numbers. Let $f : V \to Y$ be a function of a graph $G = (V, E)$ which assigns to each $v \in V$ a value in Y. The set Y is called the *weight set*

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of f. Let $f(S) = \sum_{u \in S} f(u)$ for any subset S of V. The *weight* of f is $f(V)$. Let k be a positive integer. A function $f : V \to \{-1, 1\}$ is a *signed k-independence function* of G if $f(N_G[v]) \leq k-1$ for every $v \in V$. The *signed k-independence number* of G, denoted by $\alpha_s^k(G)$, is the *maximum weight* of a signed k-independence function of G. The signed k-independence problem is to find a signed k-independence function of G of maximum weight. The signed k-independence number was introduced by Volkmann [11]. The special case $k = 2$ was introduced by Zelinka in [14] as a certain dual to *the signed domination number* of a graph. The upper and lower bounds on the signed 2-independence number or signed k-independence number can be found in [6, 10, 11, 13, 14]. From the algorithmic point of view, the signed 1-independence problem is NP-complete for general graphs, while it is linear-time solvable for trees [7].

This paper is dedicated to the complexity of the signed k-independence problem on graphs. In Section 2, we show that the signed k-independence problem on chordal graphs and bipartite planar graphs is NP-complete for any fixed positive integer $k \geq 1$. In Section 3, we study a general framework called *$\mathcal{R}$-independence* to solve the signed k-independence problem in linear time for any strongly chordal graph with a strong elimination ordering.

Definition 1. Let ℓ, d, I_1 be fixed integers and $\ell, d > 0$. Let Y be the weight set $\{I_1, I_1+d, I_1+2d, \ldots, I_1+(\ell-1)\cdot d\}$. Suppose that $G = (V, E)$ is a graph and $\mathcal{R}$ is a function which assigns to each $v \in V$ an integer $\mathcal{R}(v)$. An *$\mathcal{R}$-independence function* of G is a function $f : V \to Y$ if $f(N_G[v]) \leq \mathcal{R}(v)$ for every vertex $v \in V$. The *$\mathcal{R}$-independence number* of G, denoted by $\alpha_{\mathcal{R}}(G)$, is the maximum weight of an $\mathcal{R}$-independence function of G. The $\mathcal{R}$-independence problem is to find an $\mathcal{R}$-independence function of G of maximum weight.

It is clear that the signed k-independence problem is a special case of the $\mathcal{R}$-independence problem. We develop a linear-time algorithm for the $\mathcal{R}$-independence problem on strongly chordal graphs. In Section 4, we develop a polynomial-time algorithm for the signed k-independence problem on distance-hereditary graphs. In Section 5, we show that even when restricted to chordal graphs or bipartite planar graphs, the signed k-independence problem, parameterized by a positive integer k and weight κ, is not *fixed parameter tractable*.

2. NP-completeness results

In this section, we present NP-completeness results for bipartite planar graphs and chordal graphs. Before presenting the NP-completeness results, we restate the domination problem and the signed k-independence problem as decision problems.

(1) **The domination problem:**
Instance: A graph $G = (V, E)$ and a positive integer κ.
Question: Is $\gamma(G) \leq \kappa$?

(2) **The signed k-independence problem:**
Instance: A graph $G = (V, E)$, a positive integer k, an integer κ.
Question: Is $\alpha_s^k(G) \geq \kappa$?

Theorem 1. *The signed* 1*-independence problem is NP-complete for bipartite planar graphs and chordal graphs.*

Proof. The signed 1-independence problem on bipartite planar graphs (respectively, chordal graphs) is clearly in NP. It is known that the domination problem is NP-complete for bipartite planar graphs [12] (respectively, chordal graphs [1]). In the following, we show the NP-completeness of the signed 1-independence problem on bipartite planar graphs (respectively, chordal graphs) by a polynomial-time reduction from the domination problem on bipartite planar graphs (respectively, chordal graphs).

Given a bipartite planar graph (respectively, chordal graph) $G = (V, E)$, we construct a graph H by adding to each vertex v of G a set of $deg_G(v)$ paths of length three, say, $v - v_{i_1} - v_{i_2} - v_{i_3}$ for $1 \leq i \leq deg_G(v)$. That is, $V(H) = V \cup (\bigcup_{v \in V}\{v_{i_1}, v_{i_2}, v_{i_3} | 1 \leq i \leq deg_G(v)\})$ and $E(H) = E \cup (\bigcup_{v \in V}\{(v, v_{i_1}), (v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}) | 1 \leq i \leq deg_G(v)\})$. Clearly, H is a bipartite planar graph (respectively, chordal graph) and the construction of H can be done in polynomial-time. Let $|V| = n$ and $|E| = m$.

Let D be a dominating set of G of $\gamma(G)$ vertices. Let f be a function of H such that for each vertex $v \in V$, $f(v) = -1$ if $v \in D$, $f(v) = 1$ if $v \in V \setminus D$, and $f(v_{i_1}) = f(v_{i_2}) = -1$ and $f(v_{i_3}) = 1$ for $1 \leq i \leq deg_G(v)$. It can be easily verified that f is a signed 1-independence function. Then, $\alpha_s^1(H) \geq (n - \gamma(G)) - \gamma(G) - (\sum_{v \in V} deg_G(v)) = n - 2m - 2\gamma(G)$.

Conversely, it can be verified by contradiction that there exists a maximum signed 1-independence function f of H such that for each vertex $v \in V$, $f(v_{i_1}) = f(v_{i_2}) = -1$ and $f(v_{i_3}) = 1$ for $1 \leq i \leq deg_G(v)$. Notice that $|N_G[v]| = deg_G(v) + 1$ and $|N_H[v]| = 2 \cdot deg_G(v) + 1$ for every vertex $v \in V$. The function f assigns the value -1 to at least one vertex of $N_G[v]$ for every $v \in V$. The set $\{v \in V | f(v) = -1\}$ is a dominating set of G. Let $k_1 = |\{v \in V | f(v) = -1\}|$. The weight of f is $\alpha_s^1(H) = (-\sum_{v \in V} deg_G(v)) + ((n - k_1) - k_1)) = n - 2m - 2k_1$. Then, $\gamma(G) \leq k_1 = \frac{n-2m-\alpha_s^1(H)}{2}$. We obtain $\alpha_s^1(H) \leq n - 2m - 2\gamma(G)$.

Following the discussion above, $\alpha_s^1(H) = n - 2m - 2\gamma(G)$. Hence, for any positive integer κ, $\gamma(G) \leq \kappa$ if and only if $\alpha_s^1(H) \geq n - 2m - 2\kappa$. □

Theorem 2. *For any fixed positive integer $k \geq 2$, the signed k-independence problem is NP-complete for chordal graphs and bipartite planar graphs.*

Proof. The signed k-independence problem on bipartite planar graphs (respectively, chordal graphs) is clearly in NP. By Theorem 1, the signed 1-independence problem is NP-complete for chordal graphs and bipartite planar graphs. In the following, we show the NP-completeness of the signed k-independence problem on bipartite planar graphs (respectively, chordal graphs) by a polynomial-time reduction from the signed $(k-1)$-independence problem on bipartite planar graphs (respectively, chordal graphs).

Let $G = (V, E)$ be a bipartite planar graph (respectively, chordal graph). We construct a graph H by creating a new vertex v' for each vertex $v \in V$ and connecting v and v' with an edge. Then, $V(H) = V \cup \{v' \mid v \in V\}$ and $E(H) = E \cup \{(v, v') \mid v \in V\}$. Clearly, H is a bipartite planar graph (respectively, chordal graph) and the construction of H can be done in polynomial time.

Let $|V| = n$ and $|E| = m$. Suppose that f is a maximum signed k-independence function of H. Let $v \in V$ be a vertex such that $f(v') = -1$. We consider the following two cases.

Case 1: $f(v) = 1$. Let $f' : V(H) \to \{-1, 1\}$ be a function of H such that $f'(v) = -1$, $f'(v') = 1$, and $f'(x) = f(x)$ for every $x \in V \setminus \{v, v'\}$. The function f' is still a maximum signed k-independence function of H.

Case 2: $f(v) = -1$. Suppose that there exists a vertex $y \in N_G(v)$ such that $f(y) = 1$. Let $f' : V(H) \to \{-1, 1\}$ be a function of H such that $f'(y) = -1$, $f'(v') = 1$, and $f'(x) = f(x)$ for every $x \in V \setminus \{y, v'\}$. The function f' is still a maximum signed k-independence function of H. Suppose that there does not exist a vertex $y \in N_G(v)$ such that $f(y) = 1$. Let $f' : V(H) \to \{-1, 1\}$ be a function of H such that $f'(v') = 1$ and $f'(x) = f(x)$ for every $x \in V \setminus \{v'\}$. It can be easily verified that f' is a signed k-independence function of H and the weight of f' is larger than that of f. It contradicts the assumption that f is a maximum signed k-independence function of H.

We therefore know that there exists a maximum signed k-independence function h of H such that $h(v') = 1$ for each vertex $v \in V$. Let $g : V \to \{-1, 1\}$ be a function of G such that $g(v) = h(v)$ for every $v \in V$. For each vertex $v \in V$, $g(N_G[v]) = h(N_G[v]) - h(v) \leq (k-1) - 1 = k - 2$. The function g is a signed $(k-1)$-independence function of G and it necessarily has the maximum weight among all signed $(k-1)$-independence functions of G. We have $\alpha_s^k(H) = \alpha_s^{k-1}(G) + n$. Hence, for any integer κ, $\alpha_s^{k-1}(G) \geq \kappa$ if and only if $\alpha_s^k(H) \geq n + \kappa$. □

3. Strongly chordal graphs

Let $G = (V, E)$ be a graph. A *clique* is a subset of pairwise adjacent vertices of V. A vertex v is *simplicial* if all vertices of $N_G[v]$ form a clique.

The ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of V is a *perfect elimination ordering* of G if for all $i \in \{1, \ldots, n\}$, v_i is a simplicial vertex of the subgraph G_i of G induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. Let $N_i[v]$ denote the closed neighborhood of v in G_i. A perfect elimination ordering is called a *strong elimination ordering* if it has the following property:

For $i \le j \le k$ if v_j and v_k belong to $N_i[v_i]$ in G_i, then $N_i[v_j] \subseteq N_i[v_k]$. Farber [5] showed that a graph is *strongly chordal* if and only if it admits a strong elimination ordering. So far, the fastest algorithm to recognize a strongly chordal graph and give a strong elimination ordering takes $O(m \log n)$ [8] or $O(n^2)$ time [9]. Strongly chordal graphs include many interesting classes of graphs such as trees, block graphs, interval graphs, and directed path graphs [2]. In the following, we give Algorithm MRI$(G, \mathcal{R}, I_1, \ell, d)$ to solve the $\mathcal{R}$-independence problem in linear-time for a strongly chordal graph G with a strong elimination ordering $(v_1, v_2, \ldots, v_n)$.

Algorithm MRI$(G, \mathcal{R}, I_1, \ell, d)$

1: **for** $i = 1$ to n **do**
2:     $f(v_i) \leftarrow I_1$;
3: **end for**
4: **for** $i = 1$ to n **do**
5:     **if** $f(N_G[v_i]) > \mathcal{R}(v_i)$ **then**
6:         stop and return the infeasibility of the problem;
7:     **end if**
8: **end for**
9: **for** $i = 1$ to n **do**
10:     $M \leftarrow \min\{\mathcal{R}(v) - f(N_G[v]) | v \in N_G[v_i]\}$;
11:     $f(v_i) \leftarrow \min\{I_1 + \lfloor \frac{M}{d} \rfloor \cdot d, I_1 + (\ell - 1) \cdot d\}$;
12: **end for**
13: **return** the function f;

Lemma 1. *If the function f initialized by Algorithm MRI in Steps 1–3 is not an $\mathcal{R}$-independence function of a strongly chordal graph $G = (V, E)$, then G has no $\mathcal{R}$-independence function.*

Proof. The function f initialized by Algorithm MRI in Steps 1–3 assigns the minimum value in Y to vertex v_i for $1 \le i \le n$. The function f therefore has the minimum weight among all $\mathcal{R}$-independence functions of G if f is an $\mathcal{R}$-independence function of G. If there is a vertex $v_i \in V$ with $f(N_G[v_i]) > \mathcal{R}(v_i)$, then f is not an $\mathcal{R}$-independence function of G and thus we cannot find any $\mathcal{R}$-independence function in G. □

Lemma 2. *The function f returned from Step 13 of Algorithm MRI is an $\mathcal{R}$-independence function of a strongly chordal graph $G = (V, E)$.*

Proof. By Algorithm MRI, the function f initialized in Steps 1–3 is an $\mathcal{R}$-independence function of G if the algorithm does not stop in Step 6. In Steps 9–12, the algorithm processes vertices in strong elimination ordering $v_1, v_2, \ldots, v_n$ to increase the weight of the function f. Obviously, the function f at the beginning of the first iteration of Steps 9–12 is an $\mathcal{R}$-independence function of G. We assume that the function f of i-th iteration of Steps 9–12 is an $\mathcal{R}$-independence function of G for $1 \le i \le n$. In the following, we show that at the end of the i-th iteration of Steps 9–12, the new function f obtained by changing the value of $f(v_i)$ in Step 11 is still an $\mathcal{R}$-independence function of G. Notice that the function f at the end of i-th iteration of Steps 9–12 is the function f at the beginning of the $(i+1)$-th iteration.

Let $\mathcal{B} = \min\{I_1 + \lfloor \frac{M}{d} \rfloor \cdot d, I_1 + (\ell - 1) \cdot d\}$. Then $\mathcal{B} \le I_1 + \lfloor \frac{M}{d} \rfloor \cdot d$. We have $\mathcal{B} \le I_1 + (\frac{M}{d}) \cdot d$ and thus $M \ge \mathcal{B} - I_1$. Since $M = \min\{\mathcal{R}(v) - f(N_G[v]) | v \in N_G[v_i]\}$, $\mathcal{R}(v) - f(N_G[v]) \ge M \ge \mathcal{B} - I_1$ for every $v \in N_G[v_i]$. We have $f(N_G[v]) - I_1 + \mathcal{B} \le \mathcal{R}(v)$ for every $v \in N_G[v_i]$. Notice that $f(v_i) = I_1$ before the execution of Step 11. Therefore, the new function f obtained by replacing the value of $f(v_i)$ with $\mathcal{B}$ in Step 11 is still an $\mathcal{R}$-independence function of G. Following the discussion above, the lemma holds. □

Lemma 3. *Let ℓ, d, I_1 be fixed integers and $\ell, d > 0$. Let Y be the weight set $\{I_1, I_1+d, I_1+2d, \ldots, I_1+(\ell-1)\cdot d\}$. The function $f : V \to Y$ returned from Algorithm MRI is a* maximum *$\mathcal{R}$-independence function of a strongly chordal graph $G = (V, E)$.*

Proof. By Lemma 2, the function f returned from Algorithm MRI is an $\mathcal{R}$-independence function of G. In the following, we show that f is a maximum $\mathcal{R}$-independence function of G. Among all maximum $\mathcal{R}$-independence functions of G, we let h be a maximum $\mathcal{R}$-independence function of G such that the number of vertices in $\{v | v \in V, f(v) = h(v)\}$ is maximum. We claim that $f(v) = h(v)$ for every $v \in V$. Assume for contrary that W is a nonempty set of all vertices w with $f(w) \neq h(w)$. Suppose that t is the smallest index such that $v_t \in W$. We consider the following cases.

Case 1: $h(v_t) > f(v_t)$. The value of $f(v_t)$ is finalized in Step 11 at t-iteration of Algorithm MRI (where an *iteration* here of the algorithm is understood as one iteration of Steps 9–12). At t-th iteration of Algorithm MRI, $M = \min\{\mathcal{R}(v) - f(N_G[v]) | v \in N_G[v_t]\}$ and $f(v_t) = \min\{I_1 + \lfloor \frac{M}{d} \rfloor \cdot d, I_1 + (\ell - 1) \cdot d\}$. We consider the following two subcases:

Case 1.1: $f(v_t) = I_1 + (\ell - 1) \cdot d$. Then $h(v_t) > f(v_t) = I_1 + (\ell - 1) \cdot d$, which contradicts the assumption that $h(v_t) \in Y$ since $I_1 + (\ell - 1) \cdot d$ is the largest value in Y.

Case 1.2: $f(v_t) = I_1 + \lfloor \frac{M}{d} \rfloor \cdot d$. Then $h(v_t) \ge f(v_t) + d = I_1 + (\lfloor \frac{M}{d} \rfloor + 1) \cdot d$. Let v_c be a vertex in $N_G[v_t]$ such that $M = \mathcal{R}(v_c) - f(N_G[v_c])$. Notice that $f(v_t) = I_1$ before the execution of Step 11 at the t-th iteration.

Therefore, $f(N_G[v_c] \setminus \{v_t\}) = \mathcal{R}(v_c) - M - I_1$ before the execution of Step 11 at the t-iteration. Since only the value of $f(v_t)$ was changed at the t-th iteration, $f(N_G[v_c] \setminus \{v_t\})$ is still equal to $\mathcal{R}(v_c) - M - I_1$ at the end of t-th iteration.

Notice that $h(v_x) = f(v_x)$ for every index $x < t$. At the end of the t-th iteration, $h(v_x) \geq f(v_x) = I_1$ for every index $x > t$. Then,

$$\begin{aligned} h(N_G[v_c]) &\geq f(N_G[v_c] \setminus \{v_t\}) + h(v_t) \\ &\geq f(N_G[v_c] \setminus \{v_t\}) + I_1 + (\lfloor \tfrac{M}{d} \rfloor + 1) \cdot d \\ &> (\mathcal{R}(v_c) - M - I_1) + I_1 + (\tfrac{M}{d}) \cdot d \\ &= \mathcal{R}(v_c) \end{aligned}$$

Hence, $h(N_G[v_c]) > \mathcal{R}(v_c)$, which contradicts the assumption that h is an $\mathcal{R}$-independence function of G.

Case 2: $f(v_t) > h(v_t)$. Let $Y = \{a_1, a_2, \ldots, a_\ell\}$ where $a_1 = I_1$, $a_2 = I_1 + d$, $\ldots, a_\ell = I_1 + (\ell - 1) \cdot d$. Let $h(v_t) = a_i$ and $f(v_t) = a_j$ for $1 \leq i < j \leq \ell$. Let $P = \{v | v \in N_G[v_t], h(N_G[v]) - a_i + a_j > \mathcal{R}(v))\}$. Clearly, $P \neq \emptyset$. Otherwise, $h(N_G[v]) - a_i + a_j \leq \mathcal{R}(v)$ for every $v \in N_G[v_t]$ and there is an $\mathcal{R}$-independence function g with $g(V) > h(V)$ by setting $g(v_t) = h(v_t) - a_i + a_j = a_j$ and $g(v) = h(v)$ for every vertex $v \in V \setminus \{v_t\}$, which contradicts the assumption that h is a maximum $\mathcal{R}$-independence function of G.

Notice that $h(v_x) = f(v_x)$ for every index $x < t$. For every vertex $v \in P$, $h(N_G[v]) - a_i + a_j > \mathcal{R}(v)$ and $f(N_G[v]) \leq \mathcal{R}(v)$. We know that $N_G[v] \cap \{v_x | v_x \in W, t < x$, and $h(v_x) > f(v_x)\} \neq \emptyset$ for every $v \in P$.

Let s be the smallest index of vertices in P. Let b be the smallest index of $N_G[v_s] \cap \{v_x | v_x \in W, t < x$, and $h(v_x) > f(v_x)\}$. Notice that $P \subseteq N_G[v_t]$. Since $h(v_t)$, $f(v_t)$, $h(v_b)$, and $f(v_b)$ are in Y, there exist two positive integers c_1 and c_2 such that $f(v_t) = h(v_t) + c_1 \cdot d$ and $h(v_b) = f(v_b) + c_2 \cdot d$. We define a function h' as follows.

(1) If $c_1 \leq c_2$, $h'(v_t) = h(v_t) + c_1 \cdot d = f(v_t)$, $h'(v_b) = h(v_b) - c_1 \cdot d$ and $h'(v) = h(v)$ for every $v \in V \setminus \{v_t, v_b\}$.

(2) If $c_1 > c_2$, $h'(v_t) = h(v_t) + c_2 \cdot d$, $h'(v_b) = h(v_b) - c_2 \cdot d = f(v_b)$, and $h'(v) = h(v)$ for every $v \in V \setminus \{v_t, v_b\}$.

Clearly, $h(V) = h'(V)$ and $|\{v | v \in V, f(v) = h'(v)\}| \geq |\{v | v \in V, f(v) = h(v)\}| + 1$. We prove $h'(N_G[v]) \geq \mathcal{R}(v)$ for every vertex $v \in V$ by showing that $P \subseteq N_G[v_b]$. There are two cases.

Case 2.1: $s \leq t$. Then $s \leq t < b$. By definition of the strong elimination ordering, $N_s[v_t] \subseteq N_s[v_b]$. Since $P \subseteq N_s[v_t]$, we have $P \subseteq N_s[v_b] \subseteq N_G[v_b]$.

Case 2.2: $s > t$. By the definition of a strong elimination ordering, $N_t[v_s] \subseteq N_t[v]$ for every vertex $v \in P$. Since $v_b \in N_t[v_s]$, $v_b \in N_t[v]$ for every vertex $v \in P$. In other words, $P \subseteq N_G[v_b]$.

Hence, h' is a maximum $\mathcal{R}$-independence function such that the number of vertices in $\{v|v \in V, f(v) = h'(v)\}$ is larger than that of vertices in $\{v|v \in V, f(v) = h(v)\}$, a contradiction to the assumption that the number of vertices in $\{v|v \in V, f(v) = h(v)\}$ is maximum.

Following the discussion above, W does not exist. Hence, f is a maximum $\mathcal{R}$-independence function of G. □

Theorem 3. *Let $G = (V, E)$ be a strongly chordal graph with $|V| = n$ and $|E| = m$. Algorithm MRI solves the $\mathcal{R}$-independence problem on G in $O(n + m)$ time if a strong elimination ordering is given.*

Proof. In a practical implementation of Algorithm MRI, we use $d(v_i)$ to keep track of $f(N_G[v_i])$ for each vertex $v_i \in V$ and use $m(v_i)$ to keep track of $\mathcal{R}(v_i) - d(v_i)$. Following the initialization of a function f in Steps 1–3, we initialize $d(v_i) = (|N_G[v_i]|) \cdot I_1$ and $m(v_i) = \mathcal{R}(v_i) - d(v_i)$.

The initialization of $d(v_i)$ and $m(v_i)$ can be done in $O(deg_G(v_i) + 1)$ time. While $f(v_i)$ is replaced by $\mathcal{B} = \min\{I_1 + \lfloor \frac{M}{d} \rfloor \cdot d, I_1 + (\ell - 1) \cdot d\}$, $d(v)$ and $m(v)$ are respectively increased by $\mathcal{B} - I_1$ for every vertex $v \in N_G[v_i]$. This can be done in $O(deg_G(v_i) + 1)$ time. At i-th iteration, $1 \le i \le n$, M can be computed in $O(deg_G(v_i) + 1)$ time by verifying $m(v)$ for every vertex $v \in N_G[v_i]$. Hence, the running time of Algorithm MRI is $O(\sum_{v_i \in V}(deg_G(v_i) + 1)) = O(n + m)$. □

4. Distance-hereditary graphs

A graph is *distance-hereditary* if any two distinct vertices have the same distance in every connected induced subgraph containing them. In 1997, Chang et al. [3] showed that distance-hereditary graphs can be defined recursively.

Theorem 4 ([3]). *Distance-hereditary graphs can be defined recursively as follows:*

1. *A graph consisting of only one vertex is distance-hereditary, and the twin set is the vertex itself.*
2. *If G_1 and G_2 are disjoint distance-hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph $G = G_1 \cup G_2$ is a distance-hereditary graph and the twin set of G is $TS(G_1) \cup TS(G_2)$. G is said to be obtained from G_1 and G_2 by a* false twin *operation.*
3. *If G_1 and G_2 are disjoint distance-hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph G obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$ is a distance-hereditary graph, and the twin set of G is $TS(G_1) \cup TS(G_2)$. G is said to be obtained from G_1 and G_2 by a* true twin *operation.*

4. *If G_1 and G_2 are disjoint distance-hereditary graphs with the twin sets $TS(G_1)$ and $TS(G_2)$, respectively, then the graph G obtained by connecting every vertex of $TS(G_1)$ to all vertices of $TS(G_2)$ is a distance-hereditary graph, and the twin set of G is $TS(G_1)$. G is said to be obtained from G_1 and G_2 by a* pendant vertex *operation.*

By Theorem 4, a distance-hereditary graph G has its own twin set $TS(G)$. The twin set $TS(G)$ is a subset of vertices of G and it is defined recursively. The construction of G from disjoint distance-hereditary graphs G_1 and G_2 as described in Theorem 4 involves only the twin sets of G_1 and G_2.

Following Theorem 4, a binary ordered decomposition tree can be obtained in linear time [3]. In this decomposition tree, each leaf is a single vertex graph, and each internal node represents one of the three operations: pendant vertex operation (labeled by P), true twin operation (labeled by T), and false twin operation (labeled by F). This ordered decomposition tree is called a PTF-tree. It has $2n-1$ tree nodes. Hence, a **PTF**-tree of a distance-hereditary graph can be obtained in linear time [3]. Let $G=(V,E)$ be a distance-hereditary graph. We assume that a PTF-tree for G is part of the input.

Definition 2. Suppose that $G=(V,E)$ is a distance-hereditary graph and $TS(G)$ is the twin set of G. Let k be a positive integer and let a, b, and c be integers such that $0 \leq a, b \leq n$ and $-n \leq c \leq n$. An (a,b,c,k)-*function* $f: V \to \{1,-1\}$ of G is a function satisfying the following three conditions.

(1) $a+b=|TS(G)|$.
(2) The function f assigns the value 1 to a vertices in $TS(G)$ and assigns the value -1 to b vertices in $TS(G)$.
(3) For a vertex $v \in V$, $f(N_G[v])+c \leq k-1$ if $v \in TS(G)$; otherwise $f(N_G[v]) \leq k-1$.

We define $\alpha_s(G,a,b,c,k) = \max\{f(V(G)) \mid f$ is an (a,b,c,k)-function of $G\}$. If there does not exist an (a,b,c,k)-function of G, then $\alpha_s(G,a,b,c,k) = -\infty$. Clearly, $\alpha_s^k(G) = \max\{\alpha_s(G,a,b,0,k) \mid 0 \leq a,b \leq |TS(G)|\}$.

We give the following lemmas to compute $\alpha_s(G,a,b,c,k)$ for a distance-hereditary graph G.

Lemma 4. *Suppose that $G=(V,E)$ is a graph of only one vertex v. Then,*

$$\alpha_s(G,a,b,c,k) = \begin{cases} 1 & \text{if } a=1,\ b=0, \text{ and } c \leq k-2; \\ -1 & \text{if } a=0,\ b=1, \text{ and } c \leq k; \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. This follows from the definition. □

Lemma 5. *Suppose that $G = (V, E)$ is formed from two disjoint distance-hereditary graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by a false twin operation. Then, $\alpha_s(G, a, b, c, k) = \max\{\alpha_s(G_1, a_1, b_1, c, k) + \alpha_s(G_2, a_2, b_2, c, k)\}$, where $a_1 + a_2 = a$ and $b_1 + b_2 = b$.*

Proof. This follows from the definition. □

Lemma 6. *Suppose that $G = (V, E)$ is formed from two disjoint distance-hereditary graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by a true twin operation. Then, $\alpha_s(G, a, b, c, k) = \max\{\alpha_s(G_1, a_1, b_1, c + a_2 - b_2, k) + \alpha_s(G_2, a_2, b_2, c + a_1 - b_1, k)\}$, where $a_1 + a_2 = a$ and $b_1 + b_2 = b$.*

Proof. In this case, the graph G is formed from G_1 and G_2 by connecting every vertex in $TS(G_1)$ to all vertices in $TS(G_2)$, and $TS(G) = TS(G_1) \cup TS(G_2)$.

Assume that f_1 is an $(a_1, b_1, c + a_2 - b_2, k)$-function of G_1 of maximum weight and f_2 is an $(a_2, b_2, c + a_1 - b_1, k)$-function of G_2 of maximum weight. We define a function f of G as follows: $f(v) = f_1(v)$ if $v \in V_1$ and $f(v) = f_2(v)$ if $v \in V_2$. Let $a = a_1 + a_2$ and $b = b_1 + b_2$. Then, $|TS(G)| = a + b$ and f assigns the value 1 (respectively, -1) to a (respectively, b) vertices in $TS(G)$.

By Definition 2, it is clear that $f(N_G[v]) = f_1(N_{G_1}[v]) \leq k - 1$ for every vertex $v \in V_1 \setminus TS(G_1)$ and $f(N_G[v]) = f_2(N_{G_2}[v]) \leq k - 1$ for every vertex $v \in V_2 \setminus TS(G_2)$.

We now consider a vertex $v \in TS(G_1)$. By Definition 2, $f_1(N_{G_1}[v]) + (c + a_2 - b_2) \leq k - 1$. Note that $N_G[v] \cap V_2 = TS(G_2)$. Then, $f(N_G[v]) = f_1(N_{G_1}[v]) + a_2 - b_2 \leq k - 1 - c$. We have $f(N_G[v]) + c \leq k - 1$. Similarly, we can prove that $f(N_G[v]) + c \leq k - 1$ for every vertex $v \in TS(G_2)$. The function f is therefore an (a, b, c, k)-function of G. Hence, $\alpha_s(G, a, b, c, k) \geq \max\{\alpha_s(G_1, a_1, b_1, c + a_2 - b_2, k) + \alpha_s(G_2, a_2, b_2, c + a_1 - b_1, k)\}$, where $a_1 + a_2 = a$ and $b_1 + b_2 = b$.

Conversely, we let f be an (a, b, c, k)-function of G of maximum weight. Let f_1 be the function of G_1 defined by $f_1(v) = f(v)$ for every vertex $v \in V_1$ and let f_2 be the function of G_2 defined by $f_2(v) = f(v)$ for every vertex $v \in V_2$. Note that $TS(G) = TS(G_1) \cup TS(G_2)$. Let $a_1 + a_2 = a$ and $b_1 + b_2 = b$ such that $|TS(G_1)| = a_1 + b_1$ and $|TS(G_2)| = a_2 + b_2$. Assume that f assigns the value 1 to a_1 (respectively, a_2) vertices in $TS(G_1)$ (respectively, $TS(G_2)$) and assigns the value -1 to b_1 (respectively, b_2) vertices in $TS(G_1)$ (respectively, $TS(G_2)$).

Clearly, $f_1(N_{G_1}[v]) = f(N_G[v]) \leq k - 1$ for every vertex $v \in V_1 \setminus TS(G_1)$ and $f_2(N_{G_2}[v]) = f(N_G[v]) \leq k$ for every vertex $v \in V_2 \setminus TS(G_2)$.

We now consider a vertex $v \in TS(G_1)$. Since $N_G[v] \cap V_2 = TS(G_2)$, we have

$$\begin{aligned} f(N_G[v]) + c &= f(N_{G_1}[v] \cup TS(G_2)) + c \\ &= f(N_{G_1}[v]) + a_2 - b_2 + c \\ &= f_1(N_{G_1}[v]) + c + a_2 - b_2 \\ &\leq k - 1 \end{aligned}$$

Therefore, f_1 is an $(a_1, b_1, c + a_2 - b_2, k)$-function of G_1. Similarly, we can prove that f_2 is an $(a_2, b_2, c + a_1 - b_1, k)$-function of G_2. Hence, $\alpha_s(G, a, b, c, k) \leq \max\{\alpha_s(G_1, a_1, b_1, c + a_2 - b_2, k) + \alpha_s(G_2, a_2, b_2, c + a_1 - b_1, k)\}$, where $a_1 + a_2 = a$ and $b_1 + b_2 = b$.

Following the discussion above, the lemma holds. □

Lemma 7. *Suppose that $G = (V, E)$ is formed from two disjoint distance-hereditary graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ by a pendant vertex operation. Then, $\alpha_s(G, a, b, c, k) = \max\{\alpha_s(G_1, a, b, c + a_2 - b_2, k) + \alpha_s(G_2, a_2, b_2, a - b, k)\}$, where $a_2 + b_2 = |TS(G_2)|$.*

Proof. In this case, the graph G is formed from G_1 and G_2 by connecting every vertex in $TS(G_1)$ to all vertices in $TS(G_2)$, but $TS(G) = TS(G_1)$. By using the arguments similar to those for proving Lemma 6, the lemma holds. □

Theorem 5. *For any positive integer k, the signed k-independence problem on distance-hereditary graphs can be solved in polynomial time.*

Proof. Following Lemmas 4–7 and the recursive definition of distance-hereditary graphs in Theorem 4, we can compute the signed k-independence number of a distance-hereditary graph G in polynomial time. Moreover, it is not difficult to see that a maximum signed k-independence function can be obtained in polynomial time. □

5. The signed k-independence problem is not FPT

We have shown that the decision version of the signed k-independence problem is NP-complete, even when restricted to chordal graphs or bipartite planar graphs. One of the ways to deal with NP-complete problems is to consider the *parameterized complexity* of the problems [4]. Many NP-complete problems have the following general form: given an object x and an integer κ, does x have some property that depends on κ. In parameterized complexity theory, κ is called the *parameter*. A problem may be associated with several parameters. If some parameters of a problem instance are small, we might have some hope in finding a polynomial-time solution to that instance.

Definition 3 ([4]). Let i be a positive integer. A problem is *fixed parameter tractable* (FPT) with respect to i parameters $k_1, k_2, \ldots, k_i$ if there exists a solution running in $O(f(k_1, k_2, \ldots, k_i)n^c)$ time, where c is a constant and f is a function of $k_1, k_2, \ldots, k_i$ which are independent of n.

However, not all NP-complete problems are FPT (see [4]). In this section, we show that the signed k-independence problem, parameterized by k and weight κ, is not FPT. Consider the following problem.

Problem: The zero signed 1-independence problem.
Instance: A graph $G = (V, E)$
Question: Does G have a signed 1-independence function of weight at least 0?

Lemma 8. *The zero signed 1-independence problem is NP-complete for chordal graphs and bipartite planar graphs.*

Proof. The zero signed 1-independence problem on chordal graphs (respectively, bipartite planar graphs) is clearly in NP. By Theorem 1, we know that the signed 1-independence problem on chordal graphs (respectively, bipartite planar graphs) is NP-complete. In the following, we show the NP-completeness of the zero signed 1-independence problem on chordal graphs (respectively, bipartite planar graphs) by a polynomial-time reduction from the signed 1-independence problem on chordal graphs (respectively, bipartite planar graphs).

Let P_3 be a path of 3 vertices. Clearly, $\alpha_s^1(P_3) = -1$. Let $G = (V, E)$ be a chordal graph (respectively, bipartite planar graph) and let H be the union of G and j disjoint copies of P_3. Obviously, H is a chordal graph (respectively, bipartite planar graph) and we have

$$\alpha_s^1(H) = \alpha_s^1(G) + j \cdot \alpha_s^1(P_3) = \alpha_s^1(G) - j.$$

Hence, $\alpha_s^1(G) \geq j$ if and only if $\alpha_s^1(H) \geq 0$. □

Theorem 6. *Even when restricted to chordal graphs and bipartite planar graphs, the signed k-independence problem, parameterized by k and weight κ, is not FPT, unless P = NP.*

Proof. Assume that there exists an algorithm which runs in $O(f(k, \kappa)n^c)$ time and that determines if a graph G has a signed k-independence function of weight at least κ. Then the zero signed 1-independence problem would be solvable in polynomial time. By Lemma 8, we know that the signed k-independence problem, parameterized by k and weight κ, is not FPT, unless P = NP. □

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