# On $\alpha$-labeling of disconnected graphs

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**Abstract**

In this paper, we give a general result which enlarge the class of graphs known to have $\alpha$-labeling.

**Keywords**: Graceful labeling; $\alpha$-labeling.

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## 1. Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notation and terminology of graph theory as in [3]. Most graph labeling methods trace their origin to one introduced by Rosa[15] in 1967, or one given by Graham and Sloane[10] in 1980.Despite the unabated procession of papers, there are few general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods. In this paper we give a general result which enlarge the class of graphs known to have $\alpha$-labeling. Terms and notations not defined in the paper can be found in [7].This reference surveyed the known results to all variations of graph labelings appearing in this paper.

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Rosa [15] called a function $f$ a $\beta$-valuation of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0,1,\ldots,q\}$ such that, when each edge $xy$ is assigned the label $|f(x)-f(y)|$, the resulting edge labels are distinct. Golomb [8] subsequently called such labeling graceful and this is now the popular term. Rosa[15] has shown that if every vertex has even degree and the number of edges is congruent to 1 or 2 (mod 4) then the graph is not graceful. In particular, the cycles $C_{4n+1}$ and $C_{4n+2}$ are not graceful. There are few general results on graceful labeling. In particular the following results have been obtained $K_n$ is graceful if and only if $n \le 4$, $K_{m,n}$ is graceful for all $m$ and $n$. Rosa [15] conjectured that all trees are graceful. Among the trees known to be graceful are: caterpillars (a caterpillar is a tree with the property that the removal of its endpoints leaves a path); trees with at most 4 end-vertices and ; trees with diameter at most 5; symmetrical trees (i.e., a rooted tree in which every level contains vertices of the same degree). Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3 (see [7]). In 1966 Rosa [15] defined an $\alpha$-labeling (or $\alpha$-valuation) as a graceful labeling $f$ with the additional property that there exists an integer $k$ so that for each edge $xy$ either $f(x) \le k < f(y)$ or $f(y) \le k < f(x)$. (Other names for such labeling are balanced, interlaced, and strongly graceful.) It follows that such a $k$ must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an $\alpha$-labeling is necessarily bipartite and therefore cannot contain a cycle of odd length. Graphs with $\alpha$-labelings have proved to be useful in the development of the theory of graph decompositions.

Harmonious graphs naturally arose in the study by Graham and Sloane [10] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph $G$ with $q$ edges to be harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that when each edge $xy$ is assigned the label $(f(x)+f(y)) (\mathrm{mod}\, q)$, the resulting edge labels are distinct. Analogous to the necessity condition for graceful graphs, Graham and Sloane [10] proved that if a harmonious graph has an even number of edges $q$ and the degree of every vertex is divisible by $2^k$ then $q$ is divisible by $2^{k+1}$.

Lee, Schmeichel and Shee in [14] gave a generalization of harmonious labelings called felicitous labelings. An injective function $f$ from the vertices of a graph $G$ with $q$ edges to the set $\{0,1,\ldots,q\}$ is called felicitous if the edge labels induced by $(f(x)+f(y))(\bmod q)$ for each edge $xy$ are distinct. In contrast to the situation for felicitous labelings, we remark that $C_{4n}$ and $K_{m,n}$ where $m,n>1$ are felicitous but not harmonious.

In 1981 Chang, Hsu and Rogers [2] call a graph $G$ with $q$ edges strongly $c$-elegant if there exists an injective function $f$ from $V(G)$ to the set $\{0,1,\ldots,q\}$ such the induced function $f^*: E(G) \to \{c, c+1,\ldots,c+q-1\}$ defined as $f^*(xy) = f(x)+f(y)$ for each edge $xy \in E(G)$ is a bijection for some positive integer $c$. We will call a strongly $c$-elegant labeling a $c$-consecutive labeling. By taking the edge labels of a $c$-consecutive labeled graph with $q$ edges modulo $q$, we obviously obtain a felicitously labeled graph. It is not known if there is a graph that can be felicitously labeled but not $c$-consecutively labeled. Figueroa-Centeno, Ichishima and Muntaner-Batle [5] define a felicitous graph to be strongly felicitous if there exists an integer $k$ so that for every edge $uv$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$.

For $(p,q)$-graph $G$ with $p=q+1$, Frucht [6] invented a variation of graceful labeling. He calls a graph $G$ pseudograceful if there exists an injective function $f: V(G) \to \{0,1,2,\ldots,q-1,q+1\}$, called a pseudograceful labeling of $G$, such that the induced function $f^*: E(G) \to \{0,1,2,\ldots,q\}$ defined by $f^*(xy) = |f(x)-f(y)|$ for all $xy \in E(G)$ is a bijection. The image of $f$ ($=\mathrm{Im}(f)$) is called the corresponding set of vertex labels . Seoud and Youssef [16] extended the definition of pseudograceful to all graphs with $p \leq q+1$. They proved that $K_m$ is pseudograceful if and only if $m=1,3$ or $4$; $K_{m,n}$ is pseudograceful when $m,n \geq 2$. They also proved that if $G$ is pseudograceful, then $G \cup K_{m,n}$ is graceful for $m,n \geq 2$ and $G \cup K_{m,n}$ is pseudograceful for $m,n \geq 2, (m,n) \neq (2,2)$. They ask if $G \cup K_{2,2}$ is pseudograceful whenever $G$ is, and observed that if $G$ is a pseudograceful Eulerian graph with $q$ edges, then $q \equiv 0$ or $3 \pmod 4$. Youssef [19] has shown that $C_n$ is pseudograceful if and only if $n \equiv 0$ or $3 \pmod 4$, and for $n > 8$ and $n \equiv 0$ or $3 \pmod 4$, $C_n \cup K_{p,q}$ is pseudograceful for all $p,q \geq 2$ except $(p,q) = (2,2)$. Youssef [18] has shown that if $H$ is pseudograceful and $G$ has an $\alpha$-labeling with $k$ being the smaller vertex label of the edge labeled with 1 and if either $k+2$ or $k-1$ is not a vertex label of $G$ ,then $G \cup H$ is graceful.

**2. Pseudo $\alpha$-labelings**

Analogue of graphs admit an $\alpha$-labeling we call $f$ a pseudo $\alpha$-labeling of a graph $G$ if $f$ is a pseudograceful labeling of $G$ such that there exists an integer $k_f$ so that for each edge $xy$ of $G$ either $f(x) \leq k_f < f(y)$ or $f(y) \leq k_f < f(x)$. It follows that such a $k_f$ must be the smaller of the two vertex labels yield the edge label 1. Also, a graph with a pseudo $\alpha$-labeling is necessarily bipartite with vertex partition $\{x \in V(G) : f(x) \leq k_f\}$ and $\{x \in V(G) : f(x) > k_f\}$. We note that any graph of size $q$ that has an $\alpha$-labeling $f$ such that either $q-1$ or $1 \notin \mathrm{Im}(f)$ has a pseudo $\alpha$-labeling $\overline{f} = f+1$ or $\overline{f} = q+1-f$ respectively.

In this section, we generalize the results of Youssef [18] and give a new families of disconnected graphs that having an $\alpha$-labeling. The following result enlarge the class of disconnected graphs that admit an $\alpha$-labeling and as well as pseudo $\alpha$-labeling.

**Theorem 2.1.** Let $G$ be a graph of size $q$ and having an $\alpha$-labeling $f$ and let $H$ be a graph of size $t$ and having a pseudo $\alpha$-labeling $g$ .

(i) If $k_f + 2$ or $k_f - 1 \notin \mathrm{Im}(f)$, then $G \cup H$ has an $\alpha$-labeling, and

(ii) If ($\{k_f + 2, q-1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f + 2 \neq q-1$ ) or

($\{1, k_f - 1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f - 1 \neq 1$ ), then $G \cup H$ has a pseudo $\alpha$-labeling.

**Proof** Since $f$ is an $\alpha$-labeling of $G$, then $V(G)$ can be partitioned into two independent sets $V_1 = \{u \in V(G) : f(u) \leq k_f\}$ and $V_2 = \{v \in V(G) : f(v) > k_f\}$ and similarly as $g$ is a pseudo $\alpha$-labeling of $H$, then $V(H)$ can be partitioned into two independent sets

$$U_1 = \{u \in V(H) : g(u) \leq k_g\} \text{ and } U_2 = \{v \in V(H) : g(v) > k_g\}.$$

(i) If $k_f + 2 \notin \mathrm{Im}(f)$. Define a labeling function
$h : V(G \cup H) \to \{0, 1, \ldots, q + t\}$
as follows:

$$h\big|_{V_1} = f\big|_{V_1}$$
$$h\big|_{V_2} = f\big|_{V_2} + t$$
$$h\big|_{V(H)} = k_f + 1 + g$$

We note that as $k_f + 2 \notin \mathrm{Im}(f)$, then $k_f + t + 2 \notin h(V_2)$ also as $t \notin \mathrm{Im}(g)$ then $k_f + t + 1 \notin h(V(H))$. Now the vertex labels of $G \cup H$ is partitioned into two independent subsets of $\{0, 1, \ldots, k_f, \ldots, k_f + k_g + 1\}$ and $\{k_f + k_g + 2, \ldots, k_f + t + 1, k_f + t + 2, \ldots, q + t\}$. Hence $h$ is injective and $h^*(V(H)) = \{1, 2, \ldots, t\}$, $h^*(V(G)) = \{t + 1, t + 2, \ldots, t + q\}$ and $h$ is an $\alpha$-labeling.

If $k_f - 1 \notin \mathrm{Im}(f)$, then $\bar{f} = q - f$ is an $\alpha$-labeling of $G$ with $k_{\bar{f}} = q - k_f - 1$ so that $k_{\bar{f}} + 2 \notin \mathrm{Im}(\bar{f})$ and $G \cup H$ has an $\alpha$-labeling by the above argument.

(ii) If $\{k_f + 2, q - 1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f + 2 \neq q - 1$. Define a labeling function

$h : V(G \cup H) \to \{0, 1, \ldots, q + t - 1, q + t + 1\}$
as

$$h\big|_{V_1} = f\big|_{V_1} + 1$$
$$h\big|_{V_2} = f\big|_{V_2} + t + 1$$
$$h\big|_{V(H)} = k_f + g + 2$$

Note that $h$ and $h^*$ are both injectives and $q + t + 1 \notin \mathrm{Im}(h^*)$ as in (i) since $k_f + 2 \notin \mathrm{Im}(f)$. Observe also that $q + t \notin h(V(G))$ since $q - 1 \notin \mathrm{Im}(f)$ and $q + t \notin h(V(H))$ since $q - 1 \neq k_f + 2$ and $k_f \neq q - 1$, hence $q + t \notin \mathrm{Im}(h)$ and $h$ is a pseudo $\alpha$-labeling of $G \cup H$ as desired.

If $\{1, k_f - 1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f - 1 \neq 1$, then as in (i) $\bar{f} = q - f$ is an $\alpha$-labeling of $G$ and $k_{\bar{f}} = q - k_f - 1$ so that $\{k_{\bar{f}} + 2, q - 1\} \cap \mathrm{Im}(\bar{f}) = \Phi$ and $k_{\bar{f}} + 2 \neq q - 1$. Then $G \cup H$ has a pseudo $\alpha$-labeling by the above argument. □

**Corollary 2.1.** In the above theorem, if moreover $k_g + 2$ or $k_g - 1 \notin \mathrm{Im}(g)$, then $G \cup pH$ has an $\alpha$-labeling for every positive integer $p$.

Again, in the above theorem part (i) we have the following result

**Corollary 2.2.** If( $\{k_f + 2, q-1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f + 2 \neq q - 1$ ) or ( $\{1, k_f - 1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f - 1 \neq 1$ ), then $pG$ has an $\alpha$-labeling for every positive integer $p$.

**Proof** If ( $k_f + 2$ and $q - 1 \notin \mathrm{Im}(f)$ ), then $g = f + 1$ is a pseudo $\alpha$-labeling of $G$, and by induction $pG$ has an $\alpha$-labeling. Similarly the other case, it will be $g = q + 1 - f$ is a pseudo $\alpha$-labeling of $G$. □

In [11] Ichishima and Oshima proved that if $m, s$ and $t$ are integers with $m \geq 1, s \geq 2$ and $t \geq 2$, then the graph $mK_{s,t}$ has an $\alpha$-labeling if and only if $(m, s, t) \neq (3,2,2)$. However we have the following corollary.

**Corollary 2.3.** $mK_{s,t}$ has an $\alpha$-labeling for $s, t \geq 2, m \geq 1$ and $(s,t) \neq (2,2)$.

We excluded the case $(s,t) = (2,2)$ because in any $\alpha$-labeling $f$ of $K_{2,2}$ we have either $k_f + 2 = q - 1 = 3$ or $k_f - 1 = 1$ and of course does not satisfy the hypotheses of the corollary. The case $mK_{2,2} = mC_4$ is already known to have $\alpha$-labeling if and only if $m \geq 1, m \neq 3$ by Abrham and Kotzig [1]

There are many families of graphs satisfy the conditions of our above results specially Corollary 2.2. From which, Jungreis and Reid [12] showed that $C_{4m} \times P_n$, $C_{4m} \times C_{4n}$ and $C_{4m} \times C_{4n+2}$ has an $\alpha$-labeling $f$ and $\{k_f + 2, q - 1\} \cap \mathrm{Im}(f) = \Phi$ and $k_f + 2 \neq q - 1$, where $q$ is the size of the graph in each case. Kaneria and Makadia [13] showed that the graph $P_m \times P_n \cup P_r \times P_s$ is graceful, but their proof showing that the graph has an $\alpha$-labeling and we may deduce that $P_m \times P_n$ has an $\alpha$-labeling $f$ with $q - 1 \notin \mathrm{Im}(f)$ and either $k_f - 1$ or $k_f + 2 \notin \mathrm{Im}(f)$. For other families of graphs having $\alpha$-labeling see [7].

## 3. $c$-consecutive labeling

The following result is similar to one given by Grace [9] for proving that if a tree $T$ has an $\alpha$-labeling then $T$ is sequential, the sequential labeling is the same as the c-consecutive labeling except for the vertex label is from $\{0,1,\ldots,q-1\}$ instead of $\{0,1,\ldots,q\}$, and another given by Figueroa-Centeno, Ichishima and Muntaner-Batle [5]. For a graph with $p$ vertices and $q$ edges with $q \geq p-1$ they show that $G$ is strongly felicitous if and only if $G$ has an $\alpha$-labeling.

**Lemma 3.1.** Every graph which has an $\alpha$-labeling is $c$-consecutive.

**Proof** Let $G$ be a graph of size $q$ and has an $\alpha$-labeling $f$, then there exists an integer $k$ such that $G$ is a bipartite with partition $V_1 = \{u \in V(G) : f(u) \leq k\}$ and $V_2 = \{v \in V(G) : f(v) > k\}$. Define a labeling $g : V(G) \to \{0,1,\ldots,q\}$ as $g\big|_{V_1} = f\big|_{V_1}$ and $g\big|_{V_2} = q+1+k-f$. Cleary that $g$ is injective and moreover

$g(u)+g(v) = q+1+k-(f(v)-f(u))$ where $uv \in E(G)$, $u \in V_1, v \in V_2$ and as $1 \leq f(v)-f(u) \leq q$, then $k+1 \leq g(u)+g(v) \leq k+q$ and $G$ is $(k+1)$-consecutive. □

Given two bipartite graphs $G_1$ and $G_2$ with partite sets $H_1$, $L_1$ for $G_1$ and $H_2$, $L_2$ for $G_2$. Snevily [17] defines their weak tensor product $G_1 \bar{\otimes} G_2$ as the bipartite graph with vertex set $(H_1 \times H_2, L_1 \times L_2)$ and with edge $(h_1,h_2)(l_1,l_2)$ if $h_1 l_1 \in E(G_1)$ and $h_2 l_2 \in E(G_2)$. He proves that if $G_1$ and $G_2$ have $\alpha$-labelings then so does $G_1 \bar{\otimes} G_2$. This result considerably enlarges the class of graphs known to have $\alpha$-labelings. According to Lemma 3.1 again, we have the following result.

**Corollary 3.1** If $G_1$ and $G_2$ have $\alpha$-labelings then $G_1 \bar{\otimes} G_2$ is $c$-consecutive.

El-Zanati, Kenig, and Vanden Eynden [4] called $f$ a near $\alpha$-labeling of a graph $G$ if $f$ is a graceful labeling of $G$ such that $V(G)$ has a partition $V_1, V_2$ with the property that each edge of $G$ has the form $uv$ where $u \in V_1$, $v \in V_2$ and $f(u) < f(v)$. They further prove that if $G$ and $H$ have near $\alpha$-labelings, then so does their weak tensor product. We define a similar definition for $c$-consecutive labeling. We call $f$ a sharply (or strictly ) $c$-consecutive labeling if $f$ is a $c$-consecutive labeling of $G$ such that $V(G)$ has a partition $V_1, V_2$ with the property that each edge of $G$ has the form $uv$ where $u \in V_1, v \in V_2$ and $f(u) < f(v)$. Clearly that $G$ is a bipartite graph. Note that the graphs which admit $\alpha$-labelings are admit near $\alpha$-labelings, but the graphs have sharply $c$-consecutive labelings are having $c$-consecutive labelings. Although we could not find a relationship between near $\alpha$-labeling and sharply $c$-consecutive labeling but the result of El-Zanati et al. [4] still works in case of sharply $c$-consecutive labelings. We show a similar result for sharply $c$-consecutive labelings.

**Theorem 3.1.** If G and $H$ have sharply $c$-consecutive labelings, then so does their weak tensor product.

**Proof** Let $f$ and $g$ be sharply $c_1$-consecutive and $c_2$-consecutive labelings of $G$ and $H$ respectively and then G and $H$ are bipartite graphs with vertex bipartitions $V_1, V_2$ and $W_1, W_2$ respectively.

Define a labeling $h : V(G \bar{\otimes} H) \to \{0, 1, \ldots, q_1 q_2\}$ where $q_1 = |E(G)|$, $q_2 = |E(H)|$ as follows: $h(v_1, w_1) = q_1 g(w_1) + f(v_1)$ for $(v_1, w_1) \in V_1 \times W_1$ and $h(v_2, w_2) = q_1(g(w_2) - 1) + f(v_2)$ for $(v_2, w_2) \in V_2 \times W_2$. The range of $h$ is a subset of $\{0, 1, \ldots, q_1 q_2\}$ since $0 \le f(u) < f(v) \le q_1$ for all $u \in V_1, v \in V_2$ such that $uv \in E(G)$ and $0 \le g(s) < g(t) \le q_2$ for all $s \in W_1$, $t \in W_2$ such that $st \in E(H)$ so $0 \le h(u, s)$ and $h(v, t) \le q_1(q_2 - 1) + q_1 = q_1 q_2$.

First, we show that $h$ is injective. Suppose $h(v_1, w_1) = h(v_1', w_1')$ where $(v_1, w_1), (v_1', w_1') \in V_1 \times W_1$, then $q_1 g(w_1) + f(v_1) = q_1 g(w_1') + f(v_1')$ and $f(v_1) - f(v_1') = q_1(g(w_1') - g(w_1))$. So $q_1 | f(v_1) - f(v_1')$ and we get $f(v_1) = f(v_1')$ and as $f$ injective we get $v_1 = v_1'$ and also we can get $w_1 = w_1'$. Similarly for the second branch of $h$. Also we can show that $h(v_1, w_1) \neq h(v_2, w_2)$ for every $(v_1, w_1) \in V_1 \times W_1$ and $(v_2, w_2) \in V_2 \times W_2$. Second, we show that $h^*$ is injective. Suppose $h^*((v_1, w_1)(v_2, w_2)) = h^*((v_1', w_1')(v_2', w_2'))$, then $q_1(g(w_1)+g(w_2)-1)+f(v_1)+f(v_2)=q_1(g(w_1')+g(w_2')-1)+f(v_1')+f(v_2')$

and $f(v_1)+f(v_2)-(f(v_1')+f(v_2'))=q_1(g(w_1')+g(w_2')-(g(w_1)+g(w_2)))$. Then $q_1 \mid f(v_1)+f(v_2)-(f(v_1')+f(v_2'))$ and as $f(v_1)+f(v_2)<2q_1$, $f(v_1')+f(v_2')>0$, then $f(v_1)+f(v_2)-(f(v_1')+f(v_2'))<2q_1$, so $f(v_1)+f(v_2)-(f(v_1')+f(v_2'))$ either equal 0 or $q_1$. If $f(v_1)+f(v_2)-(f(v_1')+f(v_2'))=q_1$, we get a contradiction, then $f(v_1)+f(v_2)-(f(v_1')+f(v_2'))=0$ that is $f^*(v_1v_2)=f^*(v_1'v_2')$ and $v_1v_2=v_1'v_2'$. Similarly we get $w_1w_2=w_1'w_2'$. Hence all edge labels of $G\bar{\otimes}H$ are distinct. Finally, as $G$ is sharply $c_1$-consecutive and $H$ is sharply $c_2$-consecutive. We can check this inequality easily $q_1(c_2-1)+c_1 \le h^*(e) \le q_1(c_2-1)+c_1+q_1q_2-1$ for each edge $e\in E(G\bar{\otimes}H)$ and hence $G\bar{\otimes}H$ is sharply $q_1(c_2-1)+c_1$-consecutive □

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