# On Schwenk-like formulas for $Q$-characteristic polynomials of graphs

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**Abstract** Two Schwenk-like formulas about the signless Laplacian matrix of a graph are given, and thus it gives new tools for computing $Q$-characteristic polynomials of graphs directly. As an application, we give the $Q$-characteristic polynomial of lollipop graphs and reprove the known result that no two non-isomorphic lollipop graphs are $Q$-cospectral by a simple manner.

**Keywords:** $Q$-characteristic polynomial; Schwenk-like formulas; Lollipop graphs

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## 1 Introduction

Let $G$ be a simple graph with vertex set $V(G) = \{v_1, \dots, v_n\}$, edge set $E(G) = \{e_1, \dots, e_m\}$ and *adjacency matrix* $A(G) = (a_{ij})_{n\times n}$ where $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$, and $a_{ij} = 0$ otherwise. Let $D(G)$ be the diagonal matrix $\mathrm{diag}(d_1, \dots, d_n)$, where $d_i$ denotes the degree of vertex $v_i$ ($i = 1, \dots, n$). The *signless Laplacian matrix* $Q(G)$ is the matrix $D(G) + A(G)$. The polynomials $P(G; x) = \det(xI - A(G)) = \sum_{i=0}^{n} a_i x^{n-i}$, $\varphi(G; x) = \det(xI - Q(G)) = \sum_{i=0}^{n} q_i x^{n-i}$ where $I$ is the identity matrix, are defined

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as *characteristic polynomial* and *Q-characteristic polynomial* of the graph $G$, respectively. For brevity, write $P(G;x) = P(G)$, $\varphi(G;x) = \varphi(G)$ provided the omitted variable is obvious from the context. Since $A(G)$ and $Q(G)$ are real symmetric, their eigenvalues are all real numbers. The *A-spectrum (Q-spectrum)* of a graph $G$ consists of the eigenvalues of $A(G)$ ($Q(G)$). Two graphs $G$ and $H$ are said to be *A-cospectral (Q-cospectral)* if they share the same $A$-spectrum ($Q$-spectrum).

It is meaningful to study the property of characteristic polynomial for spectral graph theory. Since calculating the spectrum of a graph is a fundamental work and determining the coefficients of the characteristic polynomial and then finding its roots is one way for computing the eigenvalues of a graph. In addition, as cospectral graphs have the same characteristic polynomials, several graph operations and modifications in which the corresponding characteristic polynomials are known, can be used to construct cospectral graphs. For instance, by examining the characteristic polynomial of what he called the coalescence of two graphs, Schwenk [10] proved the famous theorem which stated that with respect to adjacency matrix, almost all trees are not determined by their spectra. On the other hand, comparing the exponents and coefficients of two characteristic polynomials is a frequently-used and efficient method for determining the non-cospectrality of two graphs, see [4, 8, 9, 12–14] etc.

There are several formulas for determining characteristic polynomials of graphs derived from that of their subgraphs by certain operations or modifications. The first is due to Heilbronner [7], he gave a formula which expresses the characteristic polynomial of graph $GuvH$ (the graph obtained from $G$ and $H$ by adding a bridge $uv$, where $u \in V(G)$ and $v \in V(H)$) by the characteristic polynomials of graphs $G, H, G-u$ and $H-v$. The most frequently encountered formulas in the literature are those given by Schwenk [11]. His two results display respectively the relations between the characteristic polynomial of a graph $G$ and the polynomials of $G$ minus a vertex and $G$ minus an edge. In the monograph [2] Cvetković et al. presented a equation of the characteristic polynomial of the corona $G \circ H$ in terms of that of the graph $G$ and the regular graph $H$. Recently, Belardo et al. [1] extended some well-known formulas about characteristic polynomial to weighted (di)graphs. All the above formulas are involved in adjacency matrix, however, when it comes to signless Laplacian matrix, little are known.

In this paper, by the definition of the semiedge vertex, $Q$-induced subgraph and $Q$-elementary graph, two Schwenk-like formulas for signless Laplacian matrix are established. Thus it supplies new tools for evaluating $Q$-characteristic polynomials of graphs. As one application, the $Q$-characteristic polynomial of lollipop graphs is presented and the known result that no two non-isomorphic lollipop graphs are $Q$-cospectral are reproved by a simple manner.

## 2 Main results

Let $G$ be a graph without isolated vertex. Its signless Laplacian matrix $Q(G)$ has nonzero diagonal entries. Then the principal submatrix of $Q(G)$ will not be a signless Laplacian matrix of the induced subgraph of $G$ since the two matrices do not have the identical diagonal entries. In order to avoid this difficult, we introduce the definition of semiedge vertex. For signless Laplacian matrix $Q(G)$, we view each vertex $v$ and the semiedges incident to it indivisible. We define it as *semiedge vertex* and denote by $v^-$. For example, the thick lines shown in Fig. 1 are semiedges while the thin lines are edges. The edge of $G$ is independent of semiedge vertex. By the symbol $e = u^-v^-$, we mean that there is an edge $e$ joining one semiedge of $u^-$ and one semiedge of $v^-$. The *degree of a semiedge vertex* $v^-$ is the number of semiedges incident to $v^-$. If we delete a semiedge vertex from a graph $G$, then the edges incident to it should be also removed, however, all of semiedge vertices remain the same when an edge is deleted (see for example, $\theta(2,0,2)-v_3^-$ and $\theta(2,0,2)-e_7$ in Fig. 1). A *Q-induced subgraph* $G_Q[V(Z)]$ is a subgraph induced by the semiedge vertex set $V(Z)$ and the edges whose two end semiedge vertices belong to $V(Z)$. In a *Q-elementary graph*, each component is either a semiedge vertex, or a $P_2$, or a cycle. We give all the $Q$-elementary spanning subgraph of $\theta(2,0,2)$ in Example 1.

**Example 1.** *$\theta(2,0,2)$ has 27 Q-elementary spanning subgraphs, we list them according to the number of edges.* $H_1 = \{v_1^-, v_2^-, v_3^-, v_4^-, v_5^-, v_6^-\}$; $H_2 = \{v_1^-v_2^-, v_3^-, v_4^-, v_5^-, v_6^-\}$, $H_3 = \{v_4^-v_5^-, v_1^-, v_2^-, v_3^-, v_6^-\}$, $H_4 = \{v_3^-v_6^-, v_1^-, v_2^-, v_4^-, v_5^-\}$, $H_5 = \{v_1^-v_6^-, v_2^-, v_3^-, v_4^-, v_5^-\}$, $H_6 = \{v_2^-v_3^-, v_1^-, v_4^-, v_5^-, v_6^-\}$, $H_7 = \{v_3^-v_4^-, v_1^-, v_2^-, v_5^-, v_6^-\}$, $H_8 = \{v_5^-v_6^-, v_1^-, v_2^-, v_3^-, v_4^-\}$; $H_9 = \{v_1^-v_2^-, v_3^-v_6^-, v_4^-, v_5^-\}$, $H_{10} = \{v_1^-v_6^-, v_2^-v_3^-, v_4^-, v_5^-\}$, $H_{11} = \{v_1^-v_6^-, v_3^-v_4^-, v_2^-, v_5^-\}$, $H_{12} = \{v_2^-v_3^-, v_5^-v_6^-, v_1^-, v_4^-\}$, $H_{13} = \{v_3^-v_6^-, v_4^-v_5^-, v_1^-, v_2^-\}$, $H_{14} = \{v_3^-v_4^-, v_5^-v_6^-, v_1^-, v_2^-\}$, $H_{15} = \{v_1^-v_2^-, v_3^-v_4^-, v_5^-, v_6^-\}$, $H_{16} = \{v_1^-v_2^-, v_5^-v_6^-, v_3^-, v_4^-\}$, $H_{17} = \{v_1^-v_6^-, v_4^-v_5^-, v_2^-, v_3^-\}$, $H_{18} = \{v_2^-v_3^-, v_4^-v_5^-, v_1^-, v_6^-\}$, $H_{19} = \{v_1^-v_2^-, v_4^-v_5^-, v_3^-, v_6^-\}$; $H_{20} = \{v_1^-v_2^-, v_3^-v_6^-, v_4^-v_5^-\}$, $H_{21} = \{v_1^-v_6^-, v_2^-v_3^-, v_4^-v_5^-\}$, $H_{22} = \{v_1^-v_2^-, v_3^-v_4^-, v_5^-v_6^-\}$; $H_{23} = \{v_1^-v_2^-v_3^-v_6^-v_1^-, v_4^-, v_5^-\}$, $H_{24} = \{v_3^-v_4^-v_5^-v_6^-v_3^-, v_1^-, v_2^-\}$; $H_{25} = \{v_1^-v_2^-v_3^-v_6^-v_1^-, v_4^-v_5^-\}$, $H_{26} = \{v_3^-v_4^-v_5^-v_6^-v_3^-, v_1^-v_2^-\}$; $H_{27} = \{v_1^-v_2^-v_3^-v_4^-v_5^-v_6^-v_1^-\}$.

Harary [5] first gave a structural interpretation of determinant of the adjacency matrix $A(G)$ by the linear subgraphs whose components are paths $P_2$ or cycles. The following theorem is a similar result concerning signless Laplacian matrix $Q(G)$.

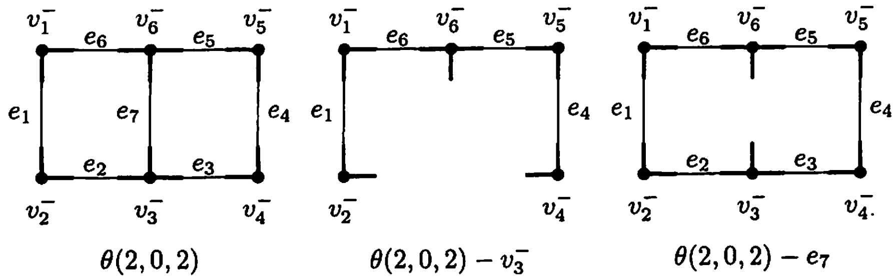


Fig. 1: Illustration of semiedge vertex

**Theorem 2.1.** *Let $G$ be a graph with $n$ vertices and no isolated vertices. Suppose that $Q(G)$ is the signless Laplacian matrix of $G$, then*

$$det(Q(G)) = (-1)^n \sum_{H \in \mathscr{H}} (-1)^{\omega(H)} \Big(\prod_{k=1}^{i(H)} d(v_k^-)\Big) 2^{c(H)},$$

*where $\mathscr{H}$ is the set of $Q$-elementary spanning subgraphs of $G$, $v_1^-, \ldots, v_{i(H)}^-$ denote the semiedge vertex components of $H$, $\omega(H)$ denotes the number of components of $H$ and $c(H)$ denotes the number of cycles in $H$.*

**Proof.** Let $Q = [q_{ij}]_{n\times n}$ be the signless Laplacian matrix of the graph $G$. Consider a term $\text{sgn}(\pi) q_{1,\pi(1)} q_{2,\pi(2)} \cdots q_{n,\pi(n)}$ in the expansion of $\det(Q)$. If this term is non-zero then $j = \pi(j)$ or $j \sim \pi(j)$. Thus $\pi$ can be expressed as a composition of disjoint cyclic permutations: $\alpha_1 \cdots \alpha_{i(H)} \beta_1 \cdots \beta_{t(H)} \gamma_1 \cdots \gamma_{c(H)}$ where $\alpha_1, \ldots, \alpha_{i(H)}$ are the fixed points; $\beta_1, \ldots, \beta_{t(H)}$ are transpositions and $\gamma_1, \ldots, \gamma_{c(H)}$ are cycles of length at least three. This expression determines an $Q$-elementary spanning subgraph $H$ in which the semiedge vertices components $v_k^-$ $(k = 1, \ldots, i(H))$ are determined by the fixed points $\alpha_1, \ldots, \alpha_{i(H)}$, the components isomorphic to $P_2$ are determined by the transpositions $\beta_1, \ldots, \beta_{t(H)}$ and the cycles are determined by the remaining $\gamma_1 \ldots \gamma_{c(H)}$. Let $\ell(\gamma_k)$ $(k = 1 \ldots, c(H))$ be the length of $\gamma_k$, since a fixed point and a transposition has length 1, 2, respectively. The sign of $\pi$ is

$$\begin{aligned}\text{sgn}(\pi) &= \sum_{k=1}^{i(H)} (\ell(\alpha_k) - 1) + \sum_{k=1}^{t(H)} (\ell(\beta_k) - 1) + \sum_{k=1}^{c(H)} (\ell(\gamma_k) - 1) \\ &= t(H) + \sum_{k=1}^{c(H)} \ell(\gamma_k) - c(H).\end{aligned}$$

In view of $i(H)+t(H)+c(H)=\omega(H)$ and $i(H)+2t(H)+\sum_{k=1}^{c(H)}\ell(\gamma_k)=n$, we have $\mathrm{sgn}(\pi)=n-\omega(H)$. Thus the term $\mathrm{sgn}(\pi)q_{1,\pi(1)}q_{2,\pi(2)}\cdots q_{n,\pi(n)}$ contributes $(-1)^{n-\omega(H)}\prod_{k=1}^{i(H)}(d(v_k^-))$ to $\det(Q(G))$. Finally, $H$ arises from $2^{c(H)}$ permutations: $\alpha_1\cdots\alpha_{i(H)}\beta_1\cdots\beta_{t(H)}\gamma_1^{\pm 1}\cdots\gamma_{c(H)}^{\pm 1}$ with the same sign and identical contribution to $\det(Q)$ as $\pi$. □

We compute the determinant of $Q(\theta(2,0,2))$ according to Theorem 2.1 and the data of $Q$-elementary spanning subgraphs obtained from Example 1 in the following.

**Example 2.**

$$
\begin{aligned}
&det(Q(\theta(2,0,2)))\\
&=(-1)^6\sum_{i=1}^{27}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)},\\
&=(-1)^{\omega(H_1)}(\prod_{k=1}^{i(H_1)}d(v_k^-))2^{c(H_1)}+\sum_{i=2}^{3}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}\\
&\quad+(-1)^{\omega(H_4)}(\prod_{k=1}^{i(H_4)}d(v_k^-))2^{c(H_4)}+\sum_{i=5}^{8}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}\\
&\quad+\sum_{i=9}^{14}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}+\sum_{i=15}^{18}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}\\
&\quad+(-1)^{\omega(H_{19})}(\prod_{k=1}^{i(H_{19})}d(v_k^-))2^{c(H_{19})}+\sum_{i=20}^{22}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}\\
&\quad+\sum_{i=23}^{24}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}+\sum_{i=25}^{26}(-1)^{\omega(H_i)}(\prod_{k=1}^{i(H_i)}d(v_k^-))2^{c(H_i)}\\
&\quad+(-1)^{\omega(H_{27})}(\prod_{k=1}^{i(H_{27})}d(v_k^-))2^{c(H_{27})}\\
&=(-1)^6\times 2^4\times 3^2+2\times(-1)^5\times 2^2\times 3^2+(-1)^5\times 2^4+4\times(-1)^5\times 2^3\times 3\\
&\quad+6\times(-1)^4\times 2^2+4\times(-1)^4\times 2\times 3+(-1)^4\times 3^2+3\times(-1)^3\\
&\quad+2\times(-1)^3\times 2^2\times 2+2\times(-1)^2\times 2+(-1)\times 2\\
&=144-184+57-3-16+4-2\\
&=0.
\end{aligned}
$$

*It is not hard to calculate the determinant of $Q(\theta(2,0,2))$ directly, we see that the two results are coincident.*

The next corollary interprets the coefficients of $Q$-characteristic polynomial of a graph by its $Q$-elementary subgraphs.

**Corollary 2.2.** *Let $\varphi(G;x) = x^n + q_1x^{n-1} + \cdots + q_{n-1}x + q_n$ be the $Q$-characteristic polynomial of $G$, and let $\mathscr{H}_s$ be the set of $Q$-elementary subgraphs of $G$ with $s$ semiedge vertices. Then*

$$q_s = \sum_{H\in\mathscr{H}_s} (-1)^{\omega(H)}\Big(\prod_{k=1}^{i(H)} d(v_k^-)\Big)2^{c(H)} \quad (s = 1, \ldots, n).$$

**Proof.** Since $(-1)^s q_s$ is the sum of all $s \times s$ principal minors of $Q(G)$, and each such minor is the determinant of the signless Laplacian matrix of a $Q$-induced subgraph on $s$ semiedge vertices. A $Q$-elementary subgraph with $s$ semiedge vertices is contained in exactly one such $Q$-induced subgraph, and so the result follows by applying Theorem 2.1 to each $s \times s$ principal minors of $Q(G)$. □

For the signless Laplacian matrix $Q(G)$, we use $\varphi(G \setminus V(Z); x)$ to denote the $Q$-characteristic polynomial of the $Q$-induced subgraph $G_Q[V(G)\setminus V(Z)]$ (or equivalently, the characteristic polynomial of the principal submatrix of $Q(G)$ whose rows and columns correspond to the semiedge vertices in $V(G) \setminus V(Z)$). In particular, if $V(Z) = \{u^-\}$ or $V(Z) = \{u^-, v^-\}$, we write $G - u^-$ and $G - u^- - v^-$ to abbreviate $G_Q[V(G) \setminus V(Z)]$. Now we give the Schwenk-like formulas for $Q$-characteristic polynomial of a graph $G$ in the following theorem.

**Theorem 2.3.** *(a) For any semiedge vertex $u^-$ of a graph $G$,*

$$\begin{aligned}\varphi(G;x) = (x - d(u^-))\varphi(G - u^-; x) - \sum_{v^-\sim u^-} \varphi(G - u^- - v^-; x) \\ - 2 \sum_{Z\in C(u^-)} \varphi(G \setminus V(Z); x),\end{aligned} \tag{1}$$

*where the first summation goes over all semiedge vertices $v^-$ adjacent to $u^-$ and $C(u^-)$ denotes the set of all cycles containing $u^-$.*

*(b) For any edge $u^-v^-$ of the graph $G$,*

$$\begin{aligned}\varphi(G;x) = \varphi(G - u^-v^-; x) - \varphi(G - u^- - v^-; x) \\ - 2 \sum_{Z\in C(u^-v^-)} \varphi(G \setminus V(Z); x),\end{aligned} \tag{2}$$

*where $C(u^-v^-)$ denotes the set of all cycles containing $u^-v^-$.*

**Proof.** (*a*) Recall that $\varphi(G;x) = \sum_{i=0}^{n} q_i x^{n-i}$, and Corollary 2.2 expresses $q_i$ in terms of the $i$-point $Q$-elementary subgraphs. We now present a one-to-one correspondence between those $Q$-elementary subgraphs contributing to $c_i$ on the left and those contributing to one of the terms on the right. Namely, letting $H$ be an $i$-point $Q$-elementary subgraph of $G$, we have four possibilities:

(i) If $u^- \notin H$, let $H'$ be the same $Q$-elementary subgraph, only now viewed as a $Q$-induced subgraph of $G - u^-$.

(ii) If $u^-$ is one of the semiedge vertex component of $H$, let $H'$ be $H - u^-$ viewed as a $Q$-induced subgraph of $G - u^-$.

(iii) If $u^- \in P_2 \subset H$, let $H'$ be $H - V(P_2)$ viewed as a $Q$-induced subgraph of $G - V(P_2)$.

(iv) If $u^- \in C_k \subset H$, let $H'$ be $H - V(C_k)$ viewed as a $Q$-induced subgraph of $G - V(C_k)$.

It is easy to see that this does indeed establish a one-to-one correspondence. Moreover, if $H$ contributes an amount $s$ toward the coefficient $x^{n-i}$ on the left, we observe that on the right $H'$ also contributes $s$ in each case as we demonstrate:

(i) Since $H' \cong H$, we see that $H'$ contributes $s$ to the coefficient of $x^{n-1-i}$ in $\varphi(G - u^-; x)$, and thus supplies $s$ toward the coefficient of $x^{n-i}$ in $x\varphi(G - u^-; x)$.

(ii) Now $H' \cong H - u^-$, so $H'$ contributes

$$(-1)^{\omega(H')}\Big(\prod_{k=1}^{i(H')} d(v_k^-)\Big)2^{c(H')} = -\frac{1}{d(u^-)}(-1)^{\omega(H)}\Big(\prod_{k=1}^{i(H)} d(v_k^-)\Big)2^{c(H)}$$
$$= -\frac{s}{d(u^-)}$$

to $x^{n-1-(i-1)} = x^{n-i}$ to the $Q$-induced subgraph $G - u^-$. Thus it contributes $sx^{n-i}$ to $-d(u^-)\varphi(G - u^-; x)$.

(iii) In this case $H' \cong H - V(P_2)$, so $H'$ contributes

$$(-1)^{\omega(H')}\Big(\prod_{k=1}^{i(H')} d(v_k^-)\Big)2^{c(H')} = -(-1)^{\omega(H)}\Big(\prod_{k=1}^{i(H)} d(v_k^-)\Big)2^{c(H)} = -s$$

to $x^{n-2-(i-2)} = x^{n-i}$ to $\varphi(G - u^- - v^-; x)$. Hence, it supplies $sx^{n-i}$ to $-\varphi(G - u^- - v^-; x)$.

(iv) Finally, we have $H' \cong H - V(C_k)$, so $H'$ contributes

$$(-1)^{\omega(H')}(\prod_{k=1}^{i(H')} d(v_k^-))2^{c(H')} = -\frac{1}{2}(-1)^{\omega(H)}(\prod_{k=1}^{i(H)} d(v_k^-))2^{c(H)} = -\frac{s}{2}$$

to $x^{n-k-(i-k)} = x^{n-i}$ in $\varphi(G \setminus V(C_k); x)$. Thus $H'$ contributes $sx^{n-i}$ in $-2\varphi(G \setminus V(C_k); x)$.

Therefore, the contribution of each $Q$-elementary subgraph $H$ to the left side is matched by a corresponding contribution on the right side of the $Q$-elementary subgraph $H'$.

For $(b)$, we have three cases, the method of proof is identical to that of (a) and is omitted. This completes the proof of the theorem. □

As an illustration of Theorem 2.3, we give the following example.

**Example 3.** *Let $G \cong \theta(2,0,2)$ (see Fig. 1), $C_1 = v_1^- v_2^- v_3^- v_6^- v_1^-$, $C_2 = v_3^- v_4^- v_5^- v_6^- v_3^-$, $C_3 = v_1^- v_2^- v_3^- v_4^- v_5^- v_6^- v_1^-$. Applying Theorem 2.3 at semivertex $v_3^-$ and edge $v_3^- v_6^-$, respectively, we have*

$$\varphi(G) = x^6 - 14x^5 + 74x^4 - 184x^3 + 213x^2 - 90x,$$

$$\varphi(G - v_3^-) = x^5 - 11x^4 + 44x^3 - 78x^2 + 59x - 15,$$
$$\varphi(G - v_2^- - v_3^-) = \varphi(G - v_3^- - v_4^-) = x^4 - 9x^3 + 27x^2 - 31x + 11,$$
$$\varphi(G - v_3^- - v_6^-) = x^4 - 8x^3 + 22x^2 - 24x + 9,$$
$$\varphi(G - v_3^- v_6^-) = x^6 - 14x^5 + 75x^4 - 192x^3 + 239x^2 - 130x + 21,$$
$$\varphi(G \setminus V(C_1)) = \varphi(G \setminus V(C_2)) = x^2 - 4x + 3,$$
$$\varphi(G \setminus V(C_3)) = 1.$$

*It follows that*

$$\begin{aligned}\varphi(G) =&(x-3)\varphi(G - v_3^-) - (\varphi(G - v_2^- - v_3^-) + \varphi(G - v_3^- - v_4^-) \\ &+ \varphi(G - v_3^- - v_6^-) - 2(\varphi(G - V(C_1)) + \varphi(G \setminus V(C_2)) \\ &+ \varphi(G - V(C_3)),\end{aligned}$$

$$\varphi(G) = \varphi(G - v_3^- v_6^-) - \varphi(G - v_3^- - v_6^-) - 2(\varphi(G - V(C_1)) + \varphi(G \setminus V(C_2)).$$

## 3 Application

The *lollipop graph* $L_{a,b}$ $(a \geq 3, b \geq 1)$ is obtained by appending a cycle $C_a$ to a pendant vertex of a path $P_b$. By showing no two non-isomorphic line graphs of lollipop graphs have the same adjacency spectrum [6, 15], the authors conclude that no two non-isomorphic lollipop graphs are $Q$-cospectral. In what follows, we will give a direct proof for the above result. Some symbols follow from the paper [3].

Let $B_n$ be the $Q$-induced subgraph obtained from $P_{n+1}$ by deleting one semiedge vertex of degree one and $H_n$ be the $Q$-induced subgraph obtained from $P_{n+2}$ by deleting two semiedge vertices of degree one. Then the $Q$-characteristic polynomials of $B_n$, $P_n$ and $H_n$ have following relations.

**Lemma 3.1.** *Set* $\varphi(P_0) = 0$, $\varphi(B_0) = 1$, $\varphi(H_0) = 1$. *We have*

*(i)* $\varphi(B_n) = \frac{1}{x}(\varphi(P_{n+1}) + \varphi(P_n))$;

*(ii)* $\varphi(P_{n+1}) = (x-2)\varphi(P_n) - \varphi(P_{n-1}), (n \geq 1)$;

*(iii)* $\varphi(H_n) = \frac{1}{x}\varphi(P_{n+1}), (n \geq 1)$.

**Proof.** Since

$$\varphi(B_n) = \begin{vmatrix} x-1-1 & -1 & 0 & \cdots & 0 \\ -1 & x-2 & -1 & \cdots & 0 \\ 0 & -1 & x-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x-1 \end{vmatrix} = \varphi(P_n) - \varphi(B_{n-1}).$$

Thus we have

$$\varphi(P_n) = \varphi(B_n) + \varphi(B_{n-1}). \tag{3}$$

From Theorem 2.3 (a), we have

$$\varphi(P_{n+1}) = (x-1)\varphi(B_n) - \varphi(B_{n-1}). \tag{4}$$

Substituting Eq. (3) into Eq. (4), we have

$$x\varphi(B_n) = \varphi(P_{n+1}) + \varphi(P_n), \tag{5}$$

therefore (i) follows.

For (ii), using (4), (3) and (5), we obtain

$$\begin{aligned} \varphi(P_{n+1}) &= (x-1)\varphi(B_n) - \varphi(B_{n-1}) \\ &= (x-1)(\varphi(P_n) - \varphi(B_{n-1})) - \varphi(B_{n-1}) \\ &= (x-1)\varphi(P_n) - x\varphi(B_{n-1}) \\ &= (x-1)\varphi(P_n) - (\varphi(P_n) + \varphi(P_{n-1})), \end{aligned} \tag{6}$$

hence $\varphi(P_{n+1}) = (x-2)\varphi(P_n) - \varphi(P_{n-1})$.

To show (iii), we need to verify $\varphi(P_n) = x\varphi(H_{n-1})$. Using induction on $n$, if $n = 1, 2$, the result is obvious. Suppose that $n \geq 3$. From (ii) and induction, we get

$$\begin{aligned}\varphi(P_n) &= (x-2)\varphi(P_{n-1}) - \varphi(P_{n-2}) \\ &= x(x-2)\varphi(H_{n-2}) - x\varphi(H_{n-3}) \\ &= x((x-2)\varphi(H_{n-2}) - \varphi(H_{n-3})) \\ &= x\varphi(H_{n-1}).\end{aligned}$$

□

The following lemma gives the concrete form of $Q$-characteristic polynomials of graph $P_n$, $B_n$ and $H_n$.

**Lemma 3.2.** *Let $y$ be the root of the equation $y^2-(x-2)y+1=0$ $(x \neq 4)$, then the $Q$-characteristic polynomial of the path $P_n$, $B_n$, $H_n$ are*

*(i)* $\varphi(P_n) = \frac{(y+1)(y^{2n}-1)}{(y^{n+1}-y^n)}$;

*(ii)* $\varphi(B_n) = \frac{y^{2n+1}-1}{y^{n+1}-y^n}$;

*(iii)* $\varphi(H_n) = \frac{y^{2n+2}-1}{y^{n+2}-y^n}$.

**Proof.** Lemma 3.1 (ii) gives the recurrence relation of $\varphi(P_n)$. It has characteristic equation

$$y^2 - (x-2)y + 1 = 0 \quad (x \neq 4). \tag{7}$$

Let $y$ be the root of (7), the general solution of $\varphi(P_n)$ is

$$\varphi(P_n) = c_1 y^n + c_2(\frac{1}{y})^n. \tag{8}$$

Note that $\varphi(P_0) = 0$ and $\varphi(P_1) = x$, we obtain that,

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 y + c_2 y^{-1} = x \end{cases}$$

so $c_2 = -c_1$ and

$$c_1 = \frac{x}{y - y^{-1}} = \frac{\frac{y^2+2y+1}{y}}{y - \frac{1}{y}} = \frac{y^2+2y+1}{y^2-1} = \frac{y+1}{y-1}. \tag{9}$$

Substituting (9) into (8) we have

$$\begin{aligned}\varphi(P_n) &= \frac{y+1}{y-1}y^n - \frac{y+1}{y-1}y^{-n} = \frac{y^{2n}(y+1)}{y^n(y-1)} - \frac{y+1}{y^n(y-1)} \\ &= \frac{(y+1)(y^{2n}-1)}{y^{n+1}-y^n}.\end{aligned} \tag{10}$$

Finally, By putting $x = \frac{y^2+2y+1}{y}$ and (10) into Lemma 3.1 (i), (iii), we see that Lemma 3.2 holds. □

Next by providing the $Q$-characteristic polynomial of the lollipop graph, we give a direct proof of the known result that no two non-isomorphic lollipop graphs have the same $Q$-spectrum.

**Lemma 3.3.** *The $Q$-characteristic polynomial of the lollipop graph $L_{a,b}$ is $\varphi(L_{a,b}; y) = \frac{1}{(y-1)^2}\varphi_1(L_{a,b}; y)$, where*

$$\begin{aligned}\varphi_1(L_{a,b}; y) =& y^{a+b+2} - 2y^{a+b+1} - 2y^{b+2} + 2y^{b+1} - y^{b+2-a} \\ &+ y^{a-b} + 2y^{1-b} + 2y^{-b} + y^{-a-b} - 2y^{-a-b-1}\end{aligned} \tag{11}$$

*and $y$ is the root of Eq. (7).*

**Proof.** Let $u^-$ be the unique semiedge vertex with three semiedges. By Theorem 2.3 (a), $\varphi(L_{a,b})$ can be computed as follows:

$$\begin{aligned}\varphi(L_{a,b}) =& (x-3)\varphi(H_{a-1})\varphi(B_b) - 2\varphi(H_{a-1})\varphi(B_b) \\ &- \varphi(H_{a-1})\varphi(B_{b-1}) - 2\varphi(B_b).\end{aligned} \tag{12}$$

Recall that $x = \frac{y^2+2y+1}{y}$, by Lemma 3.2 (ii), (iii), Maple direct calculation, we see that (11) holds. □

**Theorem 3.4.** *[6, 15] No two non-isomorphic lollipop graphs are $Q$-cospectral.*

**Proof.** Let $L_{a,b}$ and $L_{a',b'}$ be two $Q$-cospectral lollipop graphs, then they share the same $Q$-characteristic polynomial, thus from Lemma 3.3 $\varphi_1(L_{a,b}; y) = \varphi_1(L_{a',b'}; y)$. By (11), we see that the third leading term (term with the third highest exponent) of $\varphi_1(L_{a,b})$ is $-2y^{b+2}$ or $y^{a-b}$. Comparing this corresponding term in $\varphi_1(L_{a',b'}; y)$ leads to $a = a'$ and $b = b'$. Hence $L_{a,b}$ and $L_{a',b'}$ are isomorphic. □

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# References


[1] F. Belardo, E.M. Li Marzi, S.K. Simić, Combinatorial approach for computing the characteristic polynomial of a matrix, Linear Algebra Appl. 433 (2010) 1513–1523.

[2] D.M. Cvetković, P. Rowlinson, S.K. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge, 2010.

[3] J. Guo, A conjecture on the algebraic connectivity of connected graphs with fixed girth, Discrete Math. 308 (2008) 5702–5711.

[4] W.H. Haemers, X. Liu, Y. Zhang, Spectral characterizations of lollipop graphs, Linear Algebra Appl. 428 (2008) 2415–2423.

[5] F. Harary, The Determinant of the Adjacency Matrix of a Graph, SIAM Review, 4 (1962) 202–210.

[6] H. Hamidzadea, D. Kiani, Erratum to "The lollipop graph is determined by its Q-spectrum", Discrete Math. 310 (2010) 1649.

[7] E. Heilbronner, Das Kompositions Prinzip, Eine anshauliche Methode zur elektronen theoretischen Behaundlung nicht oder niedring symmetrischer Molekeln im Rahmen der MO-Theorie, Helv. Chim. Acta 37 (1954) 170–188.

[8] F. Liu, Q. Huang, On the spectral characterization of Π-shape trees, Linear and Multilinear Algebra, 61 (2013) 355–367.

[9] F. Ramezani, N. Broojerdian, B. Tayfeh-Rezaie, A note on the spectral characterization of $\theta$-graphs, Linear Algebra Appl. 431 (2009) 626–632.

[10] A.J. Schwenk, Almost all trees are cospectral, New Directions in the Theory of Graphs (F. Harary, ed.), Academic Press, New York, (1973) 275–307.

[11] A.J. Schwenk, Computing the characteristic polynomial of a graph, R. Bari, F. Harary (Eds.), Graphs and Combinatorics, Lecture Notes in Mathematics, vol. 406, Springer-Verlag, Berlin - Heidelberg - New York, (1974) 153–172.

[12] J. Wang, Q. Huang, F. Belardo, E.M. Li Marzi, On the spectral characterizations of $\infty$-graphs, Discrete Math. 310 (2010) 1845–1855.

[13] W. Wang, C. Xu, On the spectral characterization of T-shape trees, Linear Algebra Appl. 414 (2006) 492–501.

[14] X. Liu, S. Zhou, Spectral characterizations of propeller graphs, Electron. J. Linear Algebra, 27 (2014) 19–38.

[15] Y. Zhang, X. Liu, B. Zhang, X. Yong, The lollipop graph is determined by its Q-spectrum, Discrete Math. 309 (2009) 3364–3369.