The Signless Laplacian spectral radius of bicyclic graphs with order n and girth g

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Abstract

In this paper, we determine the unique bicyclic graph with the largest signless Laplacian spectral radius among all the bicyclic graphs with n vertices and a given girth.

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1 Introduction

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$. Denote by $d(v_i)$ the degree of the graph G, $N(v_i)$ the set of vertices which adjacent to vertex v_i. Let $A(G)$ be the adjacency matrix of the graph G. The signless Laplacian matrix of G is defined as $Q(G) = D(G) + A(G)$, where $D(G) = diag(d(v_1), d(v_2) \cdots, d(v_n))$ denotes the diagonal matrix of vertex degrees of G. It is easy to see that $Q(G)$ is a positive semi-definite matrix. Hence the eigenvalues of $Q(G)$ can be ordered as

$$q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0.$$

The largest eigenvalue $q_1(G)$ of $Q(G)$ is called the signless Laplacian spectral radius of the graph G, denoted by $q(G)$. If G is connected, then $Q(G)$ is nonegative irreducible matrix and by the Perron-Frobenius theory of

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non-negative matrices, $q(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $q(G)$. We shall refer to such an eigenvector as the Perron vector of G.

A bicyclic graph is a connected graph in which the number of vertices equals the number of edges minus one. An edge uv of G is called a pendant edge if $d(v) = 1$.

The spectral radius of the adjacency and Laplacian matrix of unicyclic and bicyclic graphs with k pendant vertices has been studied, recently (see [1][2]). And, in [3], M. Zhai, G. L. Yu and J. L. Shu determined the graph with the maximal Laplacian spectral radius among all the bicyclic graphs with given order and girth.

In this paper, we determine the bicyclic graph with the largest signless Laplacian spectral radius among all the bicyclic graphs with order n and girth g.

2 Preliminaries

Denote by C_n and P_n the cycle and the path with n vertices, respectively. Let $G-x$ or $G-xy$ denote the graph obtained from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G+xy$ is a graph that obtained from G by adding an edge xy, where $x, y \in V(G)$ and $xy \notin E(G)$. We will use $B(n, g)$ to denote the set of all bicyclic graphs with order n and girth g.

Let C_p and C_q be two vertex-disjoint cycles. Suppose that v_1 is a vertex of C_p and v_l is a vertex of C_q. Joining v_1 and v_l by a path $v_1v_2\cdots v_l$ of length $l-1$, where $l \geq 1$ and $l = 1$ means identifying v_1 with v_l, denoted by $B(p, l, q)$, is called an $\infty-$ graph (see Fig. 1). Let P_{l+1}, P_{p+1} and P_{q+1} be the three vertex-disjoint paths, where $l, p, q \geq 1$, and at most one of them is 1. Identifying the three initial vertices and the three terminal vertices of them, respectively, denoted by $P(p, q, r)$, is called a $\theta-$ graph (see Fig. 2). Obviously, $B(n, g)$ consists of two types of graph: one type, denoted by $B_1(n, g)$, are those graphs each of which is an $\infty-$ graph with trees attached; the other type, denoted by $B_2(n, g)$, are those graphs each of which is $\theta-$ graph with trees attached. Then we have $B(n, g) = B_1(n, g) \bigcup B_2(n, g)$.

Denote by $B^k_{p,q}$ the graph obtained from $B(p, 1, q)$ by attaching k pendant edges to u, where u is the vertex with degree 4 of $B(p, 1, q)$ (see Fig. 1). Denote by $P^k_{p,q,r}$ the graph obtained from $P(p, q, r)$ by attaching k pendant edges at one of the vertices with degree 3.

Lemma 1([4]). Let G be a simple graph on n vertices. Then

$$min\{d(v_i)+d(v_j)|v_iv_j \in E(G)\} \leq q(G) \leq max\{d(v_i)+d(v_j)|v_iv_j \in E(G)\}.$$

For a connected graph G, equality holds in either of these inequalities if and only if G is regular or semiregular bipartite.

Lemma 2([4,5]). Let G be a connected graph, and u, v be two vertices of G. Suppose that $v_1, v_2, \cdots, v_s \in N(v) \backslash N(u)$ $(1 \leq s \leq d(v))$ and $x = (x_1, x_2, \cdots, x_n)$ is the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $q(G) < q(G^*)$.

By Lemma 2, it is easy to get the following corollary

Corollary 1. Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N(u) \bigcap N(v) = \emptyset$. Let G^* be the graph obtained from G by deleting the edge uv, identifying u with v (Suppose the new vertex is u), and adding a pendant edge to u), then $q(G) < q(G^*)$.

Corollary 2. Let G be connected graph, uu_1, $uu_2, \cdots$, uu_s; vv_1, $vv_2, \cdots$, vv_t be pendant edges of G, where u and v are two different vertices of G. Let $G_1 = G - vv_1 - \cdots - vv_t + uv_1 + \cdots + uv_t$, $G_2 = G - uu_1 - \cdots - uu_s + vu_1 + \cdots + vu_s$. Then we have either $q(G_1) > q(G)$ or $q(G_2) > q(G)$.

Let $m(v_i) = \frac{\sum_{v_i v_j \in E} d(v_j)}{d(v_i)}$ be the average of the degrees of the vertices of G adjacent to v_i, which is called average 2-degree of vertex v_i.

From the proof of Theorem 3 of [6] and Theorem 2.10 of [7], we have the following result.

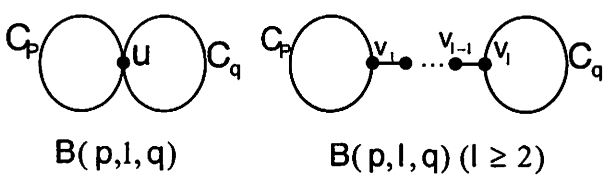
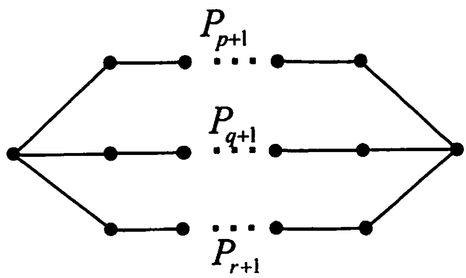


Fig. 1 B(p,1,q) and B(p,l,q) (l ≥ 2)



$P(p,q,r)$

Lemma 3. If G is a graph, then

$$q(G) \leq max\{\frac{d(u)(d(u)+m(u))+d(v)(d(v)+m(v))}{d(u)+d(v)} : uv \in E(G)\},$$

with equality if and only if G is regular or semiregular bipartite.

By similar reasoning as Theorem 2.4 of [7], we have
Lemma 4. Let G be a connected graph, then

$$q(G) \leq max\{d(v_i)+m(v_i)|v_i \in V(G)\},$$

the equality holds if and only if G is either regular or semiregular bipartite.

Lemma 5([4,8]). Let G be a simple graph on n vertices which has at least one edge. Then

$$\triangle(G)+1 \leq q(G) \leq 2\triangle(G),$$

where $\triangle(G)$ is the largest degree of G. Moreover, if G is connected, then the first equality holds if and only if G is the star $K_{1,n-1}$; and the second equality holds if and only if G is a regular graph.

3 Main results

Theorem 1. Let $G \in B_1(n,g)$. Then

$$q(G) \leq q(B_{g,g}^{n-2g+1}) < n-2g+6+\frac{4}{n-2g+5},$$

where $n \geq 2g-1$. The equality holds if and only if $G = B_{g,g}^{n-2g+1}$.

Proof: Choose $G^* \in B_1(n,g)$ such that the signless Laplacian spectral radius is as large as possible.

Let $V(G^*) = \{v_1, v_2, \cdots, v_n\}$ and $X = (x_1, x_2, \cdots, x_n)$ be the Perron vector of G^*, where x_i corresponds to the vertex v_i.

From Corollaries 1 and 2, it is easy to see that G^* is obtained from $B(g,1,q)$ $q \geq g$ by attaching $n-(g+q)+1$ pendant edges $ww_1, ww_2, \cdots, ww_{n-(g+q)+1}$ at some vertex w of $B(g,1,q)$.

Suppose $w \in V(C_q)$ and $w \neq u$. If $x_w \geq x_u$, let $G = G^* - uz_1 - uz_2 + wz_1 + wz_2$, where $uz_1, uz_2 \in E(C_g)$; If $x_w < x_u$, let $G = G^* - ww_1 \cdots - ww_{n-(g+q)+1} + uw_1 + \cdots + uw_{n-(g+q)+1}$. By Lemma 2, $q(G) > q(G^*)$, a contradiction. Hence $w = u$, where u is the vertex with degree 4 of $B(g,1,q)$. By similar reasoning as above, if $w \in V(C_g)$, then $w = u$. From Corollary 1, we have $q = g$ in G^*. By Lemma 4, we have $q(B_{g,g}^{n-2g+1}) < max\{d(v_i) + m(v_i) | v_i \in V(B_{g,g}^{n-2g+1})\} = n - 2g + 6 + \frac{4}{n-2g+5}$. $\square$

Theorem 2. Let $G \in B_2(n,g)$, $n \geq \lceil \frac{3g}{2} \rceil - 1$. Then

$$q(G) \leq q(P_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}^{n-\lceil \frac{3g}{2} \rceil+1}),$$

the equality holds if and only if $G = P_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}^{n-\lceil \frac{3g}{2} \rceil+1}$.

Proof: Choose $G^* \in B_2(n,g)$ such that the signless Laplacian spectral radius is as large as possible. From Corollaries 1 and 2, G^* is obtained from $P(p,q,r)$ by attaching $s = n-(p+q+r)+1$ pendant edges at some vertex, say u, of $P(p,q,r)$. Without loss of generality, assume $p \leq q \leq r$, then $p + q = g$.

Now we prove that $p = \lfloor \frac{g}{2} \rfloor$, $q = r = \lceil \frac{g}{2} \rceil$. If g is even, let $g = 2a$. Then $p + q = g = 2a$, and $n \geq p + q + r - 1 \geq 3a - 1$. Without loss of generality, we distinguish the following three cases.

Case 1. If $n = \frac{3g}{2} - 1 = 3a - 1$, then $p + q + r - 1 = 3a - 1$. So $r = a$, $p = q = a$.

Case 2. If $n = \frac{3g}{2} = 3a$, then $3a \leq p + q + r \leq 3a + 1$. Since $p + q = 2a$ and $p \leq q \leq r$, we have $a \leq r \leq a + 1$. $a \leq q \leq a + 1$ and $a \geq p \geq a - 1$. Hence, $(p,q,r) \in \{(a,a,a), (a,a,a+1), (a-1,a+1,a+1)\}$. If $(p,q,r) = (a-1,a+1,a+1)$, then $G^* = P(a-1,a+1,a+1)$. If $(p,q,r) = (a,a,a+1)$, then $G^* = P(a,a,a+1)$. Suppose that $(p,q,r) = (a,a,a)$. If $a = 2$, by software Matlab, we can get $q(P(2,2,3)) \approx 4.9032 < q(P_{2,2,2}^1) \approx 5.5141$, $q(P(1,3,3)) \approx 5 < q(P_{2,2,2}^1) \approx 5.5141$. If $a \geq 3$, by Lemma 1, we can get $q(G) \leq \{d(v_i) + d(v_j) | v_i v_j \in E(G)\} = 5$, however, $q(P_{a,a,a}^1) > \Delta + 1 = 5$, a contradiction. Hence, $p = q = r = a$.

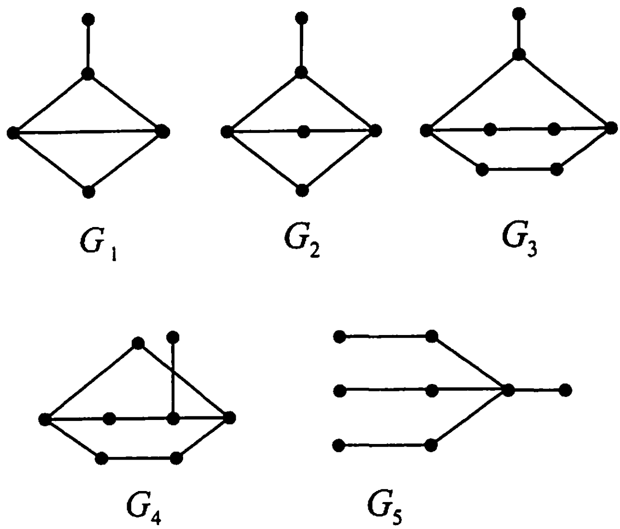


Fig. 3 $G_1 - G_5$

Case 3. Suppose that $n \geq \frac{3g}{2} + 1 = 3a + 1$.

If $r = a$, then $p = q = a$. If $r \geq a + 1$, then the number of pendant edges of G^* is $k = n - (p+q+r) + 1 \leq n - 2a - a - 1 + 1 = n - 3a$. It is easy to see that $max\{d(v_i) + m(v_i) | v_i \in V(G)\}$ attains the maxium just when $p = 1$ and k pendant edges of G^* are adjacent to a vertex with degree 3 of $P(p, q, r)$. By Lemma 4, we have

$$q(G^*) < max\{d(v_i) + m(v_i) | v_i \in V(G)\} = k + 3 + \frac{k+7}{k+3} = k + 4 + \frac{4}{k+3}.$$

Note that $k + 4 + \frac{4}{k+3}$ is increasing with nonnegative number k. Thus $q(G^*) < n - 3a + 5$, since $n \geq 3a + 1$. But, by Lemma 5, $q(B_{a,a,a}^{n-3a+1}) \geq \Delta + 1 = n - 3a + 5 > q(G^*)$, a contradiction. Hence $r = p = q = a$.

If g is odd, by similar reasoning as above, the result follows.

In the end, we prove that u is one of vertices with degree 3 of $P(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil)$. Suppose that u is some vertex with degree 2 of $P(\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil)$.

Assume that $k = n - \lceil \frac{3g}{2} \rceil + 1 = 1$. If $g \in \{3, 4, 5\}$, then G^* is one of graphs G_1, G_2, G_3 and G_4 (see Fig.3). Straigthforward calculations show that $q(G_1) \approx 5.4679 < q(P_{1,2,2}^1) \approx 5.7785$, $q(G_2) \approx 5.2361 < q(P_{2,2,2}^1) \approx 5.5141$, and $q(G_3) \approx 5.0664 < q(P_{2,3,3}^1) \approx 5.3552$, $q(G_4) \approx 4.9891 < q(P_{2,3,3}^1) \approx 5.3552$, a contradiction.

If $g \geq 6$, then by Lemma 3, we have $q(G^*) \leq 5.2$. However, when

$g \geq 6$, G_5 (see Fig.3) is a subgraph of $P^1_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}$. By Matlab, we have $q(P^1_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}) \geq q(G_5) \approx 5.2361 > 5.2$. a contradiction.

Assume that $k \geq 2$. By Lemma 4, We have $q(G^*) < k+4$. However, by Lemma 5, we have $q(P^k_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}) \geq k+4 > q(G^*)$, a contradiction. So $G^* = P^{n-\lceil \frac{3g}{2} \rceil+1}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}$. □

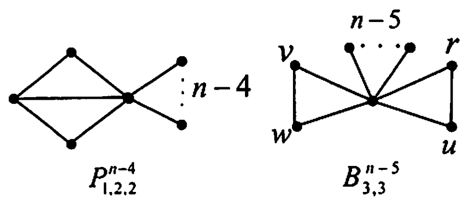


Fig. 4 $P^{n-4}_{1,2,2}$ and $B^{n-5}_{3,3}$

Theorem 3. Let $G \in B(n,g)$. Then

$$q(G) \leq q(P^{n-\lceil \frac{3g}{2} \rceil+1}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}).$$

The equality holds if and only if $G = P^{n-\lceil \frac{3g}{2} \rceil+1}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}$.

Proof: By Theorems 1-2, we only need to prove that $q(B^{n-2g+1}_{g,g}) < q(P^{n-\lceil \frac{3g}{2} \rceil+1}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil})$. Suppose that $g = 3$. Let X be a Perron vector of $B^{n-5}_{3,3}$, where $B^{n-5}_{3,3}$ is the graph shown in Fig. 4.

If $x_r \geq x_w$, let $G^* = B^{n-5}_{3,3} - vw + vr$, where u, w, v, r are vertices of $B^{n-5}_{3,3}$ with degree two, respectively (see Fig. 4); If $x_r < x_w$, let $G^* = B^{n-5}_{3,3} - ur + uw$. It is easy to see that $G^* = P^{n-4}_{1,2,2}$, where $P^{n-4}_{1,2,2}$ is the graph shown in Fig. 4. And by Lemma 2, we know that $q(B^{n-5}_{3,3}) < q(P^{n-4}_{1,2,2})$.

Suppose that $g \geq 4$. If $n < 2g-1$, then $G = P^{n-\lceil \frac{3g}{2} \rceil+1}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}$ (since $B^{n-2g+1}_{g,g}$ doesn't exist); If $n \geq 2g-1$, by Theorem 1, we know

$$q(B^{n-2g+1}_{g,g}) < n - 2g + 6 + \frac{4}{n-2g+5} \leq n - 2g + 7.$$

By Lemma 5, $q(P^{n-\lceil \frac{3g}{2} \rceil+1}_{\lfloor \frac{g}{2} \rfloor, \lceil \frac{g}{2} \rceil, \lceil \frac{g}{2} \rceil}) \geq \Delta + 1 = n - \frac{3g}{2} + 5$. When $g \geq 4$, $n - \frac{3g}{2} +$

$5-(n-2g+7)=\lfloor\frac{g}{2}\rfloor-2\geq 0$. Thus $q(B_{g,g}^{n-2g+1})<q(P_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil}^{n-\lceil\frac{3g}{2}\rceil+1})$. □

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