BI-PERIODIC INCOMPLETE LUCAS SEQUENCES

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ABSTRACT. Motivated by the recent work by Ramírez [8], related to the bi-periodic Fibonacci sequences, here we introduce the bi-periodic incomplete Lucas sequences that gives the incomplete Lucas sequence as a special case. We also give recurrence relations and the generating function of these sequences. Also, we give a relation between bi-periodic incomplete Fibonacci sequences and bi-periodic incomplete Lucas sequences.

1. INTRODUCTION

The Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$.

These numbers are famous for possessing wonderful properties. In particular, there is a combinatorial identity for Fibonacci numbers and Lucas numbers; see, e.g., [5]

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} \tag{1}$$

$$L_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i}. \tag{2}$$

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In analogy with (1) and (2), the incomplete Fibonacci and incomplete Lucas numbers were introduced by Filipponi [3], as follow

$$F_n(s) = \sum_{i=0}^{s} \binom{n-1-i}{i}, \quad n = 1,2,3,...; 0 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$L_n(s) = \sum_{i=0}^{s} \frac{n}{n-i} \binom{n-i}{i}, \quad n = 1,2,3,...; 0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Note that in the case of $s = \left\lfloor \frac{n-1}{2} \right\rfloor$, incomplete Fibonacci numbers reduce to Fibonacci numbers and in the case of $s = \left\lfloor \frac{n}{2} \right\rfloor$, incomplete Lucas numbers reduce to Lucas numbers. The generating functions of the incomplete Fibonacci numbers and the incomplete Lucas numbers were given by Pintér and Srivastava [7].

There are a lot of generalizations of Fibonacci numbers and Lucas numbers. Edson and Yayenie [2] introduced a generalization of Fibonacci numbers defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2 \tag{3}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero numbers. If we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence. Yayenie [10] gave an explicit formula of q_n:

$$q_n = a^{\xi(n-1)} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - i} \tag{4}$$

where $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Ramírez [8] defined the bi-periodic incomplete Fibonacci numbers as:

$$q_n(l) = a^{\xi(n-1)} \sum_{i=0}^{l} \binom{n-1-i}{i} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - i}, \quad 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor \tag{5}$$

by using (4), and gave the generating function for this sequence.

Similar to (3), by taking initial conditions $p_0 = 2$ and $p_1 = a$, Bilgici [1] introduced the bi-periodic Lucas numbers as:

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2. \tag{6}$$

It should also be noted that, it gives the classical Lucas sequence in the case of $a = b = 1$ in $\{p_n\}$. See also [4] and [6], for another generalization of Lucas numbers.

Following Ramírez's suggestion [8], we introduce the bi-periodic incomplete Lucas sequences, and give recurrence relations (13)-(14), the generating function (17), and the relation between bi-periodic incomplete Fibonacci sequences and bi-periodic incomplete Lucas sequences (16).

2. Bi-Periodic Incomplete Lucas Sequence

Lemma 2.1. *For $n \geq 1$, the explicit formula to bi-periodic Lucas numbers is*

$$p_n = a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i}. \tag{7}$$

Proof. We prove it by induction on n. It is clear that the result is true when $n = 1$. Assume that it is true for any m such that $1 \leq m \leq n$. Then by induction assumption, we get

$$\begin{aligned}
p_{n+1} &= a^{1-\xi(n)} b^{\xi(n)} p_n + p_{n-1} \\
&= a^{1-\xi(n)} b^{\xi(n)} a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} \\
&\quad + a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} \\
&= a^{1-\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i + \xi(n)} \\
&\quad + a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} \\
&= a^{\xi(n+1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i} \\
&\quad + a^{\xi(n+1)} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n-1}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i - 1} \\
&= a^{\xi(n+1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i} \\
&\quad + a^{\xi(n+1)} \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n-1}{n-i} \binom{n-i}{i-1} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i}
\end{aligned}$$

$$= a^{\xi(n+1)}\left((ab)^{\lfloor \frac{n+1}{2} \rfloor} + 2\xi(n)\right)$$

$$+a^{\xi(n+1)} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left[\frac{n}{n-i}\binom{n-i}{i} + \frac{n-1}{n-i}\binom{n-i}{i-1}\right] (ab)^{\lfloor \frac{n+1}{2} \rfloor - i}$$

$$= a^{\xi(n+1)}\left((ab)^{\lfloor \frac{n+1}{2} \rfloor} + 2\xi(n)\right)$$

$$+a^{\xi(n+1)} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n+1}{n+1-i}\binom{n+1-i}{i} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i}$$

$$= a^{\xi(n+1)} \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n+1}{n+1-i}\binom{n+1-i}{i} (ab)^{\lfloor \frac{n+1}{2} \rfloor - i}.$$

Thus, the given formula is true for any positive integer n. □

By using Lemma 2.1, we can give the definition of bi-periodic incomplete Lucas numbers.

Definition 2.2. *Let n be a positive integer and l be an integer such that $0 \le l \le \lfloor \frac{n}{2} \rfloor$. The bi-periodic incomplete Lucas numbers are defined as*

$$p_n(l) = a^{\xi(n)} \sum_{i=0}^{l} \frac{n}{n-i}\binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} . \tag{8}$$

For $a = b = 1$, we obtain the incomplete Lucas numbers.

By (8), it is easy to see that

$$p_n(0) = a^{\xi(n)} (ab)^{\lfloor \frac{n}{2} \rfloor} \quad , n \ge 1 \tag{9}$$

$$p_n(1) = a^{\xi(n)} (ab)^{\lfloor \frac{n}{2} \rfloor} + a^{\xi(n)} n (ab)^{\lfloor \frac{n}{2} \rfloor - 1} \quad , n \ge 2 \tag{10}$$

$$p_n\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = p_n \quad , n \ge 1 \tag{11}$$

$$p_n\left(\left\lfloor \frac{n-2}{2} \right\rfloor\right) = \begin{cases} p_n - 2, & \text{if } n \text{ is even} \\ p_n - na, & \text{if } n \text{ is odd} \end{cases} \quad , n \ge 2. \tag{12}$$

Example 2.3. *For $a = 3$, $b = 2$ and $1 \le n \le 10$, the terms of the $p_n(l)$ are listed in the following table.*

n/l	0	1	2	3	4	5
1	3					
2	6	8				
3	18	27				
4	36	60	62			
5	108	198	213			
6	216	432	486	488		
7	648	1404	1656	1677		
8	1296	3024	3744	3840	3842	
9	3888	9720	12636	13176	13203	
10	7776	20736	28296	30096	30246	30248

Proposition 2.4. *The non-linear recurrence relation of the bi-periodic incomplete Lucas numbers $p_n(l)$ is*

$$p_{n+2}(l+1) = \begin{cases} bp_{n+1}(l+1) + p_n(l), & \text{if } n \text{ is even} \\ ap_{n+1}(l+1) + p_n(l), & \text{if } n \text{ is odd} \end{cases}, 0 \leq l \leq \frac{n-1}{2} \tag{13}$$

The relation (13) can be transformed into the non-homogeneous recurrence relation

$$p_{n+2}(l) = \begin{cases} bp_{n+1}(l) + p_n(l) - \frac{n}{n-l}\binom{n-l}{l}(ab)^{\lfloor\frac{n}{2}\rfloor-l}, & \text{if } n \text{ is even} \\ ap_{n+1}(l) + p_n(l) - a\frac{n}{n-l}\binom{n-l}{l}(ab)^{\lfloor\frac{n}{2}\rfloor-l}, & \text{if } n \text{ is odd.} \end{cases} \tag{14}$$

Proof. If n is even, then $\lfloor\frac{n+1}{2}\rfloor = \lfloor\frac{n}{2}\rfloor$. By using the Definition 2.2, we can write the RHS of (13) as

$$ba^{\xi(n+1)}\sum_{i=0}^{l+1}\frac{n+1}{n-i+1}\binom{n-i+1}{i}(ab)^{\lfloor\frac{n+1}{2}\rfloor-i}$$

$$+a^{\xi(n)}\sum_{i=0}^{l}\frac{n}{n-i}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}$$

$$=\sum_{i=0}^{l+1}\frac{n+1}{n-i+1}\binom{n-i+1}{i}(ab)^{\lfloor\frac{n+1}{2}\rfloor+1-i}+\sum_{i=0}^{l}\frac{n}{n-i}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}$$

$$=\sum_{i=0}^{l+1}\frac{n+1}{n-i+1}\binom{n-i+1}{i}(ab)^{\lfloor\frac{n}{2}\rfloor+1-i}$$

$$+\sum_{i=1}^{l+1}\frac{n}{n-i+1}\binom{n-i+1}{i-1}(ab)^{\lfloor\frac{n}{2}\rfloor-i+1}$$

$$= \sum_{i=0}^{l+1} \left[\frac{n+1}{n-i+1} \binom{n-i+1}{i} + \frac{n}{n-i+1} \binom{n-i+1}{i-1} \right] (ab)^{\lfloor \frac{n}{2} \rfloor + 1 - i} - 0$$

$$= \sum_{i=0}^{l+1} \frac{n+2}{n-i+2} \binom{n-i+2}{i} (ab)^{\lfloor \frac{n}{2} \rfloor + 1 - i}$$

$$= p_{n+2}(l+1).$$

If n is odd, then $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + 1$. The proof is analogous.

Equation (14) is clear from (13). In fact, if n is even

$$\begin{aligned} p_{n+2}(l) &= bp_{n+1}(l) + p_n(l-1) \\ &= bp_{n+1}(l) + p_n(l) + (p_n(l-1) - p_n(l)) \\ &= bp_{n+1}(l) + p_n(l) - \frac{n}{n-l} \binom{n-l}{l} (ab)^{\lfloor \frac{n}{2} \rfloor - l}. \end{aligned}$$

If n is odd, the proof is completely analogous. □

Proposition 2.5. *Let* $h = \lfloor \frac{n}{2} \rfloor$, *then*

$$\sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} p_n(l) = (h+1) p_n - a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} i \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i}. \tag{15}$$

Proof.

$$\sum_{l=0}^{h} p_n(l) = p_n(0) + p_n(1) + \cdots + p_n(h)$$

$$\begin{aligned} &= a^{\xi(n)} \frac{n}{n-0} \binom{n-0}{0} (ab)^{h-0} \\ &\quad + a^{\xi(n)} \left[\frac{n}{n-0} \binom{n-0}{0} (ab)^{h-0} + \frac{n}{n-1} \binom{n-1}{1} (ab)^{h-1} \right] + \cdots \\ &\quad + a^{\xi(n)} \left[\frac{n}{n-0} \binom{n-0}{0} (ab)^{h} + \cdots + \frac{n}{n-h} \binom{n-h}{h} (ab)^{h-h} \right] \\ &= a^{\xi(n)} \left[\frac{(h+1)n}{n-0} \binom{n-0}{0} (ab)^{h} + \frac{hn}{n-1} \binom{n-1}{1} (ab)^{h-1} + \cdots \right. \\ &\quad \left. + \frac{n}{n-h} \binom{n-h}{h} (ab)^{h-h} \right] \\ &= a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (h+1-i) \frac{n}{n-i} \binom{n-i}{i} (ab)^{h-i} \\ &= (h+1) a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{h-i} \end{aligned}$$

$$-a^{\xi(n)}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} i\frac{n}{n-i}\binom{n-i}{i}(ab)^{h-i}$$

$$=(h+1)\,p_n - a^{\xi(n)}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} i\frac{n}{n-i}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}.$$

□

Now, we give a relation between bi-periodic incomplete Fibonacci numbers and bi-periodic incomplete Lucas numbers.

Proposition 2.6. *For* $1\le l\le\lfloor\frac{n}{2}\rfloor$,

$$p_n(l)=q_{n-1}(l-1)+q_{n+1}(l). \tag{16}$$

Proof. By (5) and (8), we get

$$q_{n-1}(l-1)+q_{n+1}(l)$$

$$=a^{\xi(n-2)}\sum_{i=0}^{l-1}\binom{n-2-i}{i}(ab)^{\lfloor\frac{n-2}{2}\rfloor-i}+a^{\xi(n)}\sum_{i=0}^{l}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}$$

$$=a^{\xi(n)}\sum_{i=0}^{l-1}\binom{n-2-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-1-i}+a^{\xi(n)}\sum_{i=0}^{l}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}$$

$$=a^{\xi(n)}\sum_{i=1}^{l}\binom{n-1-i}{i-1}(ab)^{\lfloor\frac{n}{2}\rfloor-i}+a^{\xi(n)}\sum_{i=0}^{l}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}$$

$$=a^{\xi(n)}\sum_{i=0}^{l}\left[\binom{n-1-i}{i-1}+\binom{n-i}{i}\right](ab)^{\lfloor\frac{n}{2}\rfloor-i}-0$$

$$=a^{\xi(n)}\sum_{i=0}^{l}\frac{n}{n-i}\binom{n-i}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-i}$$

$$=p_n(l).$$

□

Note that, if we take $l=\lfloor\frac{n}{2}\rfloor$, above result reduce to

$$p_n=q_{n-1}+q_{n+1}$$

which is given in [1, Theorem 3].

3. Generating function of the bi-periodic incomplete Lucas numbers

We need the following lemma from [8, Lemma 3.1].

Lemma 3.1. [8, Lemma 3.1] *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following non-homogeneous and non-linear recurrence relation*

$$s_n = \begin{cases} bs_{n-1} + s_{n-2} + r_n, & \text{if } n \text{ is even} \\ as_{n-1} + s_{n-2} + ar_n, & \text{if } n \text{ is odd} \end{cases}, n > 1$$

where a and b are complex numbers and $\{r_n\}_{n=0}^{\infty}$ is a given complex sequence. Then the generating function of $U(t)$ of the sequence $\{s_n\}_{n=0}^{\infty}$ is

$$U(t) = \frac{aG(t) + s_0 - r_0 + (s_1 - a(s_0 + r_1))t + (b-a)tf(t) + (1-a)R(t)}{1 - at - t^2}$$

where $G(t)$ denotes the generating function of $\{r_n\}_{n\geq 0}$, $f(t)$ denotes the generating function of $\{s_{2n+1}\}_{n=0}^{\infty}$ and $R(t)$ denotes the generating function of $\{r_{2n}\}_{n=0}^{\infty}$. Moreover,

$$f(t) = \frac{(s_1 - a(r_0 + r_1))t - (s_1 - a(s_0 + r_1))t^3 + atR(t) + a\left(1 - t^2\right)R'(t)}{1 - (ab+2)t^2 + t^4}$$

where $R'(t)$ denotes the generating function of $\{r_{2n-1}\}_{n=1}^{\infty}$.

Theorem 3.2. *The generating function of the bi-periodic incomplete Lucas numbers $p_n(l)$ is given by*

$$\begin{aligned} P_l(t) &= \sum_{n=0}^{\infty} p_n(l)\,t^n \\ &= \frac{aG(t) + p_{2l} + p_{2l-1}t + (b-a)tf(t) + (1-a)R(t)}{1 - at - t^2} \end{aligned} \tag{17}$$

where

$$\begin{aligned} G(t) &= R(t) + R'(t) \\ &= -\frac{t^2}{2}\left(\frac{(2-t) + (2-abt)(ab)^{\frac{1}{2}}}{\left(1 - (ab)^{\frac{1}{2}}t\right)^{l+1}} + \frac{(2-t) - (2-abt)(ab)^{\frac{1}{2}}}{\left(1 + (ab)^{\frac{1}{2}}t\right)^{l+1}}\right) \end{aligned}$$

$$\begin{aligned} f(t) &= \frac{p_{2l+1}t - (p_{2l+1} - ap_{2l})t^3 + atR(t) + a\left(1 - t^2\right)R'(t)}{1 - (ab+2)t^2 + t^4} \\ &= \frac{p_{2l+1}t - p_{2l-1}t^3 + atR(t) + a\left(1 - t^2\right)R'(t)}{1 - (ab+2)t^2 + t^4} \end{aligned}$$

and

$$R(t) = -\frac{(2-t)}{2}\left(\frac{t^2}{\left(1-(ab)^{\frac{1}{2}}t\right)^{l+1}}+\frac{t^2}{\left(1+(ab)^{\frac{1}{2}}t\right)^{l+1}}\right)$$

$$R'(t) = -\frac{(2-abt)}{2(ab)^{\frac{1}{2}}}\left(\frac{t^2}{\left(1-(ab)^{\frac{1}{2}}t\right)^{l+1}}-\frac{t^2}{\left(1+(ab)^{\frac{1}{2}}t\right)^{l+1}}\right)$$

Proof. Let l be a fixed positive integer. From (8) and (14),

$$\begin{aligned} p_n(l) &= 0, \text{ for } 0 \le n < 2l, \\ p_{2l}(l) &= p_{2l},\ p_{2l+2}(l) = p_{2l+2} \end{aligned}$$

and

$$p_n(l) = \begin{cases} bp_{n-1}(l) + p_{n-2}(l) - \frac{n-2}{n-l-2}\binom{n-l-2}{l}(ab)^{\left\lfloor\frac{n-2}{2}\right\rfloor - l}, & \text{if } n \text{ is even} \\ ap_{n-1}(l) + p_{n-2}(l) - a\frac{n-2}{n-l-2}\binom{n-l-2}{l}(ab)^{\left\lfloor\frac{n-2}{2}\right\rfloor - l}, & \text{if } n \text{ is odd} \end{cases}$$

Now let

$$s_0 = p_{2l}(l) = p_{2l},\ s_1 = p_{2l+1}(l) = p_{2l+1},\ s_n = p_{2l+n}(l)$$

and

$$r_0 = r_1 = 0,\ r_n = \frac{n+2l-2}{n+l-2}\binom{n+l-2}{n-2}(ab)^{\left\lfloor\frac{n}{2}\right\rfloor - 1}.$$

The generating function of the sequence $\{-r_n\}$ is

$$G(t) = -\frac{t^2}{2}\left(\frac{(2-t)+(2-abt)(ab)^{\frac{1}{2}}}{\left(1-(ab)^{\frac{1}{2}}t\right)^{l+1}}+\frac{(2-t)-(2-abt)(ab)^{\frac{1}{2}}}{\left(1+(ab)^{\frac{1}{2}}t\right)^{l+1}}\right).$$

Thus from [8, Lemma 3.1], we obtain the generating function $P_l(t)$ of the sequence $\{p_n(l)\}_{n=0}^{\infty}$. □

For the theory and application of the various methods and techniques for deriving generating functions of special functions, we may refer to the reader to a recent treatise on the subject of generating function by Srivastava and Monacha [9].

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