Total vertex irregularity strength of certain equitable complete m-partite graphs*

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Abstract

For a simple undirected graph G with vertex set V and edge set E, a total $k-$labeling $\lambda : V \cup E \to \{1, 2, \cdots, k\}$ is called a vertex irregular total $k-$labeling of G if for every two distinct vertices x and y of G their weights $wt(x)$ and $wt(y)$ are distinct where the weight of a vertex x in G is the sum of the label of x and the labels of all edges incident with the vertex x. The total vertex irregularity strength of G, denoted by $tvs(G)$, is the minimum k for which the graph G has a vertex irregular total $k-$labeling. The complete $m-$partite graph on n vertices in which each part has either $\lfloor \frac{n}{m} \rfloor$ or $\lceil \frac{n}{m} \rceil$ vertices is denoted by $T_{m,n}$. The total vertex irregularity strength of some equitable complete $m-$partite graphs, namely, $T_{m,m+1}$, $T_{m,m+2}$, $T_{m,2m}$, $T_{m,2m+1}$, $T_{m,3m-1}$ $(m \geq 4)$, $T_{m,n}$ $(n = 3m + r, r = 1, 2, \cdots, m-1)$, and equitable complete 3$-$partite graphs have been studied in this paper.

Key Words: vertex irregular total $k-$labeling; weight; total vertex irregularity strength; equitable complete 3$-$partite graph.

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1 Introduction and Preliminaries

Let us consider a simple (without loops and multiple edges) undirected graph $G = (V, E)$ with vertex set V and edge set E. For a graph G, we define a labeling $\lambda : V \cup E \to \{1, 2, \cdots, k\}$ to be a total $k-$labeling. A total $k-$labeling λ is defined to be an edge irregular total $k-$labeling of the graph G if for every two different edges xy and $x'y'$ their weights

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$\lambda(x)+\lambda(xy)+\lambda(y)$ and $\lambda(x')+\lambda(x'y')+\lambda(y')$ are distinct. Similarly, a total $k-$labeling λ is defined to be a vertex irregular total $k-$labeling of G if for every two distinct vertices x and y of G their weights $wt(x)$ and $wt(y)$ are distinct. Here, the weight $wt(x)$ in G is the sum of the label of x and the labels of all edges incident with the vertex x.

The minimum k for which the graph G has an edge irregular total $k-$labeling is called the total edge irregularity strength of G,denoted by $tes(G)$. Analogously, the minimum k for which the graph G has a vertex irregular total $k-$labeling is called the total vertex irregularity strength of G,denoted by $tvs(G)$. Combining both of these notions, C. C. Marzuki et al. in [3] introduced a new irregular total $k-$labeling of a graph G called totally irregular total $k-$labeling. The minimum k for which a graph G has a totally irregular total $k-$labeling is called the total irregularity strength of G, denoted by $ts(G)$, which is required to be at the same time both vertex and edge irregular.

A complete $m-$partite graph is a simple graph whose vertex set can be partitioned into m non-empty subsets and in which each vertex is adjacent to every vertex that is not in the same subset. The m non-empty subsets are called the parts of the complete $m-$partite graph. The complete $m-$partite graph on n vertices in which each part has either $\lceil \frac{n}{m} \rceil$ or $\lfloor \frac{n}{m} \rfloor$ vertices is denoted by $T_{m,n}$.

Let $n = mq + r$, $0 \leq r \leq m-1$. We suppose that the m parts of $T_{m,n}$ are V_1, V_2,$\cdots$,V_m, where $V_s = \{u_i^{(s)} | i = 1, 2, \cdots, \frac{n}{m}\}$ when $r = 0$ and $V_s = \{u_i^{(s)} | i = 1, 2, \cdots, \lceil \frac{n}{m} \rceil\}$, $s = 1, 2, \cdots, r$ and $V_s = \{u_i^{(s)} | i = 1, 2, \cdots, \lfloor \frac{n}{m} \rfloor\}$, $s = r+1, r+2, \cdots, m$ when $1 \leq r \leq m-1$.

Obviously, the degree of each vertex in $V(T_{m,n})$ is either δ or Δ, and Δ=δ or Δ=δ+1.

The vertex irregular total labeling and the edge irregular total labeling were introduced by M. Bača et al. in [4]. The total vertex irregularity strength of complete bipartite graphs $K_{m,n}$ for some m and n had been found by K. Wijaya et al. in [9], namely, $K_{2,n}$, $K_{n,n}$, $K_{n,n+1}$, $K_{n,n+2}$, and $K_{n,an}$. K. Wijaya and S. Slamin also showed in [8] that the values of total vertex irregularity strength of wheels W_n, fans F_n, suns M_n and friendship graphs f_n. The total vertex irregularity strength of Jahangir graphs $J_{n,2}$ for $n \geq 4$ and several types of trees were proved in [1, 5, 6]. Furthermore, the total vertex irregularity strength of several disjoint union of t copies of path with order n had been determined by N. Nurdin et al. in [7]. Recently, A. Ahmad and M. Bača determined the total vertex irregularity strength of several disjoint union of $n-$cycles in [2].

Lemma 1.1 [4]. *Let a graph G have minimum degree δ and maximum degree Δ, then* $\lceil \frac{|V|+\delta}{\Delta+1} \rceil \leq tvs(G) \leq |V| + \Delta - 2\delta + 1$.

In this paper, we determine the total vertex irregularity strength of some equitable complete $m-$partite graphs $T_{m,m+1}$, $T_{m,m+2}$, $T_{m,2m}$, $T_{m,2m+1}$, $T_{m,3m-1}$ $(m \geq 4)$, $T_{m,n}$ $(n = 3m + r, r = 1, 2, \cdots, m-1)$ and $T_{3,n}$.

2 Main Results

Note that the weight $wt(v)$ is either the sum of $\delta+1$ numbers or the sum of $\Delta+1$ numbers in $\{1,2,\cdots,k\}$ for each $v\in V(T_{m,n})$ when we consider the vertex irregular total k-labeling of $T_{m,n}$.

Theorem 2.1 *Let $m\geq 3$, then $tvs(T_{m,m+1})=2$.*

Proof: It is easy to see that $tvs(T_{m,m+1})\geq 2$. To show that $tvs(T_{m,m+1})=2$ for $m\geq 3$, we need only to give a vertex irregular total 2-labeling λ of $T_{m,m+1}$ in the following.

For $m=3$, let $\lambda(u_i^{(1)})=1$ for $i=1,2$; $\lambda(u_1^{(s)})=1$ for $s=2,3$; $\lambda(u_1^{(1)}u_1^{(s)})=1$ for $s=2,3$; $\lambda(u_2^{(1)}u_1^{(t)})=t-1$ for $t=2,3$; $\lambda(u_1^{(2)}u_1^{(3)})=2$. Thus, the weights of all vertices of $T_{3,4}$ are 3, 4, 5, 6. So $tvs(T_{3,4})=2$.

For $m=4$, let $\lambda(u_i^{(1)})=1$ for $i=1,2$; $\lambda(u_1^{(s)})=2$ for $s=2,3,4$; $\lambda(u_1^{(1)}u_1^{(t)})=1$ for $t=2,3,4$; $\lambda(u_2^{(1)}u_1^{(t)})=1$ for $t=2,3$; $\lambda(u_2^{(1)}u_1^{(4)})=2$; $\lambda(u_1^{(2)}u_1^{(t)})=1$ for $t=3,4$; $\lambda(u_1^{(3)}u_1^{(4)})=2$. Thus, the weights of all vertices of $T_{4,5}$ are 4, 5, 6, 7, 8. So $tvs(T_{4,5})=2$.

For $m\geq 5$, we consider the two cases depending on values of m.

Case 1. $m\geq 5$ and $m\equiv 1 \pmod 2$.

Let $\lambda(u_i^{(1)})=1$ for $i=1,2$; $\lambda(u_1^{(2)})=2$; $\lambda(u_1^{(s)})=1$ for $3\leq s\leq \frac{m+1}{2}$; $\lambda(u_1^{(s)})=2$ for $\frac{m+3}{2}\leq s\leq m$; $\lambda(u_1^{(1)}u_1^{(t)})=1$ for $t=2,3,\cdots,m$; $\lambda(u_2^{(1)}u_1^{(t)})=1$ for $t=2,3,\cdots,m-1$; $\lambda(u_2^{(1)}u_1^{(m)})=2$; $\lambda(u_1^{(2)}u_1^{(t)})=1$ for $t=3,4,\cdots,m$; $\lambda(u_1^{(s)}u_1^{(t)})=1$ for $3\leq s\leq\frac{m-1}{2}$, $s<t\leq m-s+1$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $3\leq s\leq\frac{m-1}{2}$, $m-s+2\leq t\leq m$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $\frac{m+1}{2}\leq s<t\leq m$. Thus, the weights of all vertices of $T_{m,m+1}$ are $m,m+1,\cdots,2m$. So, $tvs(T_{m,m+1})=2$.

Case 2. $m\geq 6$ and $m\equiv 0 \pmod 2$.

Let $\lambda(u_i^{(1)})=1$ for $i=1,2$; $\lambda(u_1^{(2)})=2$; $\lambda(u_1^{(s)})=1$ for $3\leq s\leq\frac{m}{2}$; $\lambda(u_1^{(s)})=2$ for $\frac{m}{2}+1\leq s\leq m$; $\lambda(u_1^{(1)}u_1^{(t)})=1$ for $t=2,3,\cdots,m$; $\lambda(u_2^{(1)}u_1^{(t)})=1$ for $t=2,3,\cdots,m-1$; $\lambda(u_2^{(1)}u_1^{(m)})=2$; $\lambda(u_1^{(2)}u_1^{(t)})=1$ for $t=3,4,\cdots,m$; $\lambda(u_1^{(s)}u_1^{(t)})=1$ for $3\leq s\leq\frac{m}{2}$, $s<t\leq m-s+1$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $3\leq s\leq\frac{m}{2}$, $m-s+2\leq t\leq m$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $\frac{m}{2}+1\leq s<t\leq m$. Thus, the weights of all vertices of $T_{m,m+1}$ are $m,m+1,\cdots,2m$. So, $tvs(T_{m,m+1})=2$.

Theorem 2.2 *Let $m\geq 3$, then $tvs(T_{m,m+2})=2$.*

Proof: Obviously, we have $tvs(T_{m,m+2})\geq 2$. To show that $tvs(T_{m,m+2})=2$ for $m\geq 3$, we need only to give a vertex irregular total 2-labeling λ of $T_{m,m+2}$ as follows.

For $m=3$, let $\lambda(u_i^{(s)})=s$ for $i=1,2$ and $s=1,2$; $\lambda(u_1^{(3)})=2$; $\lambda(u_1^{(1)}u_j^{(2)})=1$ for $j=1,2$; $\lambda(u_2^{(1)}u_j^{(2)})=j$ for $j=1,2$; $\lambda(u_i^{(1)}u_1^{(3)})=1$ for

$i=1,2$; $\lambda(u_i^{(2)}u_1^{(3)})=2$ for $i=1,2$. Thus, the weights of all vertices of $T_{3,5}$ are 4, 5, 6, 7, 8. So $tvs(T_{3,5})=2$.

For $m=4$, let $\lambda(u_i^{(s)})=i$ for $i=1,2, s=1,2$; $\lambda(u_1^{(s)})=s-2$ for $s=3,4$; $\lambda(u_i^{(1)}u_j^{(2)})=1$ for $i,j=1,2$; $\lambda(u_i^{(1)}u_1^{(t)})=1$ for $i=1,2, t=3,4$; $\lambda(u_i^{(2)}u_1^{(t)})=2$ for $i=1,2, t=3,4$; $\lambda(u_1^{(3)}u_1^{(4)})=2$. Thus, the weights of all vertices of $T_{4,6}$ are 5, 6, 7, 8, 9, 10. So $tvs(T_{4,6})=2$.

For $m=5$, let $\lambda(u_i^{(1)})=i$ for $i=1,2,$; $\lambda(u_i^{(2)})=1$ for $i=1,2,$; $\lambda(u_1^{(s)})=2$ for $s=3,4,5$; $\lambda(u_i^{(1)}u_j^{(2)})=1$ for $i,j=1,2$; $\lambda(u_i^{(1)}u_1^{(t)})=1$ for $i=1,2, t=3,4$; $\lambda(u_i^{(1)}u_1^{(5)})=i$ for $i=1,2$; $\lambda(u_i^{(2)}u_1^{(t)})=i$ for $i=1,2, t=3,5$; $\lambda(u_i^{(2)}u_1^{(4)})=2$ for $i=1,2$; $\lambda(u_1^{(3)}u_1^{(4)})=1$; $\lambda(u_1^{(s)}u_1^{(5)})=2$ for $s=3,4$. Thus, the weights of all vertices of $T_{5,7}$ are 6, 7, 8, 9, 10,11, 12. So $tvs(T_{5,7})=2$.

For $m=6$, let $\lambda(u_i^{(s)})=i$ for $i=1,2$ and $s=1,2$; $\lambda(u_1^{(s)})=2$ for $s=3,4,5,6$; $\lambda(u_i^{(1)}u_j^{(2)})=1$ for $i,j=1,2$; $\lambda(u_1^{(1)}u_1^{(t)})=1$ for $t=3,4,5,6$; $\lambda(u_2^{(1)}u_1^{(t)})=1$ for $t=3,4,5$; $\lambda(u_2^{(1)}u_1^{(6)})=2$; $\lambda(u_1^{(2)}u_1^{(t)})=1$ for $t=3,4,6$; $\lambda(u_1^{(2)}u_1^{(5)})=2$; $\lambda(u_2^{(2)}u_1^{(t)})=1$ for $t=3,5$; $\lambda(u_2^{(2)}u_1^{(t)})=2$ for $t=4,6$; $\lambda(u_1^{(3)}u_1^{(4)})=1$; $\lambda(u_1^{(3)}u_1^{(t)})=2$ for $t=5,6$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $4\le s<t\le 6$. Thus, the weights of all vertices of $T_{6,8}$ are $7,8,\cdots,14$. So $tvs(T_{6,8})=2$.

For $m\ge 7$, we consider the two cases depending on values of m.

Case 1. $m\ge 7$ and $m\equiv 1 \pmod 2$.

Let $\lambda(u_i^{(s)})=i$ for $i=1,2$, $s=1,2$; $\lambda(u_1^{(s)})=1$ for $s=3,4,\cdots,\frac{m-1}{2}$, $m-1$; $\lambda(u_1^{(s)})=2$ for $s=\frac{m+1}{2},\frac{m+3}{2},\cdots,m-2$, m; $\lambda(u_i^{(1)}u_j^{(2)})=1$ for $i,j=1,2$; $\lambda(u_i^{(1)}u_1^{(t)})=1$ for $i=1,2$, $t=3,4,\cdots,m$; $\lambda(u_i^{(2)}u_1^{(t)})=1$ for $i=1,2$, $t=3,4,\cdots,m-2$; $\lambda(u_i^{(2)}u_1^{(t)})=2$ for $i=1,2$, $t=m-1,m$; $\lambda(u_1^{(s)}u_1^{(t)})=1$ for $3\le s\le\frac{m-1}{2}$, $s<t\le m-s$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $3\le s\le\frac{m-1}{2}$, $m-s+1\le t\le m$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $\frac{m+1}{2}\le s<t\le m$. Thus, the weights of all vertices of $T_{m,m+2}$ are $m+1,m+2,\cdots,2m+2$. So, $tvs(T_{m,m+2})=2$.

Case 2. $m\ge 8$ and $m\equiv 0 \pmod 2$.

Let $\lambda(u_i^{(s)})=i$ for $i=1,2$, $s=1,2$; $\lambda(u_1^{(s)})=1$ for $s=3,4,\cdots,\frac{m}{2}$, $m-1$; $\lambda(u_1^{(s)})=2$ for $s=\frac{m}{2}+1,\frac{m}{2}+2,\cdots,m-2$, m; $\lambda(u_i^{(1)}u_j^{(2)})=1$ for $i,j=1,2$; $\lambda(u_i^{(1)}u_1^{(t)})=1$ for $i=1,2$, $t=3,4,\cdots,m$; $\lambda(u_i^{(2)}u_1^{(t)})=1$ for $i=1,2$, $t=3,4,\cdots,m-2$; $\lambda(u_i^{(2)}u_1^{(t)})=2$ for $i=1,2$, $t=m-1,m$; $\lambda(u_1^{(s)}u_1^{(t)})=1$ for $3\le s\le\frac{m}{2}-1$, $s<t\le m-s$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $3\le s\le\frac{m}{2}-1$, $m-s+1\le t\le m$; $\lambda(u_1^{(s)}u_1^{(t)})=2$ for $\frac{m}{2}\le s<t\le m$. Thus, the weights of all vertices of $T_{m,m+2}$ are $m+1,m+2,\cdots,2m+2$. So, $tvs(T_{m,m+2})=2$.

Theorem 2.3 *Let $m\ge 3$, then $tvs(T_{m,2m})=2$.*

Proof: Obviously, we have $\delta(T_{m,2m}) = \Delta(T_{m,2m}) = 2m-2$. To show that $tvs(T_{m,2m}) = 2$, we need only to give a vertex irregular total 2–labeling λ of $T_{m,2m}$ as follows.

Let $\lambda(u_i^{(s)}) = i$ for $i = 1,2$ and $s = 1,2,\cdots,m$; $\lambda(u_i^{(s)}u_j^{(t)}) = i$ for $i,j = 1,2$ and $1 \le s < t \le m$. Thus, the weights of all vertices of $T_{m,2m}$ are $2m-1, 2m, \cdots, 4m-3, 4m-2$. So $tvs(T_{m,2m}) = 2$.

Theorem 2.4 *Let $m \ge 3$, then $tvs(T_{m,2m+1}) = 2$.*

Proof: Obviously, we need only to give a vertex irregular total 2–labeling λ of $T_{m,2m+1}$ for $m \ge 3$.

For $m = 3$, let $\lambda(u_i^{(1)}) = 1$ for $i = 1,2$; $\lambda(u_3^{(1)}) = 2$; $\lambda(u_i^{(2)}) = i$ for $i = 1,2$; $\lambda(u_i^{(3)}) = i$ for $i = 1,2$; $\lambda(u_1^{(1)}u_j^{(t)}) = j$ for $j = 1,2$ and $t = 2,3$; $\lambda(u_i^{(1)}u_j^{(2)}) = j$ for $i = 2,3$ and $j = 1,2$; $\lambda(u_i^{(1)}u_j^{(3)}) = 2$ for $i = 2,3$ and $j = 1,2$; $\lambda(u_i^{(2)}u_j^{(3)}) = i$ for $i,j = 1,2$. Thus, the weights of all vertices of $T_{3,7}$ are 6, 7, 8, 9, 10, 11, 12. So $tvs(T_{3,7}) = 2$.

For $m \ge 4$, let $\lambda(u_i^{(1)}) = 1$ for $i = 1,2,3$; $\lambda(u_i^{(s)}) = i$ for $i = 1,2$ and $s = 2,3,\cdots,m-1$; $\lambda(u_i^{(m)}) = 2$ for $i = 1,2$; $\lambda(u_1^{(1)}u_j^{(t)}) = 1$ for $j = 1,2$ and $t = 2,3,\cdots,m$; $\lambda(u_2^{(1)}u_j^{(t)}) = 1$ for $j = 1,2$ and $t = 2,3,\cdots,m-1$; $\lambda(u_2^{(1)}u_j^{(m)}) = j$ for $j = 1,2$; $\lambda(u_3^{(1)}u_j^{(t)}) = 1$ for $j = 1,2$ and $t = 2,3,\cdots,m-1$; $\lambda(u_3^{(1)}u_j^{(m)}) = 2$ for $j = 1,2$; $\lambda(u_1^{(s)}u_j^{(t)}) = 1$ for $j = 1,2$ and $2 \le s < t \le m-1$; $\lambda(u_1^{(s)}u_j^{(m)}) = 2$ for $j = 1,2$ and $s = 2,3,\cdots,m-1$; $\lambda(u_2^{(s)}u_j^{(t)}) = 2$ for $j = 1,2$ and $2 \le s < t \le m$. Thus, the weights of all vertices of $T_{m,2m+1}$ are $2m-1, 2m, \cdots, 4m-1$. So $tvs(T_{m,2m+1}) = 2$.

Theorem 2.5 *$T_{m,3m-1}$ has no vertex irregular total 2–labeling for $m \ge 4$.*

Proof: First, we have $\delta(T_{m,3m-1}) = 3m-4$, $\Delta(T_{m,3m-1}) = 3m-3$. Assume that $T_{m,3m-1}$ has a vertex irregular total 2–labeling. The smallest weight is at least $3m-3$ and the largest weight is at least $6m-4$. And it is impossible that both $3m-3$ and $6m-4$ are the weights of two vertices of $T_{m,3m-1}$. Furthermore, the smallest weight does not exceed $3m-2$.(Otherwise, the largest weight is at least $6m-3$, a contradiction.) Suppose that the vertex with the smallest weight is u and the vertex with the largest weight is v.

Claim: If the smallest weight $wt(u)$ is $3m-3$, then
(i) the vertex u with smallest weight is of the minimum degree;
(ii) $wt(v) = 6m-5$ and v is of the maximum degree;
(iii) the vertex x with weight $6m-6$ is either in the part with u or in the part with v.

(i) and (ii) hold obviously. For (iii), suppose x is neither in the part with u nor in the part with v. Then the weight of x is the sum of $3m-3$

numbers in $\{1,2\}$, but one of the numbers is 1 (that is, the weight of edge ux). So $wt(x) \le 6m-7$, a contraction.

Based on the above Claim, we discuss by distinguishing the following four cases. Let y_i denote the vertex with weight i for $i = 3m-3, 3m-2, \cdots, 6m-5$ in case 1 and case 2 or for $i = 3m-2$, $3m-1, \cdots, 6m-4$ in case 3 and case 4 in the following discussions.

Case 1. The smallest weight is $3m-3$ (The largest weight must be $6m-5$) and the vertex x with weight $6m-6$ and the vertex u with weight $3m-3$ are in the same part.

Let $wt(u) = 3m-3$, $wt(v) = 6m-5$ and $wt(x) = 6m-6$, then the weights of vertex u and the edges incident with u are 1, the weights of vertex x and the edges incident with x are 2 and the weights of vertex v and the edges incident with v except for uv are 2.

Note that y_{3m-2} is not in the same part as $v = y_{6m-5}$. (Otherwise, y_{3m-2} is a vertex of maximum degree and the weight of $y_{3m-2}y_{6m-6}$ is 2, so the weight of y_{3m-2} is at least $3m-1$. This is a contradiction.) If y_{3m-2} and y_{3m-3}, y_{6m-6} are not in the same part, then the weights of $y_{3m-2}y_{6m-5}$ and $y_{3m-2}y_{6m-6}$ are 2. But $d(y_{3m-2}) = 3m-4$. So $wt(y_{3m-2}) \ge 3m-1$, a contradiction. Therefore, y_{3m-2} and y_{3m-3}, y_{6m-6} are in the same part. Besides, the weights of $y_{3m-3}y_{6m-7}$ and $y_{3m-2}y_{6m-7}$ are 1 ensures that y_{6m-7} and y_{6m-5} are in the same part.(Otherwise, $wt(y_{6m-7}) \le 6m-8$, a contradiction.)

Since the weights of $y_{3m-1}y_{6m-6}$ and $y_{3m-1}y_{6m-5}$ are 2, then the weights of vertex y_{3m-1} and the edges incident with y_{3m-1} except for $y_{3m-1}y_{6m-6}$, $y_{3m-1}y_{6m-5}$ are 1. And the weight of $y_{3m-1}y_{6m-7}$ is 1. The vertices y_{6m-8} and y_{3m-1} are in the same part because of the weights of $y_{6m-8}y_{3m-3}$, $y_{6m-8}y_{3m-2}$ are 1. Besides, y_{3m} and y_{3m-1}, y_{6m-8} are in the same part for the weights of $y_{3m}y_{6m-5}$, $y_{3m}y_{6m-6}$ and $y_{3m}y_{6m-7}$ are 2.

Thus, $V(T_{m,3m-1})$ has a $m-$partition $\{A_1, A_2, \cdots, A_m\}$ such that $A_1 = \{y_{6m-7}, y_{6m-5}\}$, $A_2 = \{y_{3m-3}, y_{3m-2}, y_{6m-6}\}$, $A_3 = \{y_{3m-1}, y_{3m}, y_{6m-8}\}$ and $|A_4| = |A_5| = \cdots = |A_m| = 3$. See Fig. 1.

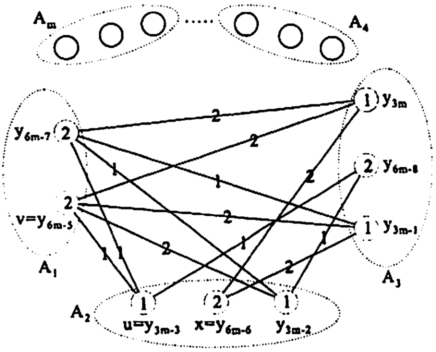

Fig. 1 $m-$partition of the vertices of $T_{m,3m-1}$

Note that for the edges that do not appear in Fig. 1, their weights are 2 if they incident with $y_{6m-5}, y_{6m-6}, y_{6m-7}$ or y_{6m-8} and 1 if they incident with $y_{3m-3}, y_{3m-2}, y_{3m-1}$ or y_{3m}, as well as in Fig. 2, Fig. 3 and Fig. 4.

According to Fig. 1, the weights of $y_{6m-9}y_{3m-3}$, $y_{6m-9}y_{3m-2}$, $y_{6m-9}y_{3m-1}$ and $y_{6m-9}y_{3m}$ are 1 wherever y_{6m-9} is in A_k $(k = 4, 5, \cdots, m)$. But $d(y_{6m-9}) = 3m - 4$, so $wt(y_{6m-9}) \leq 6m - 10$. This is a contradiction.

Case 2. The smallest weight is $3m - 3$ and the vertex x with weight $6m - 6$ and the vertex u with weight $3m - 3$ are not in the same part. By Claim, the vertices x and v are in the same part.

Let $wt(u) = 3m - 3$, $wt(v) = 6m - 5$ and $wt(x) = 6m - 6$, then the weights of vertex u and the edges incident with u are 1, the weights of vertex v and the edges incident with v except for uv are 2.

Case 2.1. The vertex y_{3m-2} with weight $3m - 2$ and u are in the same part.

Since the weight of $y_{3m-2}y_{6m-5}$ is 2, the weights of y_{3m-2} and the edges incident with y_{3m-2} except for $y_{3m-2}y_{6m-5}$ are 1. Then the weights of y_{6m-6} and the edges incident with y_{6m-6} except for $y_{6m-6}y_{3m-3}$ and $y_{6m-6}y_{3m-2}$ are 2. Assume that y_{6m-7} and y_{3m-3}, y_{3m-2} are not in the same part, then the weights of $y_{6m-7}y_{3m-3}$, $y_{6m-7}y_{3m-2}$ are 1. But $d(y_{6m-7}) = 3m - 4$, so $wt(y_{6m-7}) \leq 6m - 8$, a contradiction. Therefore, y_{6m-7} and y_{3m-3}, y_{3m-2} are in the same part.

Because the weights of $y_{3m-1}y_{6m-5}$, $y_{3m-1}y_{6m-6}$ are 2, the weights of y_{3m-1} and the edges incident with y_{3m-1} except for $y_{3m-1}y_{6m-5}$ and $y_{3m-1}y_{6m-6}$ are 1. Thus, the weight of $y_{6m-7}y_{3m-1}$ is 1 and the weights of y_{6m-7} and the edges incident with y_{6m-7} except for $y_{6m-7}y_{3m-1}$ are 2. Duing to the weights of $y_{6m-8}y_{3m-3}$, $y_{6m-8}y_{3m-2}$ are 1, y_{6m-8} and y_{3m-1} are in the same part. Moreover, y_{3m} and y_{3m-1}, y_{6m-8} are in the same part because of the weights of $y_{3m}y_{6m-5}$, $y_{3m}y_{6m-6}$, $y_{3m}y_{6m-7}$ are 2.

Thus, $V(T_{m,3m-1})$ has a $m-$partition $\{A_1, A_2, \cdots, A_m\}$ such that $A_1 = \{y_{6m-6}, y_{6m-5}\}$, $A_2 = \{y_{3m-3}, y_{3m-2}, y_{6m-7}\}$, $A_3 = \{y_{3m-1}, y_{3m}, y_{6m-8}\}$ and $|A_4| = |A_5| = \cdots = |A_m| = 3$. See Fig. 2.

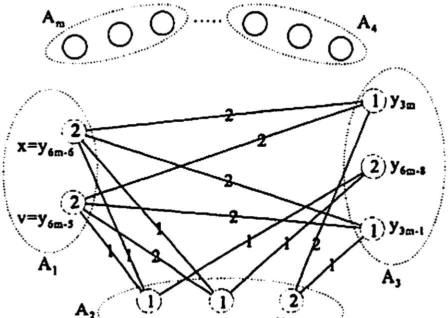

Fig. 2 $m-$partition of the vertices of $T_{m,3m-1}$

According to Fig. 2, the weights of $y_{6m-9}y_{3m-3}$, $y_{6m-9}y_{3m-2}$, $y_{6m-9}y_{3m-1}$ and $y_{6m-9}y_{3m}$ are 1 wherever y_{6m-9} is in A_k $(k = 4, 5, \cdots, m)$. But $d(y_{6m-9}) = 3m - 4$, so $wt(y_{6m-9}) \leq 6m - 10$. This is a contradiction.

Case 2.2. The vertex y_{3m-2} with weight $3m-2$ and u are not in the same part.

First, the weights of the vertex y_{6m-6} and the edges incident with y_{6m-6} except for $y_{6m-6}y_{3m-3}$, $y_{6m-6}y_{3m-2}$ are 2 and the weights of vertex y_{3m-2} and the edges incident with y_{3m-2} except for $y_{3m-2}y_{6m-5}$ are 1. Then we consider the following two cases according to the fact that the vertex y_{6m-7} is either in the same part as y_{3m-3} or in the same part as y_{3m-2}.

Case 2.2 (a). The vertex y_{6m-7} is in the same part as y_{3m-3}.

If y_{3m-1} and y_{3m-3}, y_{6m-7} are not in the same part, then the weights of $y_{3m-1}y_{6m-5}$, $y_{3m-1}y_{6m-6}$ and $y_{3m-1}y_{6m-7}$ are 2. But $d(y_{3m-1}) = 3m-4$, so $wt(y_{3m-1}) \geq 3m$, a contradiction. Thus, y_{3m-1} and y_{3m-3}, y_{6m-7} are in the same part and the weights of vertex y_{3m-1} and the edges incident with y_{3m-1} except for $y_{3m-1}y_{6m-6}$, $y_{3m-1}y_{6m-5}$ are 1. The vertices y_{6m-8} and y_{3m-2} are in the same part for the weights of $y_{6m-8}y_{3m-3}$, $y_{6m-8}y_{3m-1}$ are 1. Because the weights of $y_{3m}y_{6m-5}$, $y_{3m}y_{6m-6}$ and $y_{3m}y_{6m-7}$ are 2, y_{3m} and y_{3m-2}, y_{6m-8} are in the same part.

Thus, $V(T_{m,3m-1})$ has a $m-$partition $\{A_1, A_2, \cdots, A_m\}$ such that $A_1 = \{y_{6m-6}, y_{6m-5}\}$, $A_2 = \{y_{3m-3}, y_{3m-1}, y_{6m-7}\}$, $A_3 = \{y_{3m-2}, y_{3m}, y_{6m-8}\}$ and $|A_4| = |A_5| = \cdots = |A_m| = 3$. See Fig. 3.

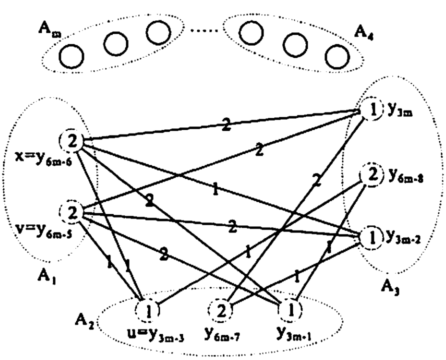

Fig. 3 $m-$partition of the vertices of $T_{m,3m-1}$

According to Fig. 3, the weights of $y_{6m-9}y_{3m-3}$, $y_{6m-9}y_{3m-2}$, $y_{6m-9}y_{3m-1}$ and $y_{6m-9}y_{3m}$ are 1 wherever y_{6m-9} is in A_k $(k = 4, 5, \cdots, m)$. But $d(y_{6m-9}) = 3m-4$, so $wt(y_{6m-9}) \leq 6m-10$. This is a contradiction.

Case 2.2 (b). The vertex y_{6m-7} is in the same part as y_{3m-2}.

The weights of vertex y_{6m-7} and the edges incident with y_{6m-7} except for $y_{6m-7}y_{3m-3}$ are 2. If y_{3m-1} and y_{3m-2}, y_{6m-7} are not in the same part, then the weights of $y_{3m-1}y_{6m-5}$, $y_{3m-1}y_{6m-6}$ and $y_{3m-1}y_{6m-7}$ are 2. But $d(y_{3m-1}) = 3m-4$, so $wt(y_{3m-1}) \geq 3m$, a contradiction. Thus, y_{3m-1} and y_{3m-2}, y_{6m-7} are in the same part. The vertices y_{6m-8} and y_{3m-3} are in the same part for the weights of $y_{6m-8}y_{3m-1}$, $y_{6m-8}y_{3m-2}$ are 1. The weights of $y_{3m}y_{6m-5}$, $y_{3m}y_{6m-6}$, $y_{3m}y_{6m-7}$ are 2, so y_{3m} and y_{3m-3}, y_{6m-8} are in the same part.

Thus, $V(T_{m,3m-1})$ has a $m-$partition $\{A_1, A_2, \cdots, A_m\}$ such that $A_1 = \{y_{6m-5}, y_{6m-6}\}$, $A_2 = \{y_{3m-3}, y_{3m}, y_{6m-8}\}$, $A_3 = \{y_{3m-2}, y_{3m-1}, y_{6m-7}\}$ and $|A_4| = |A_5| = \cdots = |A_m| = 3$. See Fig. 4.

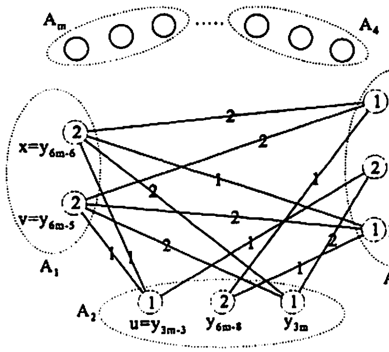

Fig. 4 $m-$partition of the vertices of $T_{m,3m-1}$

According to Fig. 4, the weights of $y_{6m-9}y_{3m-3}$, $y_{6m-9}y_{3m-2}$, $y_{6m-9}y_{3m-1}$ and $y_{6m-9}y_{3m}$ are 1 wherever y_{6m-9} is in A_k $(k = 4, 5, \cdots, m)$. But $d(y_{6m-9}) = 3m-4$, so $wt(y_{6m-9}) \leq 6m-10$. This is a contradiction.

Case 3. The vertex u with the smallest weight $3m-2$ is of the minimum degree.

Let $wt(u) = 3m-2$, $wt(v) = 6m-4$, then the weights of vertex u and the edges incident with u except for uv are 1, the weights of vertex v and the edges incident with v are 2. In addition, y_{6m-5} and y_{6m-4} are in the same part and the weights of y_{6m-5} and the edges incident with y_{6m-5} except for $y_{3m-2}y_{6m-5}$ are 2. Thus, $A_1 = \{y_{6m-5}, y_{6m-4}\}$ is a part of $T_{m,3m-1}$.

If y_{6m-6} and y_{3m-2} are not in the same part, then the weight of $y_{6m-6}y_{3m-2}$ is 1. But $d(y_{6m-6}) = 3m-4$, so $wt(y_{6m-6}) \leq 6m-7$, a contradiction. Thus, y_{6m-6} and y_{3m-2} are in the same part. Because the weights of $y_{3m-1}y_{6m-5}$, $y_{3m-1}y_{6m-4}$ are 2, y_{3m-1} and y_{3m-2}, y_{6m-6} are in the same part. So $A_2 = \{y_{3m-2}, y_{3m-1}, y_{6m-6}\}$ is a part of $T_{m,3m-1}$. Moreover, the weights of $y_{3m-2}y_{6m-7}$, $y_{3m-1}y_{6m-7}$ are 1 and $d(y_{6m-7}) = 3m-4$, so $wt(y_{6m-7}) \leq 6m-8$, a contradiction.

Case 4. The vertex u with the smallest weight $3m-2$ is of the maximum degree.

Let $wt(u) = 3m-2$, $wt(v) = 6m-4$. Since $d(u) = d(v) = 3m-3$, then the weights of vertex u and the edges incident with u are 1, the weights of vertex v and the edges incident with v are 2. But $d(y_{6m-5}) = 3m-4$ and the weight of $y_{3m-2}y_{6m-5}$ is 1, so $wt(y_{6m-5}) \leq 6m-7$, a contradiction.

Thus, we have $tvs(T_{m,3m-1}) \geq 3$ for $m \geq 4$.

Theorem 2.6 *Let $m \geq 3$, then $T_{m,n}$ doesn't have a vertex irregular total 2−labeling for $n = 3m + r$, $r = 1, 2, \cdots, m-1$.*

Proof: Suppose that $n = 3m + r$, $r = 1, 2, \cdots, m-1$. Assume that $T_{m,n}$ has a vertex irregular total 2–labeling. Since $\delta(T_{m,n}) = 3m + r - 4$ and $\Delta(T_{5,n}) = 3m+r-3$, the minimum weight is $3m+r-3$ and the maximum weight is $2(\Delta + 1) = 6m + 2r - 4$. So the weight of each vertex in $V(T_{m,n})$ is a number in the set $\{3m + r - 3, 3m + r - 2, \cdots, 6m + 2r - 4\}$. But this set has just n elements and any two different vertices have distinct weights. Thus each number in the set $\{3m + r - 3, 3m + r - 2, \cdots, 6m + 2r - 4\}$ is a weight of some vertex in $V(T_{m,n})$.

If $wt(v) = 3m+r-3$ for $v \in V(T_{m,n})$, then the degree of v is $3m+r-4$ and the weights of vertex v and the edges incident with v are 1. But there exists a vertex u with weight $6m + 2r - 4$, then the vertex u with degree $3m + r - 3$ is of the maximum degree and the weights of vertex u and the edges incident with u are 2. This is not true because the edge uv has weight 1. This is a contradiction. So $T_{m,n}$ doesn't have a vertex irregular total 2–labeling for $n = 3m + r$, $r = 1, 2, \cdots, m-1$.

Theorem 2.7 *For* $n = 4, 5, 6, 7, 8$, $tvs(T_{3,n}) = 2$.

Proof: By Theorem 2.1, Theorem 2.2, Theorem 2,3 and Theorem 2.4, we can easily obtain that $tvs(T_{3,n}) = 2$ for $n = 4, 5, 6, 7$. For $n = 8$, we need only to give a vertex irregular total 2–labeling λ of $T_{3,8}$ in the following.

Let $\lambda(u_i^{(s)}) = 1$ for $i = 1, 2$ and $s = 1, 2$; $\lambda(u_3^{(t)}) = 2$ for $t = 1, 2$; $\lambda(u_i^{(3)}) = 2$ for $i = 1, 2$; $\lambda(u_1^{(1)}u_j^{(2)}) = 1$ for $j = 1, 2, 3$; $\lambda(u_1^{(1)}u_j^{(3)}) = 1$ for $j = 1, 2$; $\lambda(u_2^{(1)}u_j^{(2)}) = 1$ for $j = 1, 2, 3$; $\lambda(u_2^{(1)}u_j^{(3)}) = j$ for $j = 1, 2$; $\lambda(u_3^{(1)}u_j^{(2)}) = 2$ for $j = 1, 2, 3$; $\lambda(u_3^{(1)}u_j^{(3)}) = 2$ for $j = 1, 2$; $\lambda(u_1^{(2)}u_j^{(3)}) = j$ for $j = 1, 2$; $\lambda(u_i^{(2)}u_j^{(3)}) = 2$ for $i = 2, 3$ and $j = 1, 2$. Thus, the weights of all vertices of $T_{3,8}$ are 6, 7, 8, 9, 10, 11, 12, 13. So $tvs(T_{3,8}) = 2$.

Theorem 2.8 *Let* $n \geq 9$, *then* $tvs(T_{3,n}) = 3$.

Proof: To prove the Theorem, we consider the following three cases depending on values of m.

Case 1. $n \geq 9$ and $n \equiv 0 \pmod 3$

For $n \geq 9$, we have $tvs(T_{3,n}) \geq \lceil \frac{n+\frac{2n}{3}}{\frac{2n}{3}+1} \rceil = \lceil \frac{5}{2+\frac{3}{n}} \rceil = 3$ by Lemma 1.1. To show that $tvs(T_{3,n}) \leq 3$, we need only to give a vertex irregular total 3–labeling λ of $T_{3,n}$ in the following.

Let $\lambda(u_i^{(t)}) = t$ for $t = 1, 2, 3$ and $i = 1, 2, \cdots, \frac{n}{3}$; for $i, j = 1, 2, \cdots, \frac{n}{3}$, $1 \leq s < t \leq 3$, let
$$\lambda(e_{ij}^{(s)(t)}) = \begin{cases} 1, & \text{for } i + j \leq \frac{n}{3}; \\ 2, & \text{for } i + j = \frac{n}{3} + 1; \\ 3, & \text{for } i + j \geq \frac{n}{3} + 2. \end{cases}$$

Thus, the weights of all vertices of $T_{3,n}$ are $wt(u_i^{(s)}) = \frac{2n}{3} + 4i + s - 2$ for $i = 1, 2, \cdots, \frac{n}{3}$, $s = 1, 2, 3$. It is easy to see that the weights at different vertices are distinct. Therefore $tvs(T_{3,n}) = 3$ for $n \geq 9$ and $n \equiv 0 \pmod 3$.

Case 2. $n \geq 10$ and $n \equiv 1 \pmod 3$

By Theorem 2.6, $T_{3,10}$ doesn't have a vertex irregular total 2−labeling. For $n > 10$, we have $tvs(T_{3,n}) \geq \lceil \frac{n+\frac{2(n-1)}{3}}{\frac{2(n-1)}{3}+1+1} \rceil = \lceil \frac{5n-2}{2n+4} \rceil = \lceil \frac{5}{2}(1-\frac{6}{n+2}) \rceil = 3$ according to Lemma 1.1.

To show that $tvs(T_{3,n}) \leq 3$, the vertex irregular total 3−labeling λ of $T_{3,n}$ when $n \geq 10$ are described as follows.

Let $\lambda(u_1^{(1)}) = 2$; $\lambda(u_k^{(1)}) = 3$ for $k = 2, 3, \cdots, \frac{n-1}{3}+1$; $\lambda(u_l^{(2)}) = 1$ for $l = 1, 2, \cdots, \frac{n-1}{3}$; $\lambda(u_m^{(3)}) = 2$ for $m = 1, 2, \cdots, \frac{n-1}{3}$; for $s = 1, t = 2, 3, i = 1, 2, \cdots, \frac{n-1}{3}+1$, $j = 1, 2, \cdots, \frac{n-1}{3}$ or $s = 2, t = 3, i, j = 1, 2, \cdots, \frac{n-1}{3}$, let $\lambda(e_{ij}^{(s)(t)}) = \begin{cases} 1, & \text{for } i+j \leq \frac{n-1}{3}+1; \\ 2, & \text{for } i+j = \frac{n-1}{3}+2; \\ 3, & \text{for } i+j \geq \frac{n-1}{3}+3. \end{cases}$

Thus, the weights of all vertices of $T_{3,n}$ are $wt(u_1^{(s)}) = \frac{2(n-1)}{3}+s+1$ for $s = 1, 2, 3$; $wt(u_i^{(s)}) = \frac{2(n-1)}{3}+4i+s-4$ for $i = 2, 3, \cdots, \frac{n-1}{3}$, $s = 1, 2, 3$; $wt(u_i^{(1)}) = 2n-1$ for $i = \frac{n-1}{3}+1$. Clearly, all the weights at different vertices are distinct. Therefore $tvs(T_{3,n}) = 3$ for $n \geq 10$ and $n \equiv 1 \pmod 3$.

Case 3. $n \geq 11$ and $n \equiv 2 \pmod 3$

By Theorem 2.6, $T_{3,11}$ doesn't have a vertex irregular total 2−labeling. For $n > 11$, we have $tvs(T_{3,n}) \geq \lceil \frac{n+\frac{2(n+1)}{3}-1}{\frac{2(n+1)}{3}+1} \rceil = \lceil \frac{5n-1}{2n+5} \rceil = \lceil \frac{5}{2} - \frac{27}{2} \cdot \frac{1}{2n+5} \rceil = 3$ according to Lemma 1.1.

To show that $tvs(T_{3,n}) \leq 3$, the vertex irregular total 3−labeling λ of $T_{3,n}$ when $n \geq 11$ are described as follows.

Let $\lambda(u_k^{(1)}) = 1$ for $k = 1, 2, \cdots, \frac{n+1}{3}$; $\lambda(u_l^{(2)}) = 2$ for $l = 1, 2, \cdots, \frac{n+1}{3}$; $\lambda(u_m^{(3)}) = 1$ for $m = 1, 2, \cdots, \frac{n-2}{3}$; and for $s = 1, t = 2, i, j = 1, 2, \cdots, \frac{n+1}{3}$ or $s = 1, 2, t = 3, i = 1, 2, \cdots, \frac{n+1}{3}, j = 1, 2, \cdots, \frac{n-2}{3}$, let $\lambda(e_{ij}^{(s)(t)}) = \begin{cases} 1, & \text{for } i+j \leq \frac{n+1}{3}; \\ 2, & \text{for } i+j = \frac{n+1}{3}+1; \\ 3, & \text{for } i+j \geq \frac{n+1}{3}+2. \end{cases}$

Thus, the weights of all vertices of $T_{3,n}$ are $wt(u_1^{(s)}) = \frac{2(n+1)}{3}+s$ for $s = 1, 2$; $wt(u_i^{(s)}) = \frac{2(n+1)}{3}+4i+s-5$ for $i = 2, 3, \cdots, \frac{n+1}{3}$, $s = 1, 2$; $wt(u_i^{(3)}) = \frac{2(n+1)}{3}+4i-1$ for $i = 1, 2, \cdots, \frac{n-2}{3}$. It is easy to see that the weights at different vertices are distinct. Therefore $tvs(T_{3,n}) = 3$ for $n \geq 11$ and $n \equiv 2 \pmod 3$.

In the Theorem 2.5 and Theorem 2.6, we prove that $T_{m,3m-1}$ $(m \geq 4)$ and $T_{m,n}$ $(n = 3m+r, r = 1, 2, \cdots, m-1)$ have no vertex irregular total 2−labeling. But we obtain that $tvs(T_{m,3m-1})$=$tvs(T_{m,n})$=3 for some m and n. So we propose the following Conjecture.

Conjecture 2.1 *Let $m \geq 4$, then $tvs(T_{m,3m-1}) = 3$.*

Conjecture 2.2 *Let $m \geq 3$, $n = 3m + r$ and $r = 1, 2, \cdots, m - 1$, then $tvs(T_{m,n}) = 3$.*

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