On Star Chromatic Number of Sunlet Graph Families

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Abstract. In this paper, we find the star chromatic number χ_s for the central graph of sunlet graphs $C(S_n)$, line graph of sunlet graphs $L(S_n)$, middle graph of sunlet graphs $M(S_n)$ and the total graph of sunlet graphs $T(S_n)$.

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1. Introduction

The notion of star chromatic number was introduced by Branko Grünbaum in 1973. A *star coloring* [1, 3, 6] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two colors have connected components that are star graphs. The star chromatic number $\chi_s(G)$ of G is the minimum number of colors needed to star color G.

Guillaume Fertin et al.[6] gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded treewidth, and cubic graphs.

Albertson et al.[1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. The problem of finding star colorings is NP-hard and remain so even for bipartite graphs [7, 8].

2. Preliminaries

The *sunlet graph* on $2n$ vertices is obtained by attaching n pendant edges to the cycle C_n and is denoted by S_n.

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non-adjacent vertices of G. The graph obtained by this process is called *central graph* [9] of G denoted by $C(G)$.

The *line graph* [2, 5] of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G.

The *middle graph* [4] of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is the vertex and the other is an edge incident with it and it is denoted by $M(G)$.

The *total graph* [2, 4, 5] of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

Additional graph theory terminology used in this paper can be found in [2, 5].

In the following sections we find the star chromatic number for the central graph of sunlet graphs $C(S_n)$, line graph of sunlet graphs $L(S_n)$, middle graph of sunlet graphs $M(S_n)$ and the total graph of sunlet graphs $T(S_n)$.

In order to prove our results, we shall use the following theorem and proof by Guillaume et.al [6].

Theorem 2.1. [6] *If C_n is a cycle with $n \geq 3$ vertices, then*

$$\chi_s(C_n) = \begin{cases} 4 & when \quad n = 5 \\ 3 & otherwise. \end{cases}$$

Proof. It can be easily checked that $\chi_s(C_5) = 4$. Now let us assume $n \neq 5$. Clearly atleast 3 colors are needed to star color C_n. We now distinguish three cases:

Case(i): If $n = 3k$, we color alternatively the vertices around the cycle by colors c_1, c_2 and c_3. Thus, for any vertex u, its two neighbours are assigned distinct colors, and consequently this is a valid star coloring. Hence $\chi_s(C_{3k}) \leq 3$.

Case(ii): If $n = 3k + 1$, in this case, let us color $3k$ vertices of C_n consequently, by repeating the sequence of colors c_1, c_2 and c_3. There remains one uncolored vertex, to which we assign color c_2. One can check easily that this is also a valid star coloring, and thus $\chi_s(C_{3k+1}) \leq 3$.

Case(iii): If $n = 3k+2$. Since the case $n = 5$ is excluded here, we can assume $k \geq 2$. Thus $n = 3(k - 1) + 5$, with $k - 1 \geq 1$. In that case, let us color $3(k - 1)$ consecutive vertices along the cycle, alternating colors c_1, c_2 and c_3.

For the 5 remaining vertices, we give the following coloring : c_2, c_1, c_2, c_3, c_2. It can be checked that this is a valid star coloring, and thus $\chi_s(C_{3k+2}) \leq 3$ for any $k \geq 2$. Globally, we have $\chi_s(C_n) = 3$ for any $n \neq 5$, and the result is proved. □

3. Star coloring on central graph of sunlet graph

Theorem 3.1. *Let S_n be a sunlet graph with $2n$ vertices, then*

$$\chi_s(C(S_n)) = n + 2, \forall n \geq 3.$$

Proof. Let $V(S_n) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \cup \{ e'_i : 1 \leq i \leq n\}$ where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. For $1 \leq i \leq n$, u_i is the pendant vertex and v_i is the adjacent vertex to u_i. By the definition of central graph $V(C(S_n)) = V(S_n) \cup E(S_n) = \{u_i : 1 \leq i \leq n\} \cup \{ v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$ where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively.
Assign the following coloring for $C(S_n)$ as star chromatic:

- For $1 \leq i \leq n$, assign the color c_i to u_i
- For $1 \leq i \leq n$, assign the color c_i to v_i
- For $1 \leq i \leq n$, assign the color c_{n+1} to u'_i
- For $1 \leq i \leq n$, assign the color c_{n+2} to v'_i

Thus, $\chi_s(C(S_n)) \leq n + 2$.

To prove $\chi_s(C(S_n)) \geq n + 2$. Assume that $\chi_s(C(S_n))$ is less than $n + 2$, say $n + 1$. We need atleast n colors say $\{c_1, c_2, \ldots, c_n\}$, since the subgraph induced by $\{u_i : 1 \leq i \leq n\}$ is a complete graph K_n and the coloring should be proper. The vertices $\{u'_i : 1 \leq i \leq n\}$ are adjacent to the vertices $\{u_i : 1 \leq i \leq n\}$ and $\{v_i : 1 \leq i \leq n\}$ needs a distinct color say c_{n+1} for proper star coloring. If we assign the same $n + 1$ colors to the vertices $\{v'_i : 1 \leq i \leq n\}$, then an easy check shows that there exists a bicolored path. A contradiction to proper star coloring. Thus, $\chi_s(C(S_n)) \geq n + 2$. Hence, $\chi_s(C(S_n)) = n + 2, \forall n \geq 3$. □

4. Star coloring on line graph of sunlet graph

Theorem 4.1. *Let S_n be a sunlet graph with $2n$ vertices and $n \geq 3$, then*

$$\chi_s(L(S_n)) = \begin{cases} 5 & if \quad n = 5 \\ 4 & otherwise. \end{cases}$$

Proof. Let $V(S_n) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(S_n) = \{e_i : 1 \leq i \leq n\} \cup \{ e'_i : 1 \leq i \leq n\}$ where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. By the definition of line graph $V(L(S_n)) = E(S_n) = \{u'_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n-1\} \cup \{v'_n\}$ where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively. Now

$\{v'_i : 1 \leq i \leq n\}$ forms a cycle C_n with n vertices.

Case(i): When $n = 5$
Assign the coloring as follows:

- For $1 \leq i \leq 4$, assign the color c_i to v'_i
- Assign the color c_2 to the vertex v'_5
- For $1 \leq i \leq 4$, assign the color c_5 to u'_i

Thus, $\chi_s(L(S_n)) \leq 5$.

To prove $\chi_s(L(S_n)) \geq 5$. Suppose $\chi_s(L(S_n))$ is less than 5, say 4. By theorem 2.1, the cycle C_n with vertices $\{v'_i : 1 \leq i \leq 5\}$ needs atleast 4 colors. If we assign the existing colors to the vertices $\{u'_i : 1 \leq i \leq 5\}$, then an easy check shows that there exists a bicolored path of length 3. A contradiction to proper star coloring. Thus, $\chi_s(L(S_n)) \geq 5$. Hence, $\chi_s(L(S_n)) = 5$ for $n = 5$.

Case(ii): When $n \neq 5$
Assign the coloring as follows:

- For $1 \leq i \leq n$, color the vertices v'_i by repeating the sequence of colors c_1, c_2, c_3
- For $1 \leq i \leq n$, color the vertices u'_i with color the c_4

Thus, $\chi_s(L(S_n)) \leq 4$.

To prove $\chi_s(L(S_n)) \geq 4$. Suppose $\chi_s(L(S_n))$ is less than 4, say 3. By theorem 2.1, the cycle C_n with vertices $\{v'_i : 1 \leq i \leq n\}$ needs atleast 3 colors. If we assign the existing colors to the vertices $\{u'_i : 1 \leq i \leq n\}$, then an easy check shows that there exists a bicolored path of length 3. A contradiction to proper star coloring. Thus, $\chi_s(L(S_n)) \geq 4$. Hence, $\chi_s(L(S_n)) = 4$ for $n \neq 5$. □

5. Star coloring on middle graph of sunlet graph

Theorem 5.1. *Let S_n be a sunlet graph with $2n$ vertices and $n \geq 3$, then*

$$\chi_s(M(S_n)) = \begin{cases} 6 & if \quad n = 5 \\ 5 & otherwise \end{cases}$$

Proof. Let $V(S_n) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(S_n)=\{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\}$ where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. By definition of middle graph $V(M(S_n)) = V(S_n) \cup E(S_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$ where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively. Now $\{v'_i : 1 \leq i \leq n\}$ forms a cycle C_n with n vertices .

Case(i): When $n = 5$
By theorem 2.1, $\chi_s(C_n) = 4$. Assign the color c_5 to the pendant vertices

$\{u_i : 1 \leq i \leq n\}$ and the vertices $\{v_i : 1 \leq i \leq n\}$. Assign the color c_6 to the vertices $\{u'_i : 1 \leq i \leq n\}$. Thus, $\chi_s(M(S_n)) \leq 6$.

To prove $\chi_s(M(S_n)) \geq 6$. Suppose $\chi_s(M(S_n))$ is less than 6, say 5. By theorem 2.1, we need atleast 4 colors to star color the cycle C_n with vertices $\{v'_i : 1 \leq i \leq n\}$. The vertices $\{u_i : 1 \leq i \leq n\}$ and the vertices $\{v_i : 1 \leq i \leq n\}$ needs a distinct color, say c_5 for proper star coloring. If we assign the existing colors for the vertices $\{u'_i : 1 \leq i \leq n\}$, then an easy check shows that there exists a bicolored path of length 3. A contradiction to proper star coloring. Thus, $\chi_s(M(S_n)) \geq 6$. Hence, $\chi_s(M(S_n)) = 6$.

Case(ii): When $n \neq 5$
By theorem 2.1, $\chi_s(C_n) = 3$. Assign the color c_4 to the pendant vertices $\{u_i : 1 \leq i \leq n\}$ and to the vertices $\{v_i : 1 \leq i \leq n\}$. Assign the color c_5 to the vertices $\{u'_i : 1 \leq i \leq n\}$. Thus, $\chi_s(M(S_n)) \leq 5$.

To prove $\chi_s(M(S_n)) \geq 5$. Suppose $\chi_s(M(S_n))$ is less than 5, say 4. By theorem 2.1, we need atleast 3 colors to star color the cycle C_n with vertices $\{v'_i : 1 \leq i \leq n\}$. The vertices $\{u_i : 1 \leq i \leq n\}$ and the vertices $\{v_i : 1 \leq i \leq n\}$ needs a distinct color, say c_4 for proper star coloring. If we assign the existing colors for the vertices $\{u'_i : 1 \leq i \leq n\}$, then an easy check shows that there exists a bicolored path of length 3. A contradiction to proper star coloring. Thus, $\chi_s(M(S_n)) \geq 5$. Hence, $\chi_s(M(S_n)) = 5$. □

6. Star coloring on total graph of sunlet graph

Theorem 6.1. *Let S_n be a sunlet graph with $2n$ vertices and $n \geq 3$, then*

$$\chi_s(T(S_n)) = \begin{cases} 7 & if \quad n \equiv 0(\text{mod } 5) \\ 8 & otherwise. \end{cases}$$

Proof. Let $V(S_n) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$and $E(S_n)=\{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\}$ where e_i is the edge $v_i v_{i+1} (1 \leq i \leq n-1)$, e_n is the edge $v_n v_1$ and e'_i is the edge $v_i u_i (1 \leq i \leq n)$. By the definition of total graph $V(T(S_n)) = V(S_n) \cup E(S_n) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \cup \{u'_i : 1 \leq i \leq n\}$ where v'_i and u'_i represents the edge e_i and e'_i, $(1 \leq i \leq n)$ respectively.

Case(i): When $n \equiv 0(\text{mod } 5)$
We color the vertices $\{v_i : 1 \leq i \leq n\}$ of the cycle C_n with color sequence c_1, c_2, c_3, c_4 and c_5 and the vertices $\{v'_i : 1 \leq i \leq n\}$ of cycle C'_n with color sequence c_4, c_5, c_1, c_2 and c_3, alternatively. Assign color c_6 to vertices $\{u'_i : 1 \leq i \leq n\}$ and c_7 to the pendant vertices $\{u_i : 1 \leq i \leq n\}$. The coloring is a valid star coloring. Thus, $\chi_s(T(S_n)) \leq 7$ for $n \equiv 0(\text{mod } 5)$.

To prove $\chi_s(T(S_n)) \geq 7$. Suppose that $\chi_s(T(S_n))$ is less than 7, say 6. The vertices of the cycles C_n and C'_n are colored with 5 colors c_1, c_2, c_3, c_4 and c_5. We assign the color c_6 to the vertices $\{u'_i : 1 \leq i \leq n\}$ for proper star coloring. If we assign one of the existing colors to the pendant vertices

$\{u_i : 1 \leq i \leq n\}$, then an easy check shows that there exists a bicolored path of length 3. A contradiction, star coloring with 6 colors is not possible. Thus, $\chi_s(T(S_n)) \geq 7$. Hence, $\chi_s(T(S_n)) = 7$.

Case(ii): When $n \not\equiv 0 (\text{mod } 5)$
Assign the coloring as follows:
Subcase(i): When $n = 3k$, we color alternatively the vertices $\{v_i : 1 \leq i \leq n\}$ around the cycle C_n by colors c_1, c_2 and c_3 and the vertices $\{v'_i : 1 \leq i \leq n\}$ around the cycle C'_n by colors c_4, c_5 and c_6. Assign color c_7 to $\{u'_i : 1 \leq i \leq n\}$ and c_8 to the pendant vertices $\{u_i : 1 \leq i \leq n\}$.

Subcase(ii): When $n = 3k + 1$. Let us color $3k$ vertices of cycle C_n and C'_n consecutively by repeating the sequence of colors c_1, c_2, c_3 and c_4, c_5, c_6 respectively. There remains one uncolored vertex in C_n and C'_n, to them we assign colors c_2 and c_5 respectively. Assign color c_7 to $\{u'_i : 1 \leq i \leq n\}$ and c_8 to the pendant vertices $\{u_i : 1 \leq i \leq n\}$.

Subcase(iii): When $n = 3k + 2$. Here $n \not\equiv 0(\text{mod } 5)$ are excluded. Thus $n = 3(k-1)+5$ with $k \neq 5i-4, i = 1,2,3,\ldots$ and $k \geq 2$. Let us color $3(k-1)$ consecutive vertices along the cycles C_n and C'_n with alternating sequence of colors c_1, c_2, c_3 and c_4, c_5, c_6 respectively. For the remaining 5 vertices we assign the following coloring: c_2, c_1, c_2, c_3, c_2 in cycle C_n and c_5, c_4, c_5, c_6, c_4 in cycle C'_n. It can be checked that this is a valid star coloring. Assign color c_7 to $\{u'_i : 1 \leq i \leq n\}$ and c_8 to the pendant vertices $\{u_i : 1 \leq i \leq n\}$.

In all the subcases above, $\chi_s(T(S_n)) \leq 8$ for $n \not\equiv 0(\text{mod } 5)$.
To prove $\chi_s(T(S_n)) \geq 8$. Suppose, $\chi_s(T(S_n))$ is less than 8, say 7. We have assigned the colors c_1, c_2, c_3, c_4, c_5 and c_6 for the vertices of the cycles C_n and C'_n as given above. We assign the color c_7 for the vertices $\{u'_i : 1 \leq i \leq n\}$ for proper star coloring. If we assign one of the existing colors to the pendant vertices $\{u_i : 1 \leq i \leq n\}$, then an easy check shows that there exists a bicolored path of length 3. A contradiction, star coloring with 7 colors is not possible. Thus, $\chi_s(T(S_n)) \geq 8$. Hence, $\chi_s(T(S_n)) = 8$ for $n \not\equiv 0(\text{mod } 5)$. □

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