Anti-Ramsey numbers of small graphs

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Abstract

The anti-Ramsey number $AR(n, G)$, for a graph G and an integer $n \geq |V(G)|$, is defined to be the minimal integer r such that in any edge-colouring of K_n by at least r colours there is a multicoloured copy of G, namely, a copy of G whose edges have distinct colours. In this paper we determine the anti-Ramsey numbers of all graphs having at most four edges.

Keywords: Anti-Ramsey, Multicoloured, Rainbow.

1 Introduction

Definition. *A subgraph of an edge-coloured graph is called* **multicoloured** *if all its edges have distinct colours.*

Let G be a (simple) graph. For any integer $n \geq |V(G)|$, let $AR(n, G)$ be the minimal integer r such that in any edge-colouring of K_n by at least r colours there is a multicoloured copy of G.

Remark 1.1. *It is easy to see that $AR(n, G)$ is also the minimal integer r such that in any edge-colouring of K_n by* **exactly** *r colours there is a multicoloured copy of G.*

$AR(n, G)$ was determined for various graphs G. We mention some of the results, which are relevant to our paper.

For $K_{1,k}$, a star of size $k \geq 2$, Jiang showed ([5]) that for any $n \geq k+1$,

$$AR(n, K_{1,k}) = \left\lfloor \frac{k-2}{2} n \right\rfloor + \left\lfloor \frac{k-2}{n-k+2} \right\rfloor + 2 + \\ + (n \bmod 2)(k \bmod 2)\left(\left\lfloor \frac{2k-4}{n-k+2} \right\rfloor \bmod 2 \right). \quad (1)$$

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For P_{k+1}, a path of length $k \geq 2$, Simonovits and Sós showed ([9]) that for large enough n ($n \geq \frac{5}{4}k + c$ for some universal constant c),

$$AR(n, P_{k+1}) = (\lfloor k/2 \rfloor - 1)\left(n - \frac{\lfloor k/2 \rfloor}{2}\right) + 2 + k \bmod 2. \tag{2}$$

For C_k, a cycle of length k, Erdős, Simonovits and Sós noted in [2], where anti-Ramsey numbers were first introduced, that for any $n \geq 3$

$$AR(n, C_3) = n, \tag{3}$$

showed that for any $n \geq k \geq 3$,

$$AR(n, C_k) \geq \binom{k-1}{2} \left\lfloor \frac{n}{k-1} \right\rfloor + \left\lceil \frac{n}{k-1} \right\rceil + \binom{n \bmod (k-1)}{2},$$

and conjectured this lower bound to be always tight. This conjecture was confirmed, first for $k = 4$ by Alon who proved ([1]) that for any $n \geq 4$,

$$AR(n, C_4) = \left\lfloor \frac{4}{3}n \right\rfloor, \tag{4}$$

and thirty years later for any k, by Montellano-Ballesteros and Neumann-Lara ([7]).

For tP_2, the disjoint union of t paths, each of length 1, i.e., a matching of size t, Schiermeyer first showed ([8]) that $AR(n, tP_2) = (t-2)\left(n - \frac{t-1}{2}\right) + 2$ for any $t \geq 2$, $n \geq 3t + 3$. Then Fujita, Kaneko, Schiermeyer and Suzuki proved ([3]) that for any $t \geq 2$, $n \geq 2t + 1$,

$$AR(n, tP_2) = \begin{cases} (t-2)(2t-3) + 2 & n \leq \frac{5t-7}{2} \\ (t-2)\left(n - \frac{t-1}{2}\right) + 2 & n \geq \frac{5t-7}{2}. \end{cases} \tag{5}$$

Finally, the remaining case $n = 2t$ was settled by Haas and Young ([6]) who confirmed the conjecture made in [3], that

$$AR(2t, tP_2) = \begin{cases} (t-2)\frac{3t+1}{2} + 2 & 3 \leq t \leq 6 \\ (t-2)(2t-3) + 3 & t \geq 7. \end{cases} \tag{6}$$

The results just mentioned cover many of the graphs with up to four edges. In this paper we complete the computation of $AR(n, G)$ for any $n \geq |V(G)|$ for all graphs G having at most four edges. The resulting anti-Ramsey numbers are summarized in Table 1 below.

Table 1: Anti-Ramsey numbers of all graphs having at most four edges. For the definitions of the graphs Y and Q, see Definitions 4.3 ad 5.1, respectively.

G	$AR(n,G)$	Reference
P_2	1	Trivial
P_3	2	Obvious
$2P_2$	4 $n=4$ 2 $n\geq 5$	[8] (for $n \geq 9$), [3] (for $n \geq 5$), Lemma 3.1
P_4	4 $n=4$ 3 $n\geq 5$	[9] for large enough n (see (2) above), Proposition 3.2
$P_3 \cup P_2$	3	Proposition 3.3
$K_{1,3}$	$\lfloor \frac{n}{2} \rfloor + 2$	[5] (see (1) above, see also Remark 4.2)
Y	$\max\{\lfloor \frac{n}{2} \rfloor + 2, 5\}$	Proposition 4.4
$K_{1,3} \cup P_2$	$\max\{\lfloor \frac{n}{2} \rfloor + 2, 6\}$	Proposition 4.5
C_3	n	[2] (see (3) above)
Q	n	Proposition 5.2
$3P_2$	$n+1$	[8] (for $n \geq 12$), [3] (for $n \geq 7$, see (5) above), [6] (for $n = 6$, see (6) above)
$P_3 \cup 2P_2$	$n+1$	Proposition 6.1
$C_3 \cup P_2$	$\max\{n+1, 7\}$	Proposition 6.2
$P_4 \cup P_2$	$n+1$	Proposition 6.3
P_5	$n+1$	[9] for large enough n (see (2) above), Proposition 6.4
$2P_3$	$\max\{n+1, 8\}$	Proposition 6.6
$K_{1,4}$	$n+2$	[5] (see (1) above)
C_4	$\lfloor \frac{4}{3} n \rfloor$	[1] (see (4) above)
$4P_2$	$2n-1$	[8] (for $n \geq 15$), [3] (for $n \geq 9$, see (5) above), [6] (for $n = 8$, see (6) above)

We remark that Proposition 3.3, Proposition 6.1, Proposition 6.2, Proposition 6.3 and Proposition 6.6, respectively, are used in [4] to deduce that for any integers $t \geq 1$, $k \geq 2$ and large enough n,

$$AR(n, P_2 \cup tP_3) = (t-1)\left(n - \frac{t}{2}\right) + 3,$$

$$AR(n, kP_2 \cup tP_3) = (k+t-2)\left(n - \frac{k+t-1}{2}\right) + 2,$$

$$AR(n, C_3 \cup tP_2) = t\left(n - \frac{t+1}{2}\right) + 2,$$

$$AR(n, P_4 \cup tP_2) = t\left(n - \frac{t+1}{2}\right) + 2,$$

$$AR(n, tP_3) = (t-1)\left(n - \frac{t}{2}\right) + 2.$$

2 Notation

- The complete graph on a vertex set V will be denoted K^V.
- For any (not necessarily disjoint) sets $A, B \subseteq V$ let $E(A, B) := \{uv \mid u \neq v, u \in A, v \in B\}$.
- Let c be an edge-colouring of a K^V.
  1. We denote by $c(uv)$ the colour on the edge uv.
  2. For any $v \in V$ let $C(v) := \{c(vw) \mid w \in V - \{v\}\}$ and $d_c(v) := |C(v)|$.
  3. For any colour a, let $N_c(v; a) := \{w \in V - \{v\} \mid c(vw) = a\}$.

3 Small graphs for which the anti-Ramsey number is a constant

Lemma 3.1. $AR(n, 2P_2) = \begin{cases} 4 & n = 4 \\ 2 & n \geq 5. \end{cases}$

Proof. The graph K_4 contains exactly three copies of $2P_2$, and they are edge-disjoint, so clearly $AR(4, 2P_2) = 4$. For $n \geq 5$, obviously $AR(n, 2P_2) \geq |E(2P_2)| = 2$. On the other hand, for any edge-colouring c of K_n by at least two colours, take two edges e_1, e_2 with different colours. If they are disjoint, they form a multicoloured copy of $2P_2$. Otherwise, since $n \geq 5$, there is an edge e_3 which is disjoint to both e_1 and e_2. The colour of e_3 is different than either $c(e_1)$ or $c(e_2)$ (or both), say $c(e_3) \neq c(e_2)$. Then the edges e_2, e_3 form a multicoloured copy of $2P_2$. □

Proposition 3.2. $AR(n, P_4) = \begin{cases} 4 & n = 4 \\ 3 & n \geq 5 . \end{cases}$

Proof. Clearly $AR(4, P_4) \geq AR(4, 2P_2)$, so by Lemma 3.1, $AR(4, P_4) \geq 4$. On the other hand, in any edge-colouring of K_4 by at least 4 colours there are, again by Lemma 3.1, two disjoint edges e_1, e_2, coloured by different colours. There are at least two other edges coloured differently than e_1 and e_2, and each of them completes e_1 and e_2 to a multicoloured copy of P_4.

For $n \geq 5$, obviously $AR(n, P_4) \geq |E(P_4)| = 3$. For the upper bound, let c be any edge-colouring of K_n by exactly (see Remark 1.1) 3 colours. Take a vertex v such that $d_c(v) > 1$.

If $d_c(v) = 2$, take an edge xy, such that $N_c(v; c(xy)) = \emptyset$. If $c(vx) = c(vy)$, take a vertex u such that $c(vu) \neq c(vx)$ and then $(uvxy)$ is a multicoloured copy of P_4. If $c(vx) \neq c(vy)$, take some vertex $w \notin \{v, x, y\}$. Since $d_c(v) = 2$, either $c(vw) = c(vx)$ or $c(vw) = c(vy)$, and then either $(xyvw)$ or $(yxvw)$, respectively, is a multicoloured copy of P_4.

If $d_c(v) = 3$ then since $n \geq 5$ there is a colour a for which $|N_c(v; a)| \geq 2$. Take x_1, x_2, y_1, y_2 such that $c(vx_1) = c(vx_2) = a$ and $a \neq c(vy_1) \neq c(vy_2) \neq a$. Since c uses only 3 colours, $c(x_1y_1)$ is either $c(vy_1)$, $c(vy_2)$ or a and then either $(y_1x_1vy_2)$, $(x_1y_1vx_2)$ or $(x_1y_1vy_2)$, respectively, is a multicoloured copy of P_4. □

Proposition 3.3. $AR(n, P_3 \cup P_2) = 3$ *for any* $n \geq 5$.

Proof. Obviously, $AR(n, P_3 \cup P_2) \geq |E(P_3 \cup P_2)| = 3$. For the upper bound, observe that in any edge-colouring c of K_n by at least 3 colours, there is, by Lemma 3.1, a multicoloured copy of $2P_2$, i.e., four distinct vertices x_1, x_2, y_1, y_2 such that $c(x_1x_2) \neq c(y_1y_2)$. Take an edge z_1z_2 with another colour. We divide the rest of the proof to three cases.

Case 1. $|\{z_1, z_2\} \cap \{x_1, x_2, y_1, y_2\}| = 1$.

In this case the three edges x_1x_2, y_1y_2, z_1z_2 form a multicoloured copy of $P_3 \cup P_2$.

Case 2. $\{z_1, z_2\} \cap \{x_1, x_2, y_1, y_2\} = \emptyset$.

$c(x_1y_1)$ is different than the colour of at least one of the edges x_1x_2, y_1y_2, say x_1x_2. If $c(x_1y_1) \neq c(z_1z_2)$ then $(y_1x_1x_2) \cup (z_1z_2)$ is a multicoloured copy of $P_3 \cup P_2$. Therefore we may assume that $c(x_1y_1) = c(z_1z_2)$, and similarly, that $c(y_1z_1) = c(x_1x_2)$ and $c(x_2z_2) = c(y_1y_2)$, but then $(x_1y_1z_1) \cup (x_2z_2)$ is a multicoloured copy of $P_3 \cup P_2$.

Case 3. $\{z_1, z_2\} \subset \{x_1, x_2, y_1, y_2\}$.

With no loss of generality assume that $\{z_1, z_2\} = \{x_2, y_2\}$. Take a vertex $u \notin \{x_1, x_2, y_1, y_2\}$. If $c(ux_1) \notin \{c(y_1y_2), c(x_2y_2)\}$ then $(x_2y_2y_1) \cup (ux_1)$ is a multicoloured copy of $P_3 \cup P_2$, and if $c(ux_1) = c(x_2y_2)$ then $(ux_1x_2) \cup (y_1y_2)$ is a multicoloured copy of $P_3 \cup P_2$. Therefore we may assume that $c(ux_1) = c(y_1y_2)$, and similaly that $c(uy_1) = c(x_1x_2)$, but then $(x_1uy_1) \cup (x_2y_2)$ is a multicoloured copy of $P_3 \cup P_2$. □

4 Small graphs for which $AR(n, G) = \lfloor n/2 \rfloor + 2$

Definition. *Let $c_{matching}$ be the edge-colouring of K_n by $\lfloor n/2 \rfloor + 1$ colours, in which all edges of some chosen maximal matching are coloured by distinct colours, and all other edges are coloured by one additional colour.*

Lemma 4.1. *If c is an edge-colouring of K_n by exactly r colours and $d_c(v) \leq 2$ for any vertex v, then $r \leq \max\{\lfloor n/2 \rfloor + 1, 3\}$.*

Proof. For any vertex v, $|C(v)| = d_c(v) \leq 2$, and for any distinct vertices v_1, v_2, $C(v_1) \cap C(v_2) \neq \emptyset$ (since $c(v_1v_2) \in C(v_1) \cap C(v_2)$). If $|\bigcap_{v \in V} C(v)| = 2$ then clearly $r = 2$. If $\bigcap_{v \in V} C(v) = \emptyset$ then it is easy to see that $r \leq 3$. Finally, if $\bigcap_{v \in V} C(v) = \{a\}$ for some colour a then edges of different colours which are not a are necessarily disjoint, hence $r \leq \lfloor n/2 \rfloor + 1$. □

Remark 4.2. *It follows immediately from Lemma 4.1 that for any $n \geq 4$, $AR(n, K_{1,3}) \leq \lfloor n/2 \rfloor + 2$. On the other hand, the colouring $c_{matching}$ of K_n shows that $AR(n, K_{1,3}) > \lfloor n/2 \rfloor + 1$, hence $AR(n, K_{1,3}) = \lfloor n/2 \rfloor + 2$, as was proved in [5] (see (1)).*

Definition 4.3. *Let Y be the graph obtained from a star $K_{1,3}$ by adding a vertex and an edge connecting it to one leaf of the star.*

Proposition 4.4. $AR(n, Y) = \max\{\lfloor n/2 \rfloor + 2, 5\}$ *for any* $n \geq 5$.

Proof. **Lower bound:** To show that $AR(n, Y) > \lfloor n/2 \rfloor + 1$, we use the colouring c_{matching} of K_n. To show that $AR(n, Y) > 4$, colour the edges of some triangle by distinct colours, and all other edges of K_n by one additional colour.

Upper bound: Let c be any edge-colouring of K_n by at least $\max\{\lfloor n/2 \rfloor + 2, 5\}$ colours. By Lemma 4.1 there is a vertex u such that $d_c(u) \geq 3$.

If $d_c(u) \geq 5$, take vertices v_1, v_2, v_3, v_4, v_5 such that $c(uv_1)$, $c(uv_2)$, $c(uv_3)$, $c(uv_4)$, $c(uv_5)$ are distinct. The colour $c(v_1v_2)$ is different than at least one of the colours $c(uv_1), c(uv_2)$ and at least two of the colours $c(uv_3), c(uv_4), c(uv_5)$. With no loss of generality, assume that $c(v_1v_2)$ is different than $c(uv_2)$, $c(uv_3)$ and $c(uv_4)$. The edges v_1v_2, uv_2, uv_3, uv_4 form a multicoloured copy of Y.

If $d_c(u) = 4$ then since $\max\{\lfloor n/2 \rfloor + 2\,,\,5\} > 4$ there is some edge v_1v_2 such that $c(v_1v_2) \notin C(u)$. Take vertices v_3, v_4 such that $c(uv_3), c(uv_4)$ are distinct and different than $c(uv_1), c(uv_2)$. The edges v_1v_2, uv_2, uv_3, uv_4 form a multicoloured copy of Y.

If $d_c(u) = 3$ then since $n \geq 5$, there is at most one edge v_1v_2 such that $|N_c(c(uv_1), u)| = |N_c(c(uv_2), u)| = 1$. Therefore, there must be at least one edge v_1v_2 such that $c(v_1v_2) \notin C(u)$ and $|N_c(c(uv_i), u)| \geq 2$ for at least one $i \in \{1, 2\}$, say $i = 1$. If $c(uv_1) = c(uv_2)$, take vertices x_1, x_2 such that $c(ux_1), c(ux_2)$ are the additional two colours in $C(u)$; The edges v_1v_2, uv_2, ux_1, ux_2 form a multicoloured copy of Y. If $c(uv_1) \neq c(uv_2)$, take $v_1 \neq y \in N_c(c(uv_1), u)$ and a vertex z such that $c(uz) \notin \{c(uv_1), c(uv_2)\}$; The edges v_1v_2, uv_2, uy, uz form a multicoloured copy of Y. □

Proposition 4.5. *$AR(n, K_{1,3} \cup P_2) = \max\{\lfloor n/2 \rfloor + 2\,,\,6\}$ for any $n \geq 6$.*

Proof. **Lower bound:** To show that $AR(n, K_{1,3} \cup P_2) > \lfloor n/2 \rfloor + 1$ we use the colouring c_{matching} of K_n. To show that $AR(n, K_{1,3} \cup P_2) > 5$ colour the edges of some cycle of length 4 by distinct colours, and all other edges of K_n by one additional colour.

Upper bound: We omit the proof for $n = 6$, which is a simple but tedious case analysis, and assume that $n \geq 7$. Let c be any edge-colouring of K_n by at least $\max\{\lfloor n/2 \rfloor + 2\,,\,6\}$ colours. By Lemma 4.1, there is a vertex u such that $d_c(u) \geq 3$.

If $d_c(u) \geq 4$, take $v_1, v_2, v_3, v_4 \neq u$ such that uv_1, uv_2, uv_3, uv_4 have different colours, and two additional vertices $w, z \notin \{u, v_1, v_2, v_3, v_4\}$. At most one of the edges uv_1, uv_2, uv_3, uv_4, say uv_4, is coloured by $c(wz)$, and then the edges uv_1, uv_2, uv_3, wz form a multicoloured copy of $K_{1,3} \cup P_2$.

If $d_c(u) = 3$ and assume, by contradiction, that there is no multicoloured copy of $K_{1,3} \cup P_2$. By what we just shown, it follows that $d_c(v) \leq 3$ for any vertex v. For any $a \in C(u)$ such that $|N_c(u; a)| \geq 3$, all edges of $K^{N_c(u;a)}$ must be coloured by colours from $C(u)$. For any $a_1, a_2 \in C(u)$ such that $|N_c(u; a_1)|, |N_c(u; a_2)| \geq 2$, all edges in $E(N_c(u; a_1), N_c(u; a_2))$ must be coloured by colours from $C(u)$. Combining all these we get, by a simple case analysis, that the total number of colours c uses is at most 6 if the multiset $\{|N_c(u; a)|\}_{a \in C(u)}$ is either $\{2, 2, 2\}$ (when $n = 7$), $\{1, 1, n-3\}$ or $\{1, 2, n - 4\}$, and at most 5 otherwise. We then immediately get a contradiction if $n \geq 10$, for which $\lfloor n/2 \rfloor + 2 > 6$. For $7 \leq n \leq 9$, a further, simple but somewhat tedious, examination of the three cases mentioned above is needed. □

5 A small graph for which $AR(n, G) = n$

Definition 5.1. *Let Q be the graph obtained from a triangle C_3 by adding a vertex and an edge connecting it to one vertex of the triangle.*

Proposition 5.2. $AR(n, Q) = n$ *for any* $n \geq 4$.

Proof. The lower bound is established by colouring each edge $\{i, j\}$ of $K^{\{1,2,\ldots,n\}}$ by the colour $\min\{i, j\}$.

Assume, by contradiction, that $AR(n, Q) > n$ for some $n \geq 4$. Take minimal such n, and an edge-colouring c of K^V, $|V| = n$, by at least n colours with no multicoloured copy of Q. Since $AR(n, C_3) = n$ ([2], see (3)), there is a multicoloured triangle $\Delta x_1x_2x_3$. Since there is no multicoloured copy of Q, $c(ux_i) \in \{c(x_1x_2), c(x_2x_3), c(x_3x_1)\}$ for any $u \in V - \{x_1, x_2, x_3\}$ and any $1 \leq i \leq 3$. We get that $K^{V-\{x_1,x_2,x_3\}}$ is edge-coloured by at least $n - 3$ colours with no multicoloured copy of Q. By the minimality of n we conclude that $n - 3 < 4$. Since the $\binom{n-3}{2}$ edges of $K^{V-\{x_1,x_2,x_3\}}$ are coloured by at least $n - 3$ colours, we must have that $n = 6$ and that the three edges of the triangle $K^{V-\{x_1,x_2,x_3\}}$ are coloured by 3 distinct colours, all different than $c(x_1x_2), c(x_2x_3), c(x_3x_1)$. Adding any edge of $E(\{x_1, x_2, x_3\}, V - \{x_1, x_2, x_3\})$ to the triangle $K^{V-\{x_1,x_2,x_3\}}$ we get a multicoloured copy of Q, and thus a contradiction. □

6 Small graphs for which $AR(n, G) = n + 1$

Definition. *Let c_{star} be the edge-colouring of K_n by n colours, in which all edges incident with some chosen vertex are coloured by distinct colours, and all other edges are coloured by one additional colour.*

Proposition 6.1. $AR(n, P_3 \cup 2P_2) = n + 1$ *for any* $n \geq 7$.

Proof. The lower bound follows by using the colouring c_{star} of K_n.

For the upper bound, let c be any edge-colouring of K_n by at least $n + 1$ colours. Take a set of maximal size of disjoint edges $\{e_i\}_{i=1}^m$ with distinct colours. Since $AR(n, 3P_2) = n + 1$ ([3], see (5)), we have that $m \geq 3$. Let B be the set of edges whose colour is not in $\{c(e_i)\}_{i=1}^m$. By the maximality of m, any $e \in B$ must have an endpoint in common with at least one of the edges $\{e_i\}_{i=1}^m$. We can now clearly get a multicoloured copy of $P_3 \cup 2P_2$, unless $m = 3$ and every $e \in B$ has both endpoints in the set S of endpoints of the edges e_1, e_2, e_3. In this case, form a graph G by taking a single edge of each colour not in $\{c(e_1), c(e_2), c(e_3)\}$. If G contains a copy H of $P_3 \cup P_2$, we get a multicoloured copy of $P_3 \cup 2P_2$ by adding to H any edge e of K_n disjoint to H (at least one of the endpoints of e is not in S, so $e \notin B$, i.e., $c(e) \in \{c(e_1), c(e_2), c(e_3)\}$). If G does not contain a copy of $P_3 \cup P_2$, then a simple case analysis shows that G has only four vertices and five edges (so necessarily $n = 7$). For any $1 \leq i \leq 3$, let x_i, y_i be the endpoints of e_i, then with no loss of generality, the edges of G are $x_1y_2, x_1y_3, y_1y_2, y_1y_3$ and y_2y_3. Let u be the only remaining vertex. If either $c(ux_2) \neq c(x_3y_3)$ or $c(ux_3) \neq c(x_2y_2)$ then either $(x_1y_2y_1) \cup (ux_2) \cup (x_3y_3)$ or $(x_1y_3y_1) \cup (ux_3) \cup (x_2y_2)$ is a multicoloured copy of $P_3 \cup 2P_2$, and if $c(ux_2) = c(x_3y_3)$ and $c(ux_3) = c(x_2y_2)$ then $(x_2ux_3) \cup (x_1y_1) \cup (y_2y_3)$ is a multicoloured copy of $P_3 \cup 2P_2$. □

Proposition 6.2. *$AR(n, C_3 \cup P_2) = \max\{n+1, 7\}$ for any $n \geq 5$.*

Proof. **Lower bound:** To show that $AR(n, C_3 \cup P_2) > n$ we use the colouring c_{star} of K_n. To show that $AR(5, C_3 \cup P_2) > 6$, let u, x_1, x_2, y_1, y_2 be the vertices of K_5; Colour each of the four edges ux_1, ux_2, uy_1, uy_2 by distinct colours, the edges x_1x_2, y_1y_2 by a fifth colour, and all other edges by a sixth colour.

Upper bound: Let c be any edge-colouring of K_n by at least $r := \max\{n+1, 7\}$ colours. Since $AR(n, C_3) = n$ ([2], see (3)), there is a multicoloured triangle $\Delta x_1x_2x_3$.

If $|\bigcup_{i=1}^{3} C(x_i)| \leq n$, then there is at least one edge e such that $c(e) \notin \bigcup_{i=1}^{3} C(x_i)$. In particular, e is disjoint to the triangle $\Delta x_1x_2x_3$ and $c(e) \notin \{c(x_1x_2), c(x_2x_3), c(x_3x_1)\}$, so $\Delta x_1x_2x_3 \cup e$ is a multicoloured copy of $C_3 \cup P_2$. If $|\bigcup_{i=1}^{3} C(x_i)| \geq n+1$ then there must be some vertex $x_4 \notin \{x_1, x_2, x_3\}$ such that $|\{c(x_ix_j)\}_{1 \leq i < j \leq 4}| \geq 5$.

If $|\{c(x_ix_j)\}_{1 \leq i < j \leq 4}| = 6$, then since $r \geq 7$ there must be at least one edge e such that $c(e) \notin \{c(x_ix_j)\}_{1 \leq i < j \leq 4}$. At most one of x_1, x_2, x_3, x_4, say x_1, is an endpoint of e and then $\Delta x_2x_3x_4 \cup e$ is a multicoloured copy of $C_3 \cup P_2$.

If $|\{c(x_ix_j)\}_{1 \leq i < j \leq 4}| = 5$, then at most one of the triangles $\Delta x_2x_3x_4$, $\Delta x_1x_3x_4$, $\Delta x_1x_2x_4$, $\Delta x_1x_2x_3$, say $\Delta x_2x_3x_4$, is not multicoloured. We now consider two cases.

Case 1. There is an edge e not incident with x_1 such that $c(e) \notin \{c(x_ix_j)\}_{1 \leq i < j \leq 4}$.

At least one of the multicoloured triangles $\Delta x_1x_3x_4, \Delta x_1x_2x_4, \Delta x_1x_2x_3$ is disjoint to e and then this triangle with the edge e form a multicoloured copy of $C_3 \cup P_2$.

Case 2. x_1 is incident with any edge e such that $c(e) \notin \{c(x_ix_j)\}_{1 \leq i < j \leq 4}$.

Since $r \geq 7$ there must be at least two such edges x_1y_1, x_1y_2 with distinct colours. If $c(y_1y_2) \in \{c(x_1y_1), c(x_1y_2)\}$ then $\Delta x_1x_2x_3 \cup y_1y_2$ is a multicoloured copy of $C_3 \cup P_2$. If $c(y_1y_2) \notin \{c(x_1y_1), c(x_1y_2)\}$, then the colour of at least one of the three edges x_2x_3, x_3x_4, x_4x_2, say x_2x_3 is not $c(y_1y_2)$, and then $\Delta x_1y_1y_2 \cup x_2x_3$ is a multicoloured copy of $C_3 \cup P_2$. □

In the proofs below we use the following notion.

Definition. *A w-colour, for a vertex w, is a colour that only appears on edges incident with w.*

Proposition 6.3. $AR(n, P_4 \cup P_2) = n + 1$ *for any* $n \geq 6$.

Proof. The lower bound follows by using the colouring c_{star} of K_n.

The upper bound is proved by induction on n. For the base case $n = 6$, consider any edge-colouring of K_6 by at least 7 colours. Since $AR(6, 3P_2) = 7$ ([6], see (6)), there is a multicoloured copy of $3P_2$. This copy together with any edge coloured in any of the remaining colours form a multicoloured copy of $P_4 \cup P_2$.

Now let $n \geq 7$, assume that $AR(n-1, P_4 \cup P_2) = n$, and consider any edge-colouring c of K_n by at least $n+1$ colours. If there exists a vertex v having less than two v-colours, then removing v along with its incident edges from the graph, at most one colour disappears from the graph, and the claim follows by the induction hypothesis. Therefore we assume that every vertex v in the graph has at least two v-colours.

Let u, x_1, x_2 be vertices such that $c(ux_1), c(ux_2)$ are two distinct u-colours, and let $y \notin \{u, x_1, x_2\}$ be some other vertex. Since there are at least two y-colours, there is some $z \neq u$ such that $c(yz)$ is a y-colour. If $z \notin \{x_1, x_2\}$, let $w \notin \{u, x_1, x_2, y, z\}$ be another vertex, then $(x_1ux_2w) \cup (yz)$ is a multicoloured copy of $P_4 \cup P_2$. If $z \in \{x_1, x_2\}$, say $z = x_1$, let $v_1, v_2 \notin \{u, x_1, x_2, y\}$ be two other vertices, then $(yx_1ux_2) \cup (v_1v_2)$ is a multicoloured copy of $P_4 \cup P_2$. □

Proposition 6.4. $AR(n, P_5) = n + 1$ *for any* $n \geq 5$.

Proof. The lower bound follows by using the colouring c_{star} of K_n.

The upper bound is proved by induction on n. For the base case $n = 5$, let c be any edge-colouring of K_5 by at least 6 colours. Since $AR(n, C_3) = n$ ([2], see (3)), there is a multicoloured triangle $\Delta x_1x_2x_3$. Let y_1, y_2 be the remaining two vertices.

If $c(y_1y_2) \notin \{c(x_1x_2), c(x_2x_3), c(x_3x_1)\}$, let x_iy_j, $1 \leq i \leq 3$, $1 \leq j \leq 2$ be an edge such that $c(x_iy_j) \notin \{c(x_1x_2), c(x_2x_3), c(x_3x_1), c(y_1y_2)\}$. With no loss of generality assume that $i = 3$ and $j = 1$, then $(x_1x_2x_3y_1y_2)$ is a multicoloured copy of P_5. If $c(y_1y_2) \in \{c(x_1x_2), c(x_2x_3), c(x_3x_1)\}$, say $c(y_1y_2) = c(x_1x_2)$, then since there are at least three edges whose colours are not in $\{c(x_1x_2), c(x_2x_3), c(x_3x_1)\}$, one of those edges must be in $E(\{x_1, x_2\}, \{y_1, y_2\})$, say x_1y_2, and then $(y_1y_2x_1x_2x_3)$ is a multicoloured copy of P_5

Now let $n \geq 6$, assume that $AR(n-1, P_5) \leq n$, and consider any edge-colouring c of K_n by at least $n+1$ colours. If there is a vertex v having less than two v-colours, then removing v along with its incident edges from the graph, at most one colour disappears from the graph, and the claim follows by the induction hypothesis. Therefore we assume that every vertex v has at least two v-colours.

Let u, x_1, x_2 be vertices such that $c(ux_1), c(ux_2)$ are two distinct u-colours, and let $y \notin \{u, x_1, x_2\}$ be some other vertex. If there is a vertex $z \notin \{u, x_1, x_2, y\}$ such that $c(yz)$ is a y-colour, then (x_1ux_2zy) is a multicoloured copy of P_5. Otherwise, all the edges having a y-colour are among the edges yu, yx_1, yx_2, so at least one of $c(yx_1), c(yx_2)$, say $c(yx_2)$, is a y-colour. Let $z \notin \{u, x_1, x_2, y\}$ be some other vertex, then (zx_1ux_2y) is a multicoloured copy of P_5. □

For the proof of our last result, Proposition 6.6 below, we need the following lemma.

Lemma 6.5. $AR(7, 2P_3) \leq 8$.

Proof. Let c be any edge-colouring of K_7 by at least 8 colours. Assume first there is a vertex u such that $d_c(u) = 6$. There must be three other vertices v_1, v_2, v_3 such that $c(v_1v_2) \neq c(v_2v_3)$ and $c(v_2v_3) \notin C(u)$. Let w_1, w_2, w_3 be the remaining vertices. At most one of the edges uw_1, uw_2, uw_3, say uw_2, is coloured by $c(v_1v_2)$, then $(v_1v_2v_3) \cup (w_1uw_3)$ is a multicoloured copy of $2P_3$.

We now assume $d_c(u) < 6$ for any vertex u. Since $AR(7, C_3 \cup P_2) = 8$, by Proposition 6.2, there is a multicoloured triangle $\Delta x_1x_2x_3$, and at least one edge disjoint to it whose colour a is not in $\{c(x_1x_2), c(x_2x_3), c(x_3x_1)\}$. Let y_1, y_2, y_3, y_4 be the remaining vertices. If not all six edges $\{y_iy_j\}_{1\leq i<j\leq 4}$ are coloured by a, then in $K^{\{y_1,y_2,y_3,y_4\}}$ there are surely two adjacent edges e_1 and e_2 such that $c(e_1) = a \neq c(e_2)$. At most one of the edges of the triangle $\Delta x_1x_2x_3$ is coloured by $c(e_2)$ and then the other two edges of the triangle, together with e_1 and e_2 form a multicoloured copy of $2P_3$. We therefore assume all six edges $\{y_iy_j\}_{1\leq i<j\leq 4}$ are coloured by a. There are at least four edges in $E(\{x_i\}_{1\leq i\leq 3}, \{y_j\}_{1\leq j\leq 4})$ having distinct colours not in $\{c(x_1x_2), c(x_2x_3), c(x_3x_1), a\}$, and since $d_c(x_i) < 6$ for any $1 \leq i \leq 3$, two of those edges must be disjoint. With no loss of generality assume these are x_2y_2 and x_3y_3, then $(x_1x_2y_2) \cup (x_3y_3y_4)$, for example, is a multicoloured copy of $2P_3$. □

Proposition 6.6. $AR(n, 2P_3) = \max\{n+1, 8\}$ *for any* $n \geq 6$.

Proof. **Lower bound:** To show that $AR(n, 2P_3) > n$, we use the colouring c_{star} of K_n. To show that $AR(n, 2P_3) > 7$, colour the edges between some four vertices by six distinct colours, and all other edges by one additional colour.

Upper bound: To show that $AR(6, 2P_3) \leq 8$, consider any edge-colouring of K_6 by at least 8 colours. Since $AR(6, 3P_2) = 7$ ([6], see (6)), there is a multicoloured copy of $3P_2$. Form a graph by adding to those three edges a single edge of each of the remaining colours. It is easy, but a bit tedious, to check that this graph must contain a copy of $2P_3$ (which is obviously multicoloured).

For $n \geq 7$, we prove that $AR(n, 2P_3) \leq n+1$ by induction on n. Lemma 6.5 takes care of the base case $n = 7$. Let $n \geq 8$, assume that $AR(n-1, 2P_3) = n$, and consider any edge-colouring c of K_n by at least $n+1$ colours. If there exists a vertex v having less than two v-colours, then removing v along with its incident edges from the graph, at most one colour disappears from the graph, and the claim follows by the induction hypothesis. Therefore we assume that every vertex v has at least two v-colours.

Let u, x_1, x_2 be vertices such that $c(ux_1), c(ux_2)$ are two distinct u-colours. If there are vertices $y, z \notin \{u, x_1, x_2\}$ such that $c(yz)$ is a y-colour, let $w \notin \{u, x_1, x_2, y, z\}$ be some other vertex, then $(x_1ux_2) \cup (yzw)$ is a multicoloured copy of $2P_3$. We therefore assume that for every vertex $y \notin \{u, x_1, x_2\}$, all the edges having a y-colour are among the edges yu, yx_1, yx_2.

Since there are at least 3 vertices other than u, x_1, x_2, and each vertex $y \notin \{u, x_1, x_2\}$ has at least two y-colours, there must be, by the pigeonhole principle, at least two vertices $y_1, y_2 \notin \{u, x_1, x_2\}$, and an $i \in 1, 2$ such that $c(y_1x_i)$ is a y_1-colour and $c(y_2x_i)$ is a y_2-colour. With no loss of generality assume that $i = 1$, and let $w \notin \{u, x_1, x_2, y_1, y_2\}$ be another vertex, then $(ux_2w) \cup (y_1x_1y_2)$ is a multicoloured copy of $2P_3$. □

Acknowledgements

We would like to thank the referee for his enlightening comments.

References

[1] N. Alon, On a conjecture of Erdős, Simonovits, and Sós concerning anti-Ramsey theorems, J. Graph Theory **7** (1983), no. 1, 91–94.

[2] P. Erdős, M. Simonovits and V. T. Sós, Anti-Ramsey theorems, in *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, 633–643. Colloq. Math. Soc. János Bolyai, 10, North-Holland, Amsterdam.

[3] S. Fujita, A. Kaneko, I. Schiermeyer and K. Suzuki, A rainbow k-matching in the complete graph with r colors, Electron. J. Combin. **16** (2009), no. 1, Research Paper 51, 13 pp.

[4] S. Gilboa and Y. Roditty, Anti-Ramsey numbers of graphs with small connected components, Graphs and Combinatorics (to appear).

[5] T. Jiang, Edge-colorings with no large polychromatic stars, Graphs Combin. **18** (2002), no. 2, 303–308.

[6] R. Haas and M. Young, The anti-Ramsey number of perfect matching, Discrete Math. **312** (2012), no. 5, 933–937.

[7] J. J. Montellano-Ballesteros and V. Neumann-Lara, An anti-Ramsey theorem on cycles, Graphs Combin. **21** (2005), no. 3, 343–354.

[8] I. Schiermeyer, Rainbow numbers for matchings and complete graphs, Discrete Math. **286** (2004), no. 1-2, 157–162.

[9] M. Simonovits and V. T. Sós, On restricted colorings of K_n, Combinatorica 4 (1984), no. 1, 101–110.