On restricted edge connectivity of strong product graphs[1]

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Abstract An explicit expression of the restricted edge connectivity of strong product of two triangle-free graphs is presented, which yields a sufficient and necessary condition for these strong product graphs to be super restricted edge connected.

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1 Introduction

A restricted edge cut is an edge cut S of a connected graph G such that $G - S$ contains no isolated vertices. The minimum cardinality $\lambda'(G)$ over all restricted edge cuts of graph G is called its restricted edge connectivity. If denote by $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$ the minimum edge degree of graph G, then $\lambda'(G) \leq \xi(G)$ holds for all connected graphs of order at least four that are not stars [7]. Graph G is called super restricted edge connected, or for short *super-λ'*, if every minimum restricted edge cut consists of edges adjacent to an edge. Super restricted edge connectivity plays an important role in reliability analysis of telenets [7,12] and draws a lot of attentions [1-3, 10, 13-16]. For details on advance of optimizing restricted edge connectivity, the readers are suggested to refer to a survey [9].

Given two graphs G_1 and G_2, the strong product $G_1 \boxtimes G_2$ has vertex set $V(G_1) \times V(G_2)$, where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1y_2 \in E(G_2)$, or $y_1 = y_2$ and $x_1x_2 \in E(G_1)$, or $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$. Occasionally one also use *strong direct product* or *symmetric composition* rather than strong product. The properties of strong product graphs are widely studied, the readers can refer to [8] and a monograph [11].

In [5], [6] and elsewhere, the authors present some basic properties on the edge connectivity of strong product graphs. This work studies the restricted edge connectivity of these product graphs. As a result, an explicit expression on the restricted edge connectivity of strong product graphs is

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presented. Sufficient conditions for these graphs to be super restricted edge connected are also obtained.

Before proceeding, let us introduce some more symbols and terminology. For two subgraphs (or two subsets) P and Q of graph G, $[P,Q]$ denotes the set of edges with one end in P and the other in Q. Let $\lambda(G)$ or simply λ indicate the edge connectivity of graph G. For other symbols and terminology not specially stated, we follow that of [4].

2 Auxiliary lemmas

In [5, 6], the authors present an explicit expression of the edge connectivity of any strong product graphs as follows.

Lemma 2.1 [5,6] Let G_i be nontrivial connected graphs with order n_i, size m_i, minimum degree δ_i and edge connectivity λ_i, $i = 1, 2$. Then $\lambda(G_1 \boxtimes G_2) = \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1 + \delta_2 + \delta_1\delta_2\}$.

Let K_2 be a complete graph with $V(K_2) = \{a, b\}$ and H be a connected graph. We define $K_2 \odot H = K_2 \boxtimes H - E(\{a\} \boxtimes H) - E(\{b\} \boxtimes H)$. It is not difficult to see that $K_2 \odot H$ is connected if and only if H is connected.

Lemma 2.2 [6] Let H be a connected graph and S be an edge cut of $K_2 \odot H$. If the vertices of $\{a\} \boxtimes H$ are in different components of $K_2 \odot H - S$ as well as $\{b\} \boxtimes H$, then $|S| \geq 2\lambda(H)$.

Lemma 2.3 [6] Let H be a connected graph and S be an edge cut of $K_2 \odot H$. If there is a vertex $x \in V(H)$ such that (a, x) and (b, x) are in different components of $K_2 \odot H - S$, then $|S| \geq \delta(H) + 1$.

Let $X \subsetneq V(G)$ be a nonempty subset and u be a vertex in X. Then $\delta(G) \leq d(u) \leq (|X| - 1) + |[X, \bar{X}]|$, where $\bar{X} = V(G) - X$ and $d(u)$ represents the degree of vertex u in G. The following lemma 2.4 follows directly from this observation.

Lemma 2.4 Let G be a connected graph. If X is a nonempty subset of $V(G)$, then $|X| + |[X, \bar{X}]| \geq \delta(G) + 1$ with the equality holding if and only if $X = \{u\}$ and $d(u) = \delta(G)$. □

Lemma 2.5 Let G be a triangle-free connected graph and A be a subset of $V(G)$. If $G[A]$ contains at least one edge, then $|A| + |[A, \bar{A}]| \geq \xi(G) + 2$.

Proof. Since G is triangle-free, it follows that $N(u) \bigcap N(v) = \emptyset$ holds for any edge $uv \in E(G[A])$. Noticing that A contains at most $|A|-2$ edges adjacent to uv, we deduce that

$$|A| - 2 + |[A, \bar{A}]| \geq d(u) + d(v) - 2 \geq \xi(G).$$

The lemma follows from above formula. □

3 Restricted edge connectivity

For convenience, we simplify $d_{G_1}(t)$, $d_{G_2}(t)$ and $d_{G_1 \boxtimes G_2}(t)$ as $d_1(t)$, $d_2(t)$ and $d(t)$ respectively in this section. For any vertex $(u, v) \in V(G_1 \boxtimes G_2)$, it's not difficult to see that $d((u,v)) = d_1(u)d_2(v) + d_1(u) + d_2(v)$.

Lemma 3.1 If G_i are nontrivial connected graphs with minimum degree δ_i and minimum edge degree ξ_i, $i = 1, 2$, then $\xi(G_1 \boxtimes G_2) = \min\{\delta_1\xi_2 + \xi_2 + 4\delta_1, \delta_2\xi_1 + \xi_1 + 4\delta_2\}$.

Proof. Let u be a minimum-degree vertex of G_1, v_1v_2 be an edge of G_2 with $d_2(v_1v_2) = \xi_2$. Then the set of edges of $G_1 \boxtimes G_2$ that are incident with $e_1 = (u, v_1)(u, v_2)$ can be partitioned into following subsets: $\{(u, v_1)(u, x) : x \in N_{G_2}(v_1) - \{v_2\}\} \cup \{(u, v_2)(u, x) : x \in N_{G_2}(v_2) - \{v_1\}\}$, $\{(u, v_1)(y, v_1) : uy \in E(G_1)\}$, $\{(u, v_2)(y, v_2) : uy \in E(G_1)\}$, $\{(u, v_1)(y, v_2) : uy \in E(G_1)\} \cup \{(u, v_2)(y, v_1) : uy \in E(G_1)\}$ and $\{(u, v_1)(y, x) : y \in N_{G_1}(u), x \in N_{G_2}(v_1) - \{v_2\}\} \cup \{(u, v_2)(y, x) : y \in N_{G_1}(u), x \in N_{G_2}(v_2) - \{v_1\}\}$. These subsets has cardinality ξ_2, δ_1, δ_1, $2\delta_1$ and $\delta_1\xi_2$ respectively. Hence, $d(e_1) = \delta_1\xi_2 + \xi_2 + 4\delta_1$. By symmetry of G_1 and G_2 in $G_1 \boxtimes G_2$, there is an edge $e_2 \in E(G_1 \boxtimes G_2)$ with $d(e_2) = \delta_2\xi_1 + \xi_1 + 4\delta_2$. In conclusion, we have

$$\xi(G_1 \boxtimes G_2) \leq \min\{\delta_1\xi_2 + \xi_2 + 4\delta_1, \delta_2\xi_1 + \xi_1 + 4\delta_2\}.$$

To prove $\xi(G_1 \boxtimes G_2) \geq \min\{\delta_1\xi_2 + \xi_2 + 4\delta_1, \delta_2\xi_1 + \xi_1 + 4\delta_2\}$, we need only show that $d(e) \geq \min\{\delta_1\xi_2 + \xi_2 + 4\delta_1, \delta_2\xi_1 + \xi_1 + 4\delta_2\}$ holds for every edge of $G_1 \boxtimes G_2$. Now, let $e = (x_1, y_1)(x_2, y_2)$ be an arbitrary edge of $G_1 \boxtimes G_2$.

If $x_1 = x_2$, then $y_1y_2 \in E(G_2)$ and

$$\begin{aligned} d(e) &= d((x_1, y_1)) + d((x_2, y_2)) - 2 \\ &= d_1(x_1)(d_2(y_1) + d_2(y_2) + 2) + d_2(y_1) + d_2(y_2) - 2 \\ &= d_1(x_1)(d_2(y_1) + d_2(y_2) - 2) + 4d_1(x_1) + (d_2(y_1) + d_2(y_2) - 2) \\ &\geq \delta_1\xi_2 + 4\delta_1 + \xi_2. \end{aligned}$$

By symmetry, we deduce that $d(e) \geq \delta_2\xi_1 + 4\delta_2 + \xi_1$ if $y_1 = y_2$. Finally, if $x_1 \neq x_2$ and $y_1 \neq y_2$, without loss of generality, assume that

$d_1(x_1) \geq d_1(x_2)$, then similarly to the case when $x_1 = x_2$ one can prove with ease that $d(e) \geq \delta_1\xi_2 + 4\delta_1 + \xi_2$. And so, the lemma follows. □

Theorem 3.2 Let G_i be nontrivial connected graphs with order n_i, size m_i, minimum degree δ_i , edge connectivity λ_i and minimum edge degree ξ_i, $i = 1, 2$. If they are triangle-free, then

$$\lambda'(G_1 \boxtimes G_2) = \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \xi(G_1 \boxtimes G_2)\}$$
$$= \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \delta_1\xi_2 + \xi_2 + 4\delta_1, \delta_2\xi_1 + \xi_1 + 4\delta_2\}.$$

Proof By lemma 3.1, it suffices to prove the first equation. Let $[X, \bar{X}]$ be a minimum edge cut of G_1. Then $[X \times V(G_2), \bar{X} \times V(G_2)]$ is a restricted edge cut of $G_1 \boxtimes G_2$. For any given vertices $u \in V(G_2)$ and $v \in V(G_1)$, let us dífine $G_2^v = \{v\} \boxtimes G_2$ and $G_1^u = G_1 \boxtimes \{u\}$. With this convention, we have $|E(G_1^u) \cap [X \times V(G_2), \bar{X} \times V(G_2)]| = \lambda_1$. For any edges $uv \in E(G_2)$ and $xy \in [X, \bar{X}]$, we have $\{(x,u)(y,v), (x,v)(y,u)\} \subseteq [X \times V(G_2), \bar{X} \times V(G_2)]$. It follows from these observations that

$$\lambda'(G_1 \boxtimes G_2) \leq |[X \times V(G_2), \bar{X} \times V(G_2)]| = \lambda_1(n_2 + 2m_2).$$

By the symmetry of G_1 and G_2 in $G_1 \boxtimes G_2$, we deduce that

$$\lambda'(G_1 \boxtimes G_2) \leq \lambda_2(n_1 + 2m_1).$$

Combining the above two formulae with the well-known observation that if a connected graph G of order at least four is not a star then $\lambda'(G) \leq \xi(G)$ [1], we obtain the following inequality.

$$\lambda'(G_1 \boxtimes G_2) \leq \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1), \xi(G_1 \boxtimes G_2)\}.$$

To prove the converse of above inequality, let $S = [F, \bar{F}]$ be a minimum restricted edge cut of $G_1 \boxtimes G_2$. For any edge $e \in E(G_1)$, define $S_e = S \cap E(e \odot G_2)$. Subgraph G_2^x is called separated by S if and only if $G_2^x \cap F \neq \emptyset \neq G_2^x \cap \bar{F}$. Let $V(G_1) = \{x_1, x_2, \ldots, x_{n_1}\}$, $S_{x_i} = S_i = S \cap E(G_2^{x_i})$, $r = |\{x \in V(G_1) : G_2^x$ is separated by S $\}|$ and $s = |\{y \in V(G_2) : G_1^y$ is separated by S $\}|$. Without loss generality, assume that $\delta_1 \leq \delta_2$.

If $r = n_1$, then $|S_i| \geq \lambda_2$ for all $1 \leq i \leq n_1$. By lemma 2.2, $|S_e| \geq 2\lambda_2$ holds for every edge $e \in E(G_1)$. Hence

$$|S| \geq \sum_{i=1}^{n_1} |S_i| + \sum_{e \in E(G_1)} |S_e| \geq n_1\lambda_2 + m_1 \cdot 2\lambda_2 = \lambda_2(n_1 + 2m_1).$$

If $r = 0$, then $s = n_2$. Similarly to the case when $r = n_1$, we have $|S| \geq \lambda_1(n_2 + 2m_2)$. By the symmetry of G_1 and G_2 in $G_1 \boxtimes G_2$, we deduce that if $s = n_2$ or 0 then $|S| \geq \min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1)\}$. And so, we may assume in what follows that $1 \leq r \leq n_1 - 1$ and $1 \leq s \leq n_2 - 1$. The first inequality of this assumption implies that there are two vertices $x_a \in V(G_1)$ and $x_b \in N_{G_1}(x_a)$ such that $G_2^{x_b}$ is separated by S but $G_2^{x_a}$ is not.

Without loss of generality, assume that $V(G_2^{x_a}) \subseteq \bar{F}$. Let $K = \{y \in V(G_2) : (x_b, y) \in F\}$, H be the maximum-order connected subgraph of G_1 such that $x_b \in V(H)$ and $\{y \in V(G_2) : (x_i, y) \in F\} = K$ holds for every vertex $x_i \in V(H)$, refer to figure 1. Define $S_H = [H, G_1 - H]$, $S_K = [K, G_2 - K]$, $s_h = |S_H|$ and $s_k = |S_K|$. From the maximality of $|H|$, we deduce that for every edge $e = uv \in S_H - \{x_a x_b\}$, there is a vertex $y_0 \in V(G_2)$ such that (u, y_0) and (v, y_0) are in the different components of $e \odot G_2 - S_e$. And so, $|S_e| \geq \delta_2 + 1$ by lemma 2.3. Noticing that $|S_{x_a x_b}| \geq \sum_{y \in K}(d_2(y) + 1) \geq |K|(\delta_2 + 1)$, we deduce that

$$\begin{aligned} |S| &\geq \sum_{x_i \in V(H)} |S_i| + \sum_{e \in E(H)} |S_e| + |S_{x_a x_b}| + \sum_{e \in S_H - \{x_a x_b\}} |S_e| \\ &\geq |H| s_k + 2|E(H)| s_k + |K|(\delta_2 + 1) + (s_h - 1)(\delta_2 + 1). \qquad (1) \end{aligned}$$

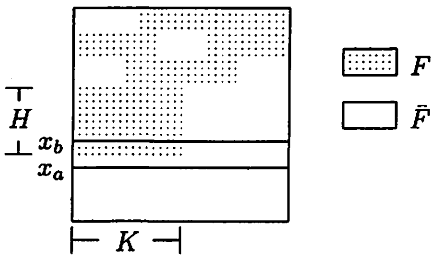

Figure 1. The sketch of case 1

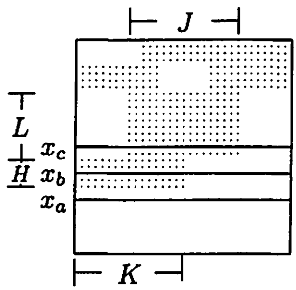

Figure 2. The sketch of case 2

Case 1. $|H| \geq 2$.

Since H is connected, it follows that $|E(H)| \geq |H| - 1$. By Lemma 2.5, we have $|H| + s_h \geq \xi_1 + 2$. If $G_2[K]$ consists of isolated vertices, then $s_k \geq \delta_2$. The combination of these observations with formula (1) implies that

$$\begin{aligned} |S| &\geq |H|\delta_2 + 2(|H| - 1)\delta_2 + s_h(\delta_2 + 1) \\ &= (|H| + s_h)(\delta_2 + 1) + 2(|H| - 1)\delta_2 - |H| \\ &\geq (\xi_1 + 2)(\delta_2 + 1) + 2(|H| - 1)\delta_2 - (|H| - 1) - 1 \\ &= \xi_1\delta_2 + \xi_1 + 2\delta_2 + 2 + (|H| - 1)(2\delta_2 - 1) - 1 \\ &\geq \xi_1\delta_2 + \xi_1 + 2\delta_2 + 2 + 2\delta_2 - 1 - 1 \\ &= \xi_1\delta_2 + \xi_1 + 4\delta_2 \geq \xi(G_1 \boxtimes G_2). \qquad (2) \end{aligned}$$

If $G_2[K]$ contains at least one edge, then $|K| + s_k \geq \xi_2 + 2$ by lemma 2.5. When $|H| \geq \delta_2 + 1$, by formula (1) and the assumption that $\delta_1 \leq \delta_2$ we have

$$\begin{aligned} |S| &\geq (\delta_2+1)s_k + 2(|H|-1)s_k + |K|(\delta_2+1) + (s_h-1)(\delta_2+1) \\ &\geq (|K| + s_k + s_h - 1)(\delta_2+1) + 2\delta_2 s_k \\ &\geq (\xi_2+2)(\delta_2+1) + 2\delta_2 \\ &> \xi_2\delta_2 + \xi_2 + 4\delta_2 \geq \xi_2\delta_1 + \xi_2 + 4\delta_1 \geq \xi(G_1 \boxtimes G_2); \end{aligned} \quad (3)$$

when $|H| \leq \delta_2$, recalling that $|K| + s_k \geq \xi_2 + 2 \geq 2\delta_2$ and noticing that $|K| + s_k \geq \delta_2 + 2$ by lemma 2.4, from (1) we deduce that

$$\begin{aligned} |S| &\geq |H|s_k + 2|E(H)|s_k + |K|(\delta_2+1) + (s_h-1)(\delta_2+1) \\ &\geq |H|s_k + 2s_k + (|K|-1)(\delta_2+1) + s_h(\delta_2+1) \\ &= |H|s_k + 2s_k + (|K|-1) + (|K|-1)\delta_2 + s_h(\delta_2+1) \\ &\geq |H|(|K| + s_k - 1) + (2s_k + |K| - 1) + s_h(\delta_2+1) \\ &\geq |H|(\delta_2+1) + (s_k + |K|) + s_h(\delta_2+1) \\ &\geq (|H| + s_h)(\delta_2+1) + 2\delta_2 \\ &\geq (\xi_1+2)(\delta_2+1) + 2\delta_2 \\ &> \xi_1\delta_2 + \xi_1 + 4\delta_2 \geq \xi(G_1 \boxtimes G_2). \end{aligned} \quad (4)$$

Case 2. $|H| = 1$.

If there is a vertex $x \in N_{G_1}(x_b)$ such that $V(G_2^x) \subseteq F$ then $s = n_2$, which contradicts our assumption that $1 \leq s \leq n_2 - 1$. If $V(G_2^x) \subseteq \bar{F}$ holds for every vertex $x \in N_{G_1}(x_b)$, from the minimality of $|S|$ we deduce that $V(F) = \{x_b\} \times K$ and that K induces a connected subgraph of order at least two. Let $G = G_1 \boxtimes G_2$. When $G_2[K]$ is an isolated edge, it's clearly that $|S| = |[\{x_b\} \times K, V(G) - \{x_b\} \times K]| \geq \xi(G_1 \boxtimes G_2)$; otherwise, by lemma 2.5 and the similar method employed in the proof of formula (1), we deduce that

$$\begin{aligned} |S| &\geq |K|\delta_1 + 2|E(G_2[K])|\delta_1 + s_k\delta_1 + s_k \\ &= (|K| + s_k)\delta_1 + (|E(G_2[K])|\delta_1 + s_k) + |E(G_2[K])|\delta_1 \\ &\geq (\xi_2+2)\delta_1 + (\xi_2+1) + 2\delta_1 \\ &> \xi(G_1 \boxtimes G_2). \end{aligned}$$

And so, we may assume in what follows that there is a vertex $x_c \in N_{G_1}(x_b)$ such that $G_2^{x_c}$ is separated by S. Define $J = \{y \in V(G_2) : (x_c, y) \in F\}$. Let L be the maximum-order connected subgraph of G_1 such that $x_c \in V(L)$ and $\{y \in V(G_2) : (x_i, y) \in F\} = J$ holds for every fixed vertex $x_i \in V(L)$, refer to figure 2.

Consider at first the case when $|L| \geq 2$. Let $Z = F - (V(H) \times K) = F \setminus (G_2^{x_b} \cap F)$, $\bar{Z} = \bar{F} \cup (G_2^{x_b} \cap F)$. Then $S' = [Z, \bar{Z}]$ contains a restricted edge cut of $G_1 \boxtimes G_2$. If regard S', L, J as S, H and K respectively and reason as in the proofs of formulae (1)-(4), one can show without difficult that

$$\begin{aligned} |S'| &\geq \sum_{x_i \in V(L)} |S_i| + \sum_{e \in E(L)} |S_e| + |S'_{x_b x_c}| + \sum_{e \in [L, G_1 - L] \setminus \{x_b x_c\}} |S_e| \\ &\geq \xi_2(G_1 \boxtimes G_2). \end{aligned}$$

Let

$$l = \sum_{x_i \in V(L)} |S_i| + \sum_{e \in E(L)} |S_e| + \sum_{e \in [L, G_1 - L] \setminus \{x_b x_c\}} |S_e|;$$

$$p = \sum_{x_i \in V(L)} |S'_i| + \sum_{e \in E(L)} |S'_e| + \sum_{e \in [L, G_1 - L] \setminus \{x_b x_c\}} |S'_e|.$$

Then $l = p$. Noticing that $|S_{x_a x_b}| = |[G_2^{x_a}, G_2^{x_b} \cap F]| = |[G_2^{x_c}, G_2^{x_b} \cap F]| = |[G_2^{x_c} \cap F, G_2^{x_b} \cap F]| + |[G_2^{x_c} \cap \bar{F}, G_2^{x_b} \cap F]|$, we deduce that

$$\begin{aligned} |S| &\geq l + |S_{x_b x_c}| + |S_{x_a x_b}| + |[G_2^{x_b} \cap F, G_2^{x_b} \cap \bar{F}]| \\ &> l + (|[G_2^{x_c} \cap F, G_2^{x_b} \cap \bar{F}]| + |[G_2^{x_c} \cap \bar{F}, G_2^{x_b} \cap F]|) \\ &\quad + (|[G_2^{x_c} \cap F, G_2^{x_b} \cap F] + [G_2^{x_c} \cap \bar{F}, G_2^{x_b} \cap F]|) \\ &\geq l + |[G_2^{x_c} \cap F, G_2^{x_b} \cap \bar{F}]| + |[G_2^{x_c} \cap F, G_2^{x_b} \cap F]| \\ &= l + |[G_2^{x_c} \cap F, G_2^{x_b}]| = p + |S'_{x_b x_c}| \\ &\geq \xi_2(G_1 \boxtimes G_2). \end{aligned}$$

Continue to consider the case when $|L| = 1$. Let $q = |S_{x_a x_b}| + |S_{x_b x_c}| + |S_b| + |S_c|$. If $|K| \geq 3$, since $|K| + |S_b| \geq \delta_2 + 2$ by lemma 2.4 and $|S_{x_b x_c}| \geq \delta_2 + 1$ by lemma 2.3, it follows that $q > |S_{x_a x_b}| + |S_{x_b x_c}| + |S_b| \geq |K|(\delta_2 + 1) + \delta_2 + 1 + |S_b| > 5\delta_2 + 1$; if $|K| = 2$, then $|S_b| \geq 2(\delta_2 - 1)$ and $q > |S_{x_a x_b}| + |S_{x_b x_c}| + |S_b| \geq |K|(\delta_2 + 1) + (\delta_2 + 1) + 2(\delta_2 - 1) = 5\delta_2 + 1$; if $|K| = 1$ and $|J| \geq 3$, then $|S_b| \geq \delta_2$ and $|S_{x_b x_c}| \geq |J|\delta_2$, hence $q > |S_{x_a x_b}| + |S_{x_b x_c}| + |S_b| \geq (\delta_2 + 1) + |J|\delta_2 + \delta_2 \geq 5\delta_2 + 1$; if $|K| = 1$ and $|J| = 1$, then $|S_{x_b x_c}| \geq 2\delta_2$. Hence

$$q = |S_{x_a x_b}| + |S_{x_b x_c}| + |S_b| + |S_c| \geq (\delta_2 + 1) + 2\delta_2 + \delta_2 + \delta_2 = 5\delta_2 + 1. \quad (5)$$

Finally, consider the case when $|K| = 1$ and $|J| = 2$, we shall prove at first that $|S_{x_b x_c}| \geq 2\delta_2 + 1$. Let $K = \{y_1\}$ and $J = \{y_2, y_3\}$. If $y_1 \notin J$, then $|S_{x_b x_c}| \geq d_2(y_2) + d_2(y_3) + |\{(x_b, y_1)(x_c, y_1)\}| \geq 2\delta_2 + 1$; if $y_1 \in J$, say $y_1 = y_2$, and $[y_2, G_2 - J] \neq \emptyset$, then $|S_{x_b x_c}| \geq d_2(y_2) + d_2(y_3) + |[(x_b, y_1), \{x_c\} \times$

$V(G_2 - J)]| \geq 2\delta_2 + 1$; if otherwise then $d_2(y_1) = d_2(y_2) = 1 = \delta_2$ and $d_2(y_3) \geq \delta_2 + 1$, and so the inequality also holds. Combining $|S_{x_b x_c}| \geq 2\delta_2+1$, $|S_b| \geq \delta_2$, $|S_{x_a x_b}| \geq \delta_2+1$ and $|S_c| \geq 2(\delta_2-1)$ by lemma 2.4 we have $q = |S_{x_a x_b}|+|S_{x_b x_c}|+|S_b|+|S_c| \geq (\delta_2+1)+(2\delta_2+1)+(\delta_2+1)+2(\delta_2-1) > 5\delta_2 + 1$.

In conclusion, $q \geq 5\delta_2+1$ when L is an isolated vertex, with the equality holding only if $|K| = |J| = 1$. Recalling that $|H| = |L| = 1$ in this case, from lemma 2.3 we deduce that $|S_e| \geq \delta_2 + 1$ holds for every edge $e \in \{ux_b : u \in N_{G_1}(x_b) - \{x_a, x_c\}\} \cup \{vx_c : v \in N_{G_1}(x_c) - \{x_b\}\}$. And so

$$\begin{aligned} |S| &\geq \sum_{e\in[x_b,N_{G_1}(x_b)]\setminus\{x_a x_b, x_b x_c\}} |S_e| + \sum_{e\in[x_c,N_{G_1}(x_c)]\setminus\{x_b x_c\}} |S_e| + q \\ &\geq (d_1(x_b) - 2)(\delta_2 + 1) + (d_1(x_c) - 1)(\delta_2 + 1) + q \\ &\geq (d_1(x_b) + d_1(x_c) - 2)\delta_2 + (d_1(x_b) + d_1(x_c) - 2) + 4\delta_2 \\ &\geq \xi_1\delta_2 + \xi_1 + 4\delta_2 \geq \xi_2(G_1 \boxtimes G_2). \end{aligned} \tag{6}$$

Theorem 3.2 follows from these discussions. □

Corollary 3.3 Let G_i be connected triangle-free graphs with order $n_i \geq 3$, size m_i, minimum degree δ_i, edge-connectivity λ_i and minimum edge degree ξ_i, $(i = 1, 2)$. Then $G_1 \boxtimes G_2$ is super-λ' if and only if $\min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1)\} > \xi(G_1 \boxtimes G_2)$.

Proof By theorem 3.2, it suffices to prove the sufficiency. For convenience, we adopt the symbols employed in the of theorem 3.2 for those not specified herein. Since $\min\{\lambda_1(n_2 + 2m_2), \lambda_2(n_1 + 2m_1)\} > \xi(G_1 \boxtimes G_2)$, it follows from theorem 3.2 that $|S| = \lambda'(G_1 \boxtimes G_2) = \xi(G_1 \boxtimes G_2)$. From the proof of theorem 3.2, we deduce that one of the following two cases occurs.

Case 1. All the inequalities of formulae (1) and (2) become equalities.

In this case, $V(F) = V(H) \times K$ since otherwise the first inequality formula (1) would strictly hold. From the first and fifth equality of formula (2), we deduce that $|K| = 1$ and $|H| = 2$. Hence, $G[F]$ is an isolated edge of $G_1 \boxtimes G_2 - S$.

Case 2. All the inequalities of formulae (5) and (6) hold become equalities.

In this case, $V(F) = (V(H) \times K) \cup (V(L) \times J)$ since otherwise the first inequality formula (6) would strictly hold. Notice that $|H| = |L| = |K| = |J| = 1$ in this case. It is easy to see that $G[F]$ is an isolated edge of $G_1 \boxtimes G_2 - S$. The corollary follows from this observation. □

Corollary 3.4. Let G_i be connected triangle-free graphs with order $n_i \geq 3$, size m_i, minimum degree δ_i, edge-connectivity λ_i and minimum edge degree ξ_i, $i = 1, 2$. If both G_1 and G_2 are maximally edge connected, then $G_1 \boxtimes G_2$ is super-λ'.

Proof. Since G_1 is maximally edge connected, it follows that $\lambda_1 = \delta_1$. Noticing that $n_2 \geq \xi_2 + 2$ and $m_2 \geq n_2 - 1$, we have

$$\lambda_1(n_2 + 2m_2) \geq \delta_1(\xi_2 + 2 + 2(\xi_2 + 1)) > \delta_1\xi_2 + \xi_2 + 4\delta_1 \geq \xi(G_1 \boxtimes G_2).$$

Similarly, $\lambda_2(n_1 + 2m_1) > \xi(G_1 \boxtimes G_2)$. The corollary follows from the combination of these two observations and corollary 3.3. □

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