Super connectivity of Kronecker products of some graphs *

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Abstract: Let G_1 and G_2 be two connected graphs. The Kronecker product $G_1 \times G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. In this paper, we show that $K_m \times K_n$ is super-κ for $n \geq m \geq 2$ and $n + m > 5$, $K_m \times P_n$ is super-κ for $n \geq m \geq 3$, and $K_m \times C_n$ is super-κ for $n \geq m \geq 3$.

Key words: Kronecker product; Super connectivity; Cut set

1 Introduction

We only consider undirected simple connected graphs without loops and multiple edges. Unless stated otherwise, we follow Bondy and Murty [3] for terminology and definitions.

There have been several proposals for measures of stability of a communication network. The first such parameters one generally encounters are connectivity and edge connectivity, which

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measure the vulnerability of a graph or network. The connectivity gives the minimum cost to disrupt the network.

Let $G = (V, E)$ be a connected graph. $S \subseteq V, x \in V, N(x)$ is the set of neighbors of x in G, $N_S(x) = N(x) \cap S$. The *connectivity* $\kappa(G)$ of a connected graph G is the least positive integer k such that there is $S \subset V, |S| = k$ and $G - S$ is disconnected or reduces to the trivial graph K_1 and such a set S is called vertex cut. A graph G is *super connected*, or simply *super-κ*, if every minimum vertex cut is the neighbors of a vertex of G, that is every minimum vertex cut isolates a vertex.

Let G_1 and G_2 be two nontrivial connected graphs. The Kronecker product (also named direct product, tensor product and cross product) $G_1 \times G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1), v_1v_2 \in E(G_2)\}$. It is known that the Kronecker product of two nontrivial graphs is connected if and only if at least one of the factors is not bipartite [5]. The Kronecker product of graphs has been extensively investigated concerning graph colorings, graph recognition and decomposition, graph embeddings, matching theory and stability in graphs (see, for example, [1] and [4], and the references therein), and this graph product has several applications, for instance, it can be used in modeling concurrency in multiprocessor system [8] and in automata theory [6].

The Kronecker product of graphs is the natural product in the category of graphs [7] and is also a widely used tool in the study of intersection networks. Moreover, it is universal in the sense that every graph is an induced subgraph of a suitable Kronecker product of complete graphs [10]. Since the Kronecker product of graphs have been widely used as the models of some practical structures used in the design of certain optimal networks (see [2] and [6]), it is of significance to consider the vulnerability parameters of this product of graphs.

2 Main results

When considering the Kronecker product $G_1 \times G_2$ of G_1 and G_2 with $|G_1| = m, |G_2| = n$, we shall always label $V_1 = V(G_1) = \{u_1, \cdots, u_m\}, V_2 = V(G_2) = \{v_1, \cdots, v_n\}$, and set $S_i = V_1 \times \{v_i\}, i = 1, \ldots, n$. Moreover, for convenience, we shall abbreviate (u_i, v_j) as w_{ij} for $i = 1, \ldots, m, j = 1, \ldots, n$. Then $S_i = \{w_{1i}, \cdots, w_{mi}\}(i = 1, 2, \ldots, n)$ is an independent set in $G_1 \times G_2$, and $V(G_1 \times G_2)$ has a partition $V_1 \times V_2 = S_1 \cup S_2 \cup \cdots \cup S_n$. For two vertices u and v in a graph G, we write $u \sim v$ if $uv \in E(G)$, and $u \nsim v$ otherwise. We denote by K_n the complete graph, C_n a cycle of n vertices, P_n a path of n vertices, respectively.

Lemma 2.1. *[9] Let m, n be integers with $n \geq m \geq 2$ and $n \geq 3$. Then $\kappa(K_m \times K_n) = (m-1)(n-1)$.*

Theorem 2.2. *Let m, n be integers with $n \geq m \geq 2$ and $n+m > 5$. Then $G = K_m \times K_n$ is super-κ.*

Proof. By contradiction. Suppose that $G = K_m \times K_n$ is not super-κ. Then there is a vertex cut S with $|S| = (m-1)(n-1)$ such that $G - S$ is not connected but has no isolated vertex. Thus each component of $G - S$ has at least two vertices.

Let w_{kt} and w_{pq} be two vertices in different components in $G-S$. Clearly $k = p$ or $t = q$. Without loss of generality, let $k = p$. Since no component is isolated, w_{kt} has a neighbor, say, $w_{k't'}$.

Since w_{kt} and w_{pq} are in different components in $G-S$, $w_{k't'}$ is not adjacent to w_{pq} and thus $t'=q$. Also w_{pq} has a neighbor, say, $w_{p'q'}$ in $G-S$ and by our assumption the only possibility is that $p'=k'$ and $q'=t$. Every other vertex is adjacent to at least one of the vertices from both of the components. And the vertices other than these four must be in S. Thus $(m-1)(n-1)=|S|\geq mn-4$ which implies that $m+n\leq 5$, a contradiction. □

Lemma 2.3. *Let m,n be integers with $n\geq m\geq 3$, $G=K_m\times P_n$. Then $\kappa(G)=m-1$.*

Proof. Obviously the set $T_1=\{w_{i2}:2\leq i\leq m\}$ is a cut set of G with $|T_1|=m-1$. Let S be a cut set of G with $|S|<m-1$. We shall prove that $G-S$ is connected.

Take arbitrary vertices, say w_{pk} and w_{qt} with $k\leq t$, in two components of $G-S$. Then $|S_l-S|\geq 2$ for any l.

Case 1. $k=t$.

If $|S_{k+1}-S|\geq 3$, then there is a (w_{pk},w_{qk})-path (if $k=n$, then take S_{k-1}). We are done. So assume that $|S_{k+1}-S|=2$, then $S=\{w_{i(k+1)}:1\leq i\leq m, i\neq p,q\}$. That is $S_{k+1}-S=\{w_{p(k+1)},w_{q(k+1)}\}$. Therefore, there is vertex $w_{jk}\overline{\in} S$ in S_k with $j\neq p,q$. Hence we have a (w_{pk},w_{qk})-path (if $k=n$, then let $S_{k+1}=S_{k-1}$). Then it is proved.

Case 2. $k<t$

Since w_{pk} and w_{qt} are nonadjacent vertices, and because $|S_l-S|\geq 2$ for any l, there is a path from w_{qt} to any S_l with $l\neq k$ and there is a path from w_{qt} to some vertex $w_{p'k}$ of S_k. Hence by Case 1, there is a path from w_{qt} to w_{pk} in $G-S$. Then the required result follows. □

Theorem 2.4. *Let m, n be integers with $n \geq m \geq 3$. Then $G = K_m \times P_n$ is super-κ.*

Proof. Assume that $G = K_m \times P_n$ is not super-κ. There is a set $S \subseteq V(G)$ with $|S| = m - 1$ such that $G - S$ is not connected but has no isolated vertex. Take arbitrary vertices, say w_{kt} and w_{pq} with $t \leq q$, in two components of $G - S$.

Case 1. $t = q$.

If there is no vertex in S_{t+1} in $G - S$, then $S_{t+1} \subseteq S$, a contradiction. If there is a vertex $w_{i(t+1)} (i \neq k, p)$ in $G - S$, then there is a (w_{kt}, w_{pt})-path, which is impossible (If $t = n$, we take $w_{i(t-1)}$). Hence $i = k$ or p. Say $i = k$. Then $S = \{w_{j(t+1)} : 1 \leq j \leq m, j \neq k\}$ or $S_{t+1} - \{w_{k(t+1)}, w_{p(t+1)}\} \subseteq S$.

(1) $S = \{w_{j(t+1)} : 1 \leq j \leq m, j \neq k\}$. If $t = 1$ or $t + 2 = n$, then there would be an isolated vertex, a contradiction. Then $t \geq 2$ and $t+2 < n$. Clearly $G-S$ is connected, a contradiction.

(2) $S_{t+1} - w_{k(t+1)} - w_{p(t+1)} \subseteq S$. Even if we remove the last vertex of S, $G - S$ is connected, a contradiction.

Case 2. $t < q$.

If there is some l such that $|S_l - S| = 1$, since there is no isolated vertex, then $G - S$ is connected. Hence $|S_l - S| \geq 2$ for any l. It is similar to Lemma 2.3 Case 2, there is a path from w_{pq} to w_{kt} in $G - S$ Then the required result follows. □

Lemma 2.5. *Let m, n be integers with $n \geq m \geq 3$, $G = K_m \times C_n$. Then $\kappa(G) = 2(m - 1)$.*

Proof. If $n = m = 3$, then by Lemma 2.1 we are done. We can easily verify $\kappa(K_4 \times C_3) = 6$. We assume that $n \geq 5$. Obviously, $T = \{w_{i2}, w_{in} : i = 2, 3, \cdots, m\}$ is a vertex cut with $|T| = 2(m-1)$. Let S be a vertex set of G with $|S| < 2(m-1)$.

We will show that $G - S$ is connected. Take arbitrary vertices w_{kt} and w_{pq} with $t \leq q$ in two components of $G - S$.

Case 1. $t = q$.

Firstly if there is a vertex $w_{k_1k_2}$ in S_{q+1} or S_{q-1} (if $q = n$, take $S_1 = S_{q+1}$) with $k_1 \neq k, p$, then there is a (w_{kq}, w_{pq})-path, $G - S$ is connected. Secondly if there are no such vertices, then $(S_{q-1} - w_{k(q-1)} - w_{p(q-1)}) \cup (S_{q+1} - w_{k(q+1)} - w_{p(q+1)}) \subseteq S$. We have $|S_{q-1}| - 2 + |S_{q+1}| - 2 = 2m - 4, |S| \leq 2m - 3$, and $|N_{S_{q-2}}(w_{k(q-1)}) \cap N_{S_{q-2}}(w_{p(q-1)})| \geq 1$, $|N_{S_{q+2}}(w_{k(q+1)}) \cap N_{S_{q+2}}(w_{p(q+1)})| \geq 1$. Although we remove the last vertex of S, there is a (w_{kq}, w_{pq})-path in $G - S$. Hence $G - S$ is also connected.

Case 2. $t < q$.

Without loss of generality, we assume that $1 < t < q < n$. Since w_{kt} and w_{pq} are nonadjacent vertices, $q - t \geq 2$.

Claim: Any connected subgraph H of $G-S$ in $\{S_i, S_{i+1}, \cdots, S_{j-1}, S_j\}$, there are edges between H and S_{i-1} or between H and S_{j+1} (If $i = 1$, then let $S_{i-1} = S_n, S_{i-2} = S_{n-1}, \cdots$. And if $j = n$, then let $S_{j+1} = S_1, S_{j+2} = S_2, \cdots$).

We prove it by contradiction. Without loss of generality, we assume that H has at least one vertex in each of $\{S_i, S_{i+1}, \cdots, S_{j-1}, S_j\}$.

(1) H has only one vertex w_{i_1i} and w_{j_1j} in S_i and S_j, respectively. Then w_{i_1i} has no neighbors in S_{i-1}, and w_{j_1j} has no neighbors in S_{j+1}. Since $|N_{S_{i-1}}(w_{i_1i})| + |N_{S_{j+1}}(w_{j_1j})| = 2(m-1) > |S|$, a contradiction.

(2) H has at least two vertices in S_i or S_j, say two vertices w_{j_1j} and w_{j_2j} in S_j. Then w_{j_1j} and w_{j_2j} have no neighbors in S_{j+1}, that is $S_{j+1} \subseteq S$. Let w_{i_1i} be a vertex of H in S_i. Then

$w_{i_1 i}$ has no neighbors in S_{i-1}, that is $N_{S_{i-1}}(w_{i_1 i}) \subseteq S$. Thus $|S_{j+1}| + |N_{S_{i-1}}(w_{i_1 i})| = 2m-1 > |S|$, which is impossible.

And w_{kt} has a neighbor in S_{t-1} or S_{t+1}. Otherwise $N(w_{kt}) \subseteq S$, but $|N(w_{kt})| = 2(m-1) > |S|$, which is a contradiction. Then by Claim there is a path P from w_{kt} to some vertex $w_{k'q}$ of S_q. Then by Case 1, there is a path from w_{kt} to w_{pq}, we are done. □

Theorem 2.6. *Let m, n be integers with $n \geq m \geq 3$, $G = K_m \times C_n$. Then G is super-κ.*

Proof. If $n = m = 3$, then by Theorem 2.2 we are done. We can easily verify $K_4 \times C_3$ is super-κ. We assume that $n \geq 5$. Assume that G is not super-κ. There is a vertex set $S \subseteq V(G)$ with $|S| = 2(m-1)$ such that $G-S$ is not connected but has no isolated vertex. Take arbitrary vertices, say w_{kt} and w_{pq} with $t \leq q$, in two components of $G-S$.

Case 1. $t = q$.

Assume that $1 < q < n$. Since $G-S$ has no isolated vertex, if there is a vertex $w_{k_1 k_2} \in S_{q+1}$ or S_{q-1} with $k_1 \neq k, p$, then there is a (w_{kt}, w_{pq})-path. That is $G-S$ is connected, a contradiction. Thus there are no such vertices. So $(S_{q-1} - w_{k(q-1)} - w_{p(q-1)}) \cup (S_{q+1} - w_{k(q+1)} - w_{p(q+1)}) \subseteq S$, and $|(S_{q-1} - w_{k(q-1)} - w_{p(q-1)}) \cup (S_{q+1} - w_{k(q+1)} - w_{p(q+1)})| = 2m-4, |S| = 2m-2$. Notice that $G-S$ has no isolated vertex, even if we remove the last two vertices of S, there is also a (w_{kt}, w_{pq})-path. That is $G-S$ is connected, which is a contradiction.

Case 2. $t < q$.

Assume that $1 < t < q < n$. Since w_{kt} and w_{pq} are nonadjacent vertices, $q - t \geq 2$.

Claim: Any connected subgraph H of $G-S$ in $\{S_i, S_{i+1}, \cdots, S_{j-1}, S_j\}$, there are edges between H and S_{i-1} or between H and S_{j+1} (If $i=1$, then let $S_{i-1}=S_n, S_{i-2}=S_{n-1}, \cdots$. And if $j=n$, then let $S_{j+1}=S_1, S_{j+2}=S_2, \cdots$).

The proof is similar to the Claim of Lemma 2.5.

Because $G-S$ has no isolated vertex, w_{kt} has a neighbor in S_{t-1} or S_{t+1}. Then by Claim there is a path P from w_{kt} to S_q with $P = w_{kt} \cdots w_{p'q}$. By Case 1, there is also a (w_{kt}, w_{pq})-path, that is $G-S$ is connected, a contradiction. □

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