On (p,1)-total labelling of special 1-planar graphs *

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Abstract

A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. A k-$(p,1)$-total labelling of a graph G is a function f from $V(G) \cup E(G)$ to the color set $\{0, 1, \cdots, k\}$ such that $|f(u) - f(v)| \geq 1$ if $uv \in E(G)$, $|f(e_1) - f(e_2)| \geq 1$ if e_1 and e_2 are two adjacent edges in G and $|f(u) - f(e)| \geq p$ if the vertex u is incident to the edge e. The minimum k such that G has a k-$(p,1)$-total labelling, denoted by $\lambda_p^T(G)$, is called the $(p,1)$-total labelling number of G. In this paper, we prove that, if a 1-planar graph G satisfies that maximum degree $\Delta(G) \geq 7p+1$ and no adjacent triangles in G or maximum degree $\Delta(G) \geq 6p+3$ and no intersecting triangles in G, then $\lambda_p^T(G) \leq \Delta + 2p - 2$, $p \geq 2$.

Key words: 1-planar graph; $(p,1)$-total labelling, minimal counterexample, discharging method.

1 Introduction

In the channel assignment problems, different frequencies are assigned to close transmitters so that they can avoid interference and communication link failure. Moreover, a sufficient separation of the frequencies assigned to two close transmitters is also necessary. The L(p,q)-labelling is a popular graph theoretic model for this problem. An *$L(p,q)$-labelling* of a graph G is a mapping from the set of vertices $V(G)$ to the set of integers $Z_k = \{0, 1, \cdots, k\}$ such that $|f(x) - f(y)| \geq p$ if x and y are adjacent and $|f(x) - f(y)| \geq q$ if x and y are at distance 2. The interested readers can refer to the surveys by Calamoneri [5] and by Yeh [18].

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The notation of $(p, 1)$-total labelling is due to the $L(2, 1)$-labelling of the incidence graph $I(G)$ of a graph G, which is obtained from G by inserting one vertex of degree 2 on each edge of G. The $L(2, 1)$-labelling of $I(G)$ can be considered as another kind of labelling, the so-called $(2, 1)$-total labelling of G, which was introduced by Havet and Yu [9, 10] and generalized to the notion of $(p, 1)$-total labelling.

A k-$(p, 1)$-*total labelling* of a graph G is a function f from $V(G) \cup E(G)$ to the color set $\{0, 1, \cdots, k\}$ such that $|f(u) - f(v)| \geq 1$ if $uv \in E(G)$, $|f(e_1) - f(e_2)| \geq 1$ if e_1 and e_2 are two adjacent edges in G and $|f(u) - f(e)| \geq p$ if the vertex u is incident to the edge e. The minimum k such that G has a k-$(p, 1)$-total labelling, denoted by $\lambda_p^T(G)$, is called the $(p, 1)$-*total labelling number* of G. When $p = 1$, the $(1, 1)$-total labelling is the total coloring of graphs, which is a classic graph coloring. Among the types of the total coloring, list total coloring and the neighbor sum distinguishing total colorings have attracted people's attention, recently. Interested readers can refer to [4, 7, 13].

The next step is to look for any Brooks-typed or Vizing-typed upper bound on the $(p, 1)$-total labelling number in terms of maximum degree and p. One can see that for any graph G with maximum degree $\Delta(G)$ satisfies that $\lambda_p^T(G) \geq \Delta(G) + p - 1$ with $p \geq 1$. Meanwhile, we can construct a $(p, 1)$-total labelling of G by properly coloring its edges with $\chi'(G)$ integers of $[0, \chi'(G) - 1]$, and its vertices with $\chi(G)$ integers of $[\chi'(G) + p - 1, \chi(G) + \chi'(G) + p - 2]$, where $\chi(G)$ and $\chi'(G)$ denote the vertex chromatic number and the edge chromatic number of G, respectively. Thus, we have the trivial bound $\Delta(G) + p - 1 \leq \lambda_p^T(G) \leq \chi(G) + \chi'(G) + p - 2$. In [10, 11], Havet and Yu posed the $(p, 1)$-Total Labelling Conjecture, which is a natural extension of the Total Coloring Conjecture.

Conjecture 1. [10, 11] Let G be a graph, Then $\lambda_p^T(G) \leq \min\{\Delta(G) + 2p - 1, 2\Delta(G) + p - 1\}$.

If $p = 1$, this conjecture is Total Coloring Conjecture, which has been extensively studied in many papers, we give some [12, 16, 17]. For $p = 2$, the $(2, 1)$-Total Labelling Conjecture has already been confirmed for all outerplanar graphs [6, 8]. In [19], Yu et al. proved that every planar graphs G with maximum degree $\Delta(G) \geq 12$ satisfies that $\Delta(G) + 1 \leq \lambda_2^T(G) \leq \Delta(G) + 2$. In general, the $(p, 1)$-Total Labelling Conjecture has been considered for planar graphs with high girth and high maximum degree [1] and graphs with a given maximum average degree [14]. Particularly, Bazzaro, Montassier and Raspaud proved the following theorem for all planar graphs [1].

Theorem 2. [1] Let G be a planar graph with maximum degree Δ. If $\Delta \geq 8p + 2$ and $p \geq 2$, then $\lambda_p^T(G) \leq \Delta + 2p - 2$.

Recently, Zhang, Yu and Liu proved the following theorem for all 1-planar graphs [20]. A graph is 1-planar if it can be drawn on the plane so that each edge is crossed by at most one other edge. This notion of 1-planar graphs was introduced by Ringle [15] while trying to simultaneously colour the vertices and faces of a plane graph G such that any pair of adjacent or incident elements receive different colors.

Theorem 3. [20] Let $p \geq 2$ and let G be a 1-planar graph with maximum degree Δ and girth g. If $\Delta \geq 8p + 4$ or $\Delta \geq 6p + 2$ and $g \geq 4$, then $\lambda_p^T(G) \leq \Delta + 2p - 2$.

For a 1-planar graph G, it is proved that $\chi(G) \leq 6$ by Borodin [3] and $\chi'(G) = \Delta(G)$ provided that $\Delta(G) \geq 10$ by Zhang et al [21]. Recall that the trivial upper bound of $\lambda_p^T(G)$ is $\lambda_p^T(G) \leq \chi(G) + \chi'(G) + p - 2$ for every graph G. So the $(p, 1)$-total labelling is better for p with $\Delta(G) + 2p - 2 \leq \Delta(G) + p + 4$, if we want to prove that $\lambda_p^T(G) \leq \Delta(G) + 2p - 2$.

In this paper, we obtain a result of 1-planar graphs by proving the following theorem.

Theorem 4. *Let $p \geq 2$ be an positive integer and let G be a 1-planar graph satisfying that the maximum degree $\Delta(G) \geq 7p+1$ and no adjacent triangles in G or maximum degree $\Delta(G) \geq 6p + 3$ and no intersecting triangles in G. Then $\lambda_p^T(G) \leq \Delta(G) + 2p - 2$.*

The following theorem is only a technical strengthening of Theorem 4. Without it we would get complications when considering a subgraph $G' \subset G$ such that $\Delta(G') < \Delta(G)$.

Theorem 5. *Let M, p be two positive integers and let G be a 1-planar graph with maximum degree $\Delta(G) \leq M$. Then $\lambda_p^T(G) \leq M + 2p - 2$ with $p \geq 2$ in the following cases:*
(1) *$M \geq 7p + 1$ and G contains no adjacent triangles;*
(2) *$M \geq 6p + 3$ and G contains no intersecting triangles.*

Some notations should be introduced. Two triangles are said to be adjacent if they have at least one common edge, and intersecting if they have at least one common vertex. A k-, k^+- and k^--vertex (resp. face) is a vertex (resp. face) of degree k, at least k and at most k, respectively. A vertex u is called a k-neighbor(resp. k^--neighbor, k^+-neighbor) of a vertex v if $uv \in E(G)$ and $d_G(u) = k$(resp. $d_G(u) \leq k$, $d_G(u) \geq k$). For other undefined notations, we refer the readers to [2].

2 Structural properties of the minimum counterexample to Theorem 5

Let G be a counterexample to Theorem 5 with $|V(G)| + |E(G)|$ being minimum. First of all, we give the following lemmas.

Lemma 6. [20] *For any edge $uv \in E(G)$, if*

$$min\{d_G(u), d_G(v)\} \leq \lfloor (M + 2p - 2)/(2p) \rfloor,$$

then $d_G(u) + d_G(v) \geq M + 2$, otherwise, $d_G(u) + d_G(v) \geq M - 2p + 3$.

Lemma 7. [20] *For any integer k satisfying $2 \leq k \leq \lfloor (M + 2p - 2)/(2p) \rfloor$, let $X_k = \{x \in V(G) : d_G(x) \leq k\}$ and $Y_k = \cup_{x \in X_k} N_G(x)$. If $X_k \neq \emptyset$, then there exists a bipartite subgraph M_k of G with partite sets X_k and Y_k such that $d_{M_k}(x) = 1$ for every $x \in X_k$ and $d_{M_k}(y) \leq k - 1$ for every $y \in Y_k$.*

Following the terms of Borodin, and Woodall in [4], in Lemma 7 we call y the *k-master* of x if $xy \in M_k$ and $x \in X_k$ and we call x the *k-dependent* of y. From Lemma 7, we can get the following useful lemma.

Lemma 8. [20] *Every i-vertex in G has a j-master, where $2 \leq i \leq j \leq \lfloor (M + 2p - 2)/(2p) \rfloor$, and every vertex in G has at most $k-1$ k-dependents, $2 \leq k \leq \lfloor (M + 2p - 2)/(2p) \rfloor$.*

The above lemmas are used to state the structural properties due to the minimality of G. In the following, we will get some other structural properties due to the 1-planarity of G. From now on, we always assume that G has been embedded on a plane such that the number of crossings is as small as possible. As for the notations of 1-planar graphs, we follow the terms of Zhang and Wu in [21], which are introduced as follows. The *associated plane graph* $G^\times$ of G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^\times$ is called *false* if it is not a vertex of G and *true* otherwise. Note that every 3-face in $G^\times$ is incident with at most one false vertex, so we call a 3-face in $G^\times$ *false* or *true* according to whether it is incident with a false vertex or not. In [21], Zhang and Wu showed some basic properties between a 1-plane graph and its associated plane graph.

Lemma 9. [21] *The following results hold for G and $G^\times$:*
(1) In $G^\times$, any two false vertices are not adjacent.
(2) If there is a 3-face $uvwu$ in $G^\times$ such that $d_G(v) = 2$, then u and w are both true vertices.
(3) If a 3-vertex u in G is adjacent to a false vertex v in $G^\times$, then uv is not incident with two 3-faces.

(4) *If a 3-vertex v in G is incident with two 3-faces and adjacent to two false vertices in $G^\times$, then v must also be incident with a 5^+-face.*
(5) *For any 4-vertex u in G, u is incident with at most three false 3-faces.*

Now, we focus on the number of 3-faces incident with a vertex v in $G^\times$ for any vertex $v \in V(G)$, which is denoted by $f_3(v)$.

Lemma 10. [22] *If G contains no adjacent triangles, then for every 5^+-vertex $v \in V(G)$, $f_3(v) \le \lfloor (4/5)d_G(v) \rfloor$ in $G^\times$.*

Lemma 11. *If G contains no intersecting triangles, then for every vertex $v \in V(G)$, we have*
(1) *If $d_G(v) = 3$, then $f_3(v) \le 2$;*
(2) *If $d_G(v) = 4$, then $f_3(v) \le 3$;*
(3) *If $d_G(v) = 5$, then $f_3(v) \le 4$;*
(4) *If $d_G(v) = 6$, then $f_3(v) \le 4$;*
(5) *If $d_G(v) = 7$, then $f_3(v) \le 5$;*
(6) *If $d_G(v) = 8$, then $f_3(v) \le 6$;*
(7) *If $d_G(v) \ge 9$, then $f_3(v) \le 4 + \lfloor 2(d_G(v) - 6)/3 \rfloor$.*

Proof. We only need to show that three consecutive adjacent 3-faces incident with a common vertex can form at least one triangle and five consecutive adjacent 3-faces incident with a common vertex can form at least two triangles. Let a vertex v in G be incident with three consecutive adjacent 3-faces f_1, f_2 and f_3, $f_i = vv_iv_{i+1}$, $i = 1, 2, 3$. If f_2 is a false 3-face, without loss of generality, assume that v_2 is a false vertex. By (1) of Lemma 9, v_1 and v_3 are true vertices. Then there is a triangle vv_1v_3. Let a vertex v in G be incident with five consecutive adjacent 3-faces f_1, f_2, f_3, f_4 and f_5, $f_i = vv_iv_{i+1}$, $i = 1, 2, 3, 4, 5$. By the result above, f_1, f_2 and f_3 can form at least one triangle, say, vv_1v_3. If v_4 is a true vertex, then there are two triangles vv_1v_3 and vv_3v_4. Otherwise, by (1) of Lemma 9, v_5 is a true vertex, and it follows that there are two triangles vv_1v_3 and vv_3v_5. □

In order to make our proof much simpler, we need the following lemma. Before stating the lemma, we divide the false 3-faces into two types: small 3-faces and big 3-faces. A false 3-face in $G^\times$ is called *small* if it is incident with a $(\lfloor (M + 2p - 2)/(2p) \rfloor + 1)^-$-vertex and *big* otherwise.

Lemma 12. *If $M \ge 7p + 1$ with $p \ge 2$ and G contains no adjacent two triangles, then for any s-vertex v in $V(G)$, the following results hold:*
(1) *If $s = 5t$, then v is incident with at most $2t$ small false 3-faces in $G^\times$;*
(2) *If $s = 5t + 1$, then v is incident with at most $2t$ small false 3-faces in $G^\times$;*
(3) *If $s = 5t + 2$, then v is incident with at most $2t + 1$ small false 3-faces in $G^\times$;*

(4) *If* $s = 5t + 3$, *then* v *is incident with at most* $2t + 2$ *small false* 3*-faces in* $G^{\times}$*;*
(5) *If* $s = 5t + 4$, *then* v *is incident with at most* $2t + 2$ *small false* 3*-faces in* $G^{\times}$.

Proof. By Lemma 10, for any vertex $v \in V(G)$, the number of consecutive adjacent 3-faces incident with a common vertex v is at most four. By minimality of G and Lemma 6, no two $(\lfloor (M + 2p - 2)/(2p) \rfloor + 1)^{-}$-vertices are adjacent in G for $M \geq 7p + 1$ with $p \geq 2$. So there are at least two big false 3-faces and at most two small false 3-faces in four consecutive adjacent 3-faces incident with a common vertex v in $G^{\times}$. □

3 The proof of Theorem 5

Firstly, we prove (1) of Theorem 5. Suppose that G is a minimum counterexample to it. It is easy to verify that G is connected. Moreover, $\delta(G) \geq 2$ by Lemma 6. All of the above results also hold in $G^{\times}$. By Euler's formula, we have that $\sum_{x \in V(G^{\times}) \cup F(G^{\times})} (d_{G^{\times}}(x) - 4) = -8$. Then we assign an initial charge $c(x) = d_{G^{\times}}(x) - 4$ to x for every element $x \in V(G^{\times}) \cup F(G^{\times})$. Now we redistribute the charges of vertices and faces in $G^{\times}$ according to the following rules and also check that the final charge $c'(x)$ of every element $x \in V(G^{\times}) \cup F(G^{\times})$ is nonnegative. We use $\tau(x_1 \to x_2)$ to denote the charge moved from x_1 to x_2. Since our rules only move charges around and do not affect the total charges, this leads to a contradiction to Euler's formula transformation in final and completes our proof.

R1. Let $f = uvw$ be a true 3-face in $G^{\times}$. If $d_{G^{\times}}(u) \geq 8$, then $\tau(u \to f) = 1/2$.

R2. Let $f = uvw$ be a false 3-face in $G^{\times}$ with a false vertex u.

- **R2.1.** If $d_{G^{\times}}(w) \geq d_{G^{\times}}(v)$ and $d_{G^{\times}}(v) \leq 5$, then $\tau(v \to f) = 1/3$ and $\tau(w \to f) = 2/3$.
- **R2.2.** If $d_{G^{\times}}(w) \geq d_{G^{\times}}(v) \geq 6$, then $\tau(v \to f) = \tau(w \to f) = 1/2$.

R3. Let uv be an edge in G such that u is an i-master of v for some $i \in \{2, 3, 4\}$. Then $\tau(u \to v) = 2/3$.

R4. Let f be a 5^{+}-face in $G^{\times}$ and v be a true 5^{-}-vertex. Then $\tau(f \to v) = 1/3$.

Let $f \in F(G^\times)$. If f is a 5^+-face, then the number of true 5^--vertices incident with f is at most $\lfloor d_{G^\times}(f)/2 \rfloor$ by Lemma 6, and it follows that $c'(f) \geq (d_{G^\times}(f) - 4) - \lfloor d_{G^\times}(f)/2 \rfloor \times (1/3) > 0$ by R4. If f is a 4-face, then $c'(f) = c(f) = 0$. Suppose that f is a 3-face in $G^\times$. If f is a false 3-face, then f is incident with two true vertices by (1) of Lemma 9, and it follows that $c'(f) \geq (3-4) + \min\{2 \times (1/2), 1/3 + 2/3\} = 0$ by R2. If f is a true 3-face, then $c'(v) \geq (3-4) + 2 \times (1/2) = 0$ by R1, since a true 3-face is incident with at least two 8^+-vertices by Lemma 6. Hence $c'(f) \geq 0$ for every face $f \in F(G^\times)$.

Let $v \in V(G^\times)$. If $d_{G^\times}(v) = k$, then we use $v_1, v_2, \cdots, v_k$ to denote its neighbors in $G^\times$ in a clockwise order and f_i to denote the face incident with vv_i and vv_{i+1} in $G^\times$, $i = 1, \cdots, k$, where the addition on subscripts are taken modulo k.

Suppose that $d_{G^\times}(v) = 2$. By Lemma 8, (2) of Lemma 9, R1, R2 and R3, $c'(v) \geq 2 - 4 + 3 \times (2/3) = 0$. Suppose that $d_{G^\times}(v) = 3$. Since $f_3(v) \leq 2$, by Lemma 8 and (4) of Lemma 9, we have $c'(v) \geq 3 - 4 - 2 \times (1/3) + 2 \times (2/3) + 1/3 = 0$ by R1, R2, R3 and R4. Suppose that $d_{G^\times}(v) = 4$ and v is a true 4-vertex. By (5) of Lemma 9, $f_3(v) \leq 3$. If $f_3(v) \leq 2$, then $c'(v) \geq 4 - 4 - 2 \times (1/3) + 2/3 = 0$ by R1, R2, R3 and Lemma 8. Otherwise, $f_3(v) = 3$, without loss of generality, we assume that f_1, f_2 and f_3 are 3-faces. If f_4 is a 4-face, then there exist two adjacent triangles, a contradiction. So f_4 is a 5^+-face, and it follows that $c'(v) \geq 4 - 4 - 3 \times (1/3) + 2/3 + 1/3 = 0$ by R1, R2, R3, R4 and Lemma 8. Suppose that $d_{G^\times}(v) = 4$ and v is a false 4-vertex. Then by rules, $c'(v) = c(v) = 4 - 4 = 0$. Suppose that $d_{G^\times}(v) = 5$. Then $f_3(v) \leq 4$ by Lemma 10. If $f_3(v) \leq 3$, then $c'(v) \geq 5 - 4 - 3 \times (1/3) = 0$ by R2. Otherwise, $f_3(v) = 4$, without loss of generality, we assume that f_1, f_2, f_3 and f_4 are 3-faces. If f_5 is a 4-face, then there exist two adjacent triangles, a contradiction. So f_5 is a 5^+-face, and it follows that $c'(v) \geq 5 - 4 - 4 \times (1/3) + 1/3 = 0$ by R1, R2 and R4. Suppose that $d_{G^\times}(v) = k$ and $6 \leq k \leq 5p - 2$. Then $c'(v) \geq k - 4 - (1/2) \times \lfloor (4/5)k \rfloor \geq 0$ by Lemma 10, R1 and R2. Suppose that $d_{G^\times}(v) = k$ and $5p - 1 \leq k \leq M - 3$. Then $c'(v) \geq k - 4 - (2/3) \times \lfloor (4/5)k \rfloor \geq 1/3$ ($k \geq 5p - 1 \geq 9$) by Lemma 10, R1 and R2. Suppose that $d_{G^\times}(v) = k = M - 2$. By Lemma 6 and Lemma 8, v has at most three 4-dependents. Then $c'(v) \geq k - 4 - 3 \times (2/3) - (2/3) \times \lfloor (4/5)k \rfloor \geq 1/3$ ($k \geq 7p - 1 \geq 13$) by Lemma 10, R1 and R2.

Suppose that $d_{G^\times}(v) = k = M - 1$. By Lemma 6 and Lemma 8, v has at most three 4-dependents and at most two 3-dependents. By using R3, the charge of v is changed to at least $k - 4 - 3 \times (2/3) - 2 \times (2/3) = M - 25/3$. By Lemma 12, if $k = 5t$, then $c'(v) \geq (5t + 1) - 25/3 - (2t) \times (1/2) - (2t) \times (2/3) = (8/3)t - 22/3 \geq 2/3$ ($t \geq 3$) by R1 and R2 ($t \geq 3$ for $5t = M - 1 \geq 7p \geq 14$, $p \geq 2$); if $k = 5t + 1$, then $c'(v) \geq (5t + 2) - 25/3 - (2t) \times (1/2) - (2t) \times (2/3) = (8/3)t - 19/3 \geq 5/3$ ($t \geq 3$) by R1 and R2; if

$k = 5t+2$, then $c'(v) \geq (5t+3)-25/3-(2t)\times(1/2)-(2t+1)\times(2/3) = (8/3)t-6 \geq 2$ $(t \geq 3)$ by R1 and R2; if $k = 5t+3$, then $c'(v) \geq (5t+4)-25/3-(2t)\times(1/2)-(2t+2)\times(2/3) = (8/3)t-17/3 \geq 7/3$ $(t \geq 3)$ by R1 and R2; if $k = 5t+4$, then $c'(v) \geq (5t+5)-25/3-(2t+1)\times(1/2)-(2t+2)\times(2/3) = (8/3)t-31/6 \geq 1/6$ $(t \geq 2)$ by R1 and R2.

Suppose that $d_{G^\times}(v) = k = M$. By Lemma 6 and Lemma 8, v has at most three 4-dependents, at most two 3-dependents and at most one 2-dependent. After only using R3, the charge of v is changed to at least $k-4-3\times(2/3)-2\times(2/3)-2/3 = M-8$. By Lemma 12, if $k = 5t$, then $c'(v) \geq 5t-8-(2t)\times(1/2)-(2t)\times(2/3) = (8/3)t-8 \geq 0$ $(t \geq 3)$ by R1 and R2 ($t \geq 3$ for $5t = M \geq 7p+1 \geq 15$, $p \geq 2$); if $k = 5t+1$, then $c'(v) \geq 5t+1-8-(2t)\times(1/2)-(2t)\times(2/3) = (8/3)t-7 \geq 1$ $(t \geq 3)$ by R1 and R2; if $k = 5t+2$, then $c'(v) \geq 5t+2-8-(2t)\times(1/2)-(2t+1)\times(2/3) = (8/3)t-20/3 \geq 4/3$ $(t \geq 3)$ by R1 and R2; if $k = 5t+3$, then $c'(v) \geq 5t+3-8-(2t)\times(1/2)-(2t+2)\times(2/3) = (8/3)t-19/3 \geq 5/3$ $(t \geq 3)$ by R1 and R2; if $k = 5t+4$, then $c'(v) \geq 5t+4-8-(2t+1)\times(1/2)-(2t+2)\times(2/3) = (8/3)t-35/6 \geq 13/6$ $(t \geq 3)$ by R1 and R2.

Now, the final charge on each vertex and each face in $G^\times$ is nonnegative, which leads to a contradiction to Euler's formula transformation. Therefore, we complete our proof of (1) of Theorem 5.

Secondly, we prove (2) of Theorem 5. Euler's formula transformation, the initial charge of an element x and each of the discharging rules are the same to those in the proof of (1), $x \in V(G^\times) \cup F(G^\times)$. Since the condition that no intersecting triangles is much stronger than that no adjacent triangles, the final charge of every 5^--vertex is nonnegative. Moreover, the final charge of every face is nonnegative by a similar proof as that of Theorem 5(1). We only consider the final charges of 6^+-vertices.

Suppose that $d_{G^\times}(v) = k$ and $6 \leq k \leq 4p$. Then by Lemma 11, R1 and R2, $c'(v) \geq 6-4-(1/2)\times 4 = 0$ for every 6-vertex v, $c'(v) \geq 7-4-(1/2)\times 5 = 1/2$ for every 7-vertex v, $c'(v) \geq 8-4-(1/2)\times 6 = 1$ for every 8-vertex v and $c'(v) \geq k-4-(1/2)\times(4+\lfloor 2(k-6)/3 \rfloor) \geq (2/3)k-4 > 0$ for every k-vertex v, $9 \leq k \leq 4p$.

Suppose that $d_{G^\times}(v) = k$ and $4p+1 \leq k \leq M-3$. Then $c'(v) \geq k-4-(2/3)\times(4+\lfloor 2(k-6)/3 \rfloor) \geq (5/9)k-4 \geq 1$ $(k \geq 4p+1 \geq 9)$ by R1, R2 and Lemma 6. Suppose that $d_{G^\times}(v) = k = M-2$. By Lemma 6 and Lemma 8, v has at most three 4-dependents. Then $c'(v) \geq k-4-3\times(2/3)-(2/3)\times(4+\lfloor 2(k-6)/3 \rfloor) \geq (5/9)k-6 \geq 11/9$ $(k \geq 6p+1 \geq 13)$ by Lemma 11, R1, R2 and R3.

Suppose that $d_{G^\times}(v) = k = M-1$. By Lemma 6 and Lemma 8, v has at most three 4-dependents and at most two 3-dependents. Then $c'(v) \geq k-4-3\times(2/3)-2\times(2/3)-(2/3)\times(4+\lfloor 2(k-6)/3 \rfloor) \geq (5/9)k-22/3 \geq 4/9$ $(k \geq 6p+2 \geq 14)$ by Lemma 11, R1, R2 and R3.

Suppose that $d_{G^\times}(v) = k = M$. By Lemma 6 and Lemma 8, v has at most three 4-dependents, at most two 3-dependents and at most one 2-dependent. Then $c'(v) \geq k - 4 - 3 \times (2/3) - 2 \times (2/3) - 2/3 - (2/3) \times (4 + \lfloor 2(k-6)/3 \rfloor) \geq (5/9)k - 8 \geq 1/3$ $(k \geq 6p + 3 \geq 15)$ by Lemma 11, R1, R2 and R3. Hence we have completed the proof of Theorem 5(2).

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