On traceable and upper traceable numbers of graphs

FUTABA FUJIE *

Abstract

For a connected graph G of order $n \geq 2$ and a linear ordering s : $v_1, v_2, \ldots, v_n$ of $V(G)$, define $d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1})$. The traceable number $t(G)$ and upper traceable number $t^+(G)$ of G are defined by $t(G) = \min\{d(s)\}$ and $t^+(G) = \max\{d(s)\}$, respectively, where the minimum and maximum are taken over all linear orderings s of $V(G)$. Consequently, $t(G) \leq t^+(G)$. It is known that $n-1 \leq t(G) \leq 2n-4$ and $n-1 \leq t^+(G) \leq \lfloor n^2/2 \rfloor - 1$ for every connected graph G of order $n \geq 3$ and, furthermore, for every pair n, A of integers with $2 \leq n-1 \leq A \leq 2n-4$ there exists a graph of order n whose traceable number equals A. In this work we determine all pairs A, B of positive integers with $A \leq B$ that are realizable as the traceable number and upper traceable number, respectively, of some graph. It is also determined for which pairs n, B of integers with $n-1 \leq B \leq \lfloor n^2/2 \rfloor - 1$ there exists a graph whose order equals n and upper traceable number equals B.

Keywords: Hamiltonian graphs, traceable graphs, traceable number, upper traceable number.

AMS subject classification: 05C12, 05C45.

1 Introduction

We refer to the book [4] for graph-theoretical notation and terminology not described in this paper. In 1974 Goodman and Hedetniemi [6] introduced the concept of a *Hamiltonian walk* in a connected graph, defined as a closed spanning walk of minimum length in the graph. Therefore, for a connected graph G of order $n \geq 3$, the length of a Hamiltonian walk of G is at least n and is equal to n if and only if G is Hamiltonian. In the 1970s and early

*futaba@math.nagoya-u.ac.jp, Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan.

1980s this concept received considerable attention (see [1, 2, 3, 7, 11, 12] for example).

This concept was studied from a different point of view in 2004 by Chartrand et al. [5], namely in terms of sequences of vertices of a graph. For a connected graph G of order $n \geq 3$ and a *cyclic ordering* $s : v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of $V(G)$ (the vertex set of G), the number $d(s)$ is defined by $d(s) = \sum_{i=1}^{n} d(v_i, v_{i+1})$, where $d(v_i, v_{i+1})$ is the distance between v_i and v_{i+1}. The *Hamiltonian number* $h(G)$ of G is defined in [5] by $h(G) = \min\{d(s)\}$, where the minimum is taken over all cyclic orderings s of $V(G)$. Thus $h(G) \geq n$ for every connected graph G of order $n \geq 3$ and $h(G) = n$ if and only if G is Hamiltonian. The Hamiltonian number of a graph G is, in fact, the length of a Hamiltonian walk in G.

A related concept was introduced in [8]. For a connected graph G of order $n \geq 2$ and a *linear ordering* $s : v_1, v_2, \ldots, v_n$ of $V(G)$, consider the number

$$d(s) = \sum_{i=1}^{n-1} d(v_i, v_{i+1}). \tag{1}$$

The *traceable number* $t(G)$ of G is defined in [8] as $t(G) = \min\{d(s)\}$, where the minimum is taken over all linear orderings s of $V(G)$. Thus, if G is a connected graph of order $n \geq 2$, then $t(G) \geq n-1$. Furthermore, $t(G) = n-1$ if and only if G is traceable, that is, G contains a Hamiltonian path. In fact, the traceable number of a connected graph G is the minimum length of a spanning walk in G. The Hamiltonian number $h(G)$ and traceable number $t(G)$ of a connected graph G therefore provide measures of traversability of G.

For a connected graph G, the *upper Hamiltonian number* $h^+(G)$ is defined also in [5] by $h^+(G) = \max\{d(s)\}$, where the maximum is taken over all cyclic orderings s of $V(G)$. As expected, the *upper traceable number* $t^+(G)$ of a nontrivial connected graph G is defined in [9] as $t^+(G) = \max\{d(s)\}$, where $d(s)$ is described in (1) and the maximum is taken over all linear orderings s of $V(G)$.

To illustrate the concepts of traceable and upper traceable numbers of graphs, let us look at the graph G of order 7 in Figure 1. Since G is not traceable while $d(s_1) = 7$ for the linear ordering $s_1 : v_1, v_2, \ldots, v_7$, it follows that $t(G) = 7$. On the other hand, the diameter of G is 3 and $d(u, v) = 3$ if and only if $u \in \{v_1, v_7\}$ and $v \in \{v_4, v_5\}$. Hence, $d(s)$ contains at most three terms equal to 3 and at least three terms equal to 1 or 2 for every linear ordering s of $V(G)$, that is, $t^+(G) \leq 3 \cdot 3 + 2 \cdot 3 = 15$. Since $d(s_2) = 15$ for the linear ordering $s_2 : v_2, v_4, v_1, v_5, v_7, v_3, v_6$, it follows that $t^+(G) = 15$.

For a connected graph G, let $\text{diam}(G)$ denote the diameter of G. Some results in [8, 9] are summarized below.

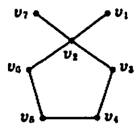

Figure 1: A graph G

Theorem 1.1 [8, 9] *If G is a connected graph of order $n \geq 2$, then*

$$n-1 \leq t(G) \leq t^+(G) \leq (n-1)\,\mathrm{diam}(G).$$

Furthermore,

(a) *$t(G) = 1$ if $n = 2$ while $n-1 \leq t(G) \leq 2n-4$ for $n \geq 3$ and*

$$t(G) = \begin{cases} n-1 & \text{if and only if } G \text{ is traceable} \\ 2n-4 & \text{if and only if } G \text{ is a triangle or a star.} \end{cases}$$

(b) *$n-1 \leq t^+(G) \leq \lfloor n^2/2 \rfloor - 1$ and*

$$t^+(G) = \begin{cases} n-1 & \text{if and only if } G \text{ is complete} \\ \lfloor n^2/2 \rfloor - 1 & \text{if and only if } G \text{ is a path.} \end{cases}$$

Proposition 1.2 [8, 9] *Suppose that G is a nontrivial graph and H is a connected spanning subgraph of G. Then $t(G) \leq t(H)$ and $t^+(G) \leq t^+(H)$.*

By Proposition 1.2 the traceable numbers of a spanning tree of a graph G are upper bounds of the corresponding traceable numbers of G. For this reason, the traceable and upper traceable numbers of nontrivial trees are studied in [8, 9]. For each edge e of a tree T, the *component number* $\mathrm{cn}(e)$ of e is defined in [5] as the minimum order of a component of $T - e$. Let $\mathrm{cn}(T) = \sum_{e \in E(T)} \mathrm{cn}(e)$.

Theorem 1.3 [8, 9] *If T is a tree of order $n \geq 2$, then $t(T) = 2n - 2 - \mathrm{diam}(T)$ and $t^+(T) = 2\,\mathrm{cn}(T) - 1$. Hence, the upper traceable number of every nontrivial tree is odd.*

By Theorem 1.1

$$n-1 \leq t(G) \leq t^+(G) \leq \lfloor n^2/2 \rfloor - 1 \tag{2}$$

for every connected graph G of order $n \geq 2$.

We saw in Theorem 1.1(a) that $n-1 \leq t(G) \leq 2n-4$ for every connected graph G of order $n \geq 3$. For each graph G, let $n(G)$ denote its order. The following realization result appears in [8].

Theorem 1.4 [8] *For a pair n, A of positive integers, there exists a graph G such that $(n(G), t(G)) = (n, A)$ if and only if $(n, A) = (2, 1)$ or $2 \leq n-1 \leq A \leq 2n-4$.*

By (2) we see that $t(G) \leq t^+(G) \leq \lfloor (t(G)+1)^2/2 \rfloor - 1$ for every nontrivial connected graph G. A pair (A, B) of positive integers A and B is said to be *realizable* if there exists a graph G such that $(t(G), t^+(G)) = (A, B)$, and *forbidden* if no such graph exists. Therefore, if (A, B) is a realizable pair, then

$$A \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1. \tag{3}$$

The rest of the paper is organized as follows: We begin Section 2 by establishing some preliminary results and determining the realizable and forbidden pairs for $1 \leq A \leq 6$. In Subsections 2.1 and 2.2, we determine all the remaining forbidden pairs for A even and odd, respectively. In Section 3 we prove the main Theorem 3.2, which gives a complete description of the set of realizable pairs, by constructing examples of graphs to show that all the pairs with $A \geq 7$ which have not been shown to be forbidden are in fact realizable. Furthermore, we establish a result that is parallel to Theorem 1.4 for upper traceable numbers, that is, we determine the set of pairs n, B of integers for which there exists a graph whose order and upper traceable number equal n and B, respectively. We end the paper in Section 4 with an open problem.

2 Some forbidden and realizable pairs

Consider a pair (A, B) of positive integers with $A \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1$. In this section we present a set of forbidden pairs, which will turn out to be *the* set of forbidden pairs later. We first state some useful results which appeared in [9, 10]. Let F_n be a tree of order $n \geq 4$ obtained from a path $(v_1, v_2, \ldots, v_{n-1})$ of order $n-1$ by attaching an end-vertex v_0 at v_2. Furthermore, let $F'_n = F_n + v_0v_1$.

Theorem 2.1 [9] *For every integer $n \geq 3$, $t^+(C_n) = \lceil (n-1)^2/2 \rceil$.*

Theorem 2.2 [10] *If G is a connected graph of order n, then $t^+(G) \neq \lfloor n^2/2 \rfloor - 2$ if $n \geq 4$ and $t^+(G) \neq \lfloor n^2/2 \rfloor - 4$ if $n \geq 6$. Also $t^+(G) = \lfloor n^2/2 \rfloor - 3$ if and only if $G \in \{F_n, F'_n\}$.*

We begin by studying those pairs (A, B) with $A \leq 4$. In order to do so let us examine the traceable and upper traceable numbers of connected graphs having small order.

If $(1, B)$ is realizable, then $B = 1$ by (3) and observe that $t(K_2) = t^+(K_2) = 1$. Similarly, if $(2, B)$ is realizable, then $B = 2, 3$. If G is a graph having traceable number 2, then G is either K_3 or P_3. Observe that $t^+(K_3) = 2$ while $t^+(P_3) = 3$.

If $(3, B)$ is realizable, then $3 \leq B \leq 7$ by (3). If G is a graph whose traceable number equals 3, then G must be a traceable graph of order 4 and so $G \in \{K_4, K_{1,1,2}, C_4, F_4', P_4\}$ and, furthermore, $t^+(G) \neq 6$ by Theorem 2.2. Since $t^+(K_4) = 3$, $t^+(K_{1,1,2}) = 4$, $t^+(C_4) = t^+(F_4') = 5$, and $t^+(P_4) = 7$, it follows that $(3, B)$ is realizable if and only if $3 \leq B \leq 7$ and $B \neq 6$.

If $(4, B)$ is realizable, then $4 \leq B \leq 11$ by (3). If G is a graph whose traceable number equals 4, then either $G = K_{1,3}$ ($= F_4$) or G is a traceable graph of order 5. It then follows from Theorem 2.2 that $t^+(K_{1,3}) = 5$ and $t^+(G) \neq 10$. All traceable graphs of order 5 are shown in Figure 2, where each graph G is labeled by $t^+(G)$. Therefore, $(4, B)$ is realizable if and only

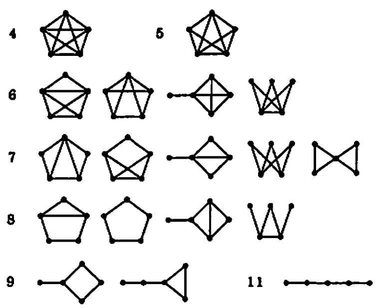

Figure 2: Graphs G of order 5 with $t(G) = 4$

if $4 \leq B \leq 11$ and $B \neq 10$.

For $1 \leq A \leq 4$ we have seen that the pair (A, B) with $A \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1$ is forbidden if and only if $(A, B) \in \{(3, 6), (4, 10)\}$. We next therefore study forbidden pairs (A, B) with $5 \leq A \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1$. More specifically, we show that

$$\text{if } \lfloor A^2/2 \rfloor + 2 \leq B \leq \lfloor (A+1)^2 \rfloor - 2 \text{ and } B \text{ is even, then } (A, B) \text{ is a forbidden pair.} \tag{4}$$

Let $A \geq 5$ and $\lfloor A^2/2 \rfloor + 2 \leq B \leq \lfloor (A+1)^2/2 \rfloor - 2$ and suppose that G is a graph for which $(t(G), t^+(G)) = (A, B)$. If G is connected but not traceable, then $n(G) \leq A$ and so $B \leq \lfloor A^2/2 \rfloor - 1$ by Theorem 1.1(b) (or by (2)), which is impossible since $B \geq \lfloor A^2/2 \rfloor + 2$. Hence, G must be traceable and so $n(G) = A + 1 \geq 6$. Furthermore, G is not Hamiltonian by Proposition 1.2 and Theorem 2.1. The following is a direct consequence of Theorem 2.2.

Theorem 2.3 *The pairs* $(A, \lfloor (A+1)^2/2 \rfloor - 4)$ *and* $(A, \lfloor (A+1)^2/2 \rfloor - 2)$ *are forbidden pairs for each integer* $A \geq 5$.

Note that for $A = 5, 6$ the statement (4) immediately follows by Theorem 2.3. Thus, if $(5, B)$ is realizable, then $5 = A \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1 = 17$ and $B \neq 14, 16$. Similarly, if $(6, B)$ is realizable, then $6 = A \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1 = 23$ and $B \neq 20, 22$. Furthermore, one can verify that the converse of each of the two statements also holds, that is, $(5, B)$ is realizable if and only if $5 \leq B \leq 17$ and $B \neq 14, 16$ while $(6, B)$ is realizable if and only if $6 \leq B \leq 23$ and $B \neq 20, 22$. To see this, let $C_{A+1} = (v_1, v_2, \ldots, v_{A+1}, v_1)$ be an $(A+1)$-cycle. For $A = 5$ consider the graphs G_B ($5 \leq B \leq 17$, $B \neq 14, 16$) such that $G_B = P_{B-4} \cup (10 - B)K_1$ for $5 \leq B \leq 10$, $G_{11} = C_6 + v_1v_3$, $G_{12} = C_6 + v_1v_4$, $G_{13} = C_6$, $G_{15} = (C_6 - v_1v_6) + v_4v_6$ $(= F_6')$, and $G_{17} = C_6 - v_1v_6$ $(= P_6)$. Then each G_B is a traceable graph of order 6 whose upper traceable number equals B. Similarly, for $A = 6$ consider the graphs G_B ($6 \leq B \leq 23$, $B \neq 20, 22$) such that $G_B = P_{B-5} \cup (12 - B)K_1$ for $6 \leq B \leq 12$, $G_{13} = C_7 + v_1v_3 + v_1v_4 + v_2v_5$, $G_{14} = C_7 + v_1v_4 + v_2v_5$, $G_{15} = C_7 + v_1v_3 + v_1v_5$, $G_{16} = C_7 + v_1v_4$, $G_{17} = C_7 + v_1v_3$, $G_{18} = C_7$, $G_{19} = (C_7 - v_1v_7) + v_4v_7$, $G_{21} = (C_7 - v_1v_7) + v_5v_7$ $(= F_7')$, and $G_{23} = C_7 - v_1v_7$ $(= P_7)$. Then each G_B is a traceable graph of order 7 whose upper traceable number equals B.

Remark 2.4 *Note that at this stage we have shown that for* $1 \leq A \leq 6$ *all pairs* (A, B) *with* $A \leq B \leq \lfloor A^2/2 \rfloor$ *are realizable and all pairs* (A, B) *with* $\lfloor A^2/2 \rfloor + 1 \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1$ *and* B *odd are realizable. We have also shown that all pairs* (A, B) *with* $\lfloor A^2/2 \rfloor + 1 \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1$ *and* B *even are forbidden.*

Finally let us assume that $A \geq 7$. Recall that if G is a graph whose traceable and upper traceable numbers are A and B, respectively, where $\lfloor A^2/2 \rfloor + 2 \leq B \leq \lfloor (A+1)^2/2 \rfloor - 2$, then G is a traceable graph of order $A+1$. Since the upper traceable number of a spanning tree of G is an upper bound for $t^+(G)$, it is useful to study the properties of trees of order $A+1$ having upper traceable number greater than or equal to $t^+(G) = B$. In particular, recall that the upper traceable number of every nontrivial tree is odd and so $\lfloor A^2/2 \rfloor + 3 \leq t^+(T) \leq \lfloor (A+1)^2/2 \rfloor - 1$ if T is a spanning tree of G whose upper traceable number is at least $\lfloor A^2/2 \rfloor + 2$.

Before continuing we make the following observation on the component numbers of edges in a tree.

Observation 2.5 *If* T *is a nontrivial tree, then no three consecutive edges in* T *have the same component number. Also, the subgraph induced by the set of edges having the maximum component number is connected. Therefore, the subgraph induced by the set of edges having the maximum component number is isomorphic to a star.*

2.1 Forbidden pairs (A, B) where A is even

In this subsection we consider those pairs (A, B) with $A^2/2 + 2 \le B \le \lfloor (A+1)^2/2 \rfloor - 2$, where $A \ge 8$ and is even. We first state the following observation without a proof.

Observation 2.6 *If T is a tree of odd order $A + 1 \ge 9$ with $t^+(T) \ge A^2/2 + 3$, then there are four consecutive edges e_1, e_2, e_3, and e_4 such that* $\text{cn}(e_1) = A/2 - 1$, $\text{cn}(e_2) = \text{cn}(e_3) = A/2$, *and* $\text{cn}(e_4) \in \{A/2 - 2, A/2 - 1\}$. *In particular, if* $\text{cn}(e_4) = A/2 - 2$, *then T is constructed from a path $(v_1, v_2, \ldots, v_A)$ of order A by adding a pendant edge at $v_{A/2-1}$. We denote this tree by T_A^*.*

The following is an immediate consequence of Observation 2.6.

Lemma 2.7 *If T is a tree of odd order $A+1 \ge 9$ with $t^+(T) \ge A^2/2+3$, then either* (i) $T \cong T_A^*$ *or* (ii) *T is obtained from two vertex-disjoint trees T_1 and T_2 of order $A/2$ by adding a new vertex x and joining x to v_1 and v_2, where v_i is an end-vertex of T_i for $i = 1, 2$. In particular, if* (ii) *occurs, then* $\text{cn}(xv_i) = A/2$ *and* $\deg x = \deg v_i = 2$ *for $i = 1, 2$. That is, the vertices incident with an edge whose component number equals $A/2$ must have degree 2.*

Let S_A be the set of trees of odd order $A + 1 \ge 9$ satisfying either (i) or (ii) described in Lemma 2.7.

Lemma 2.8 *If G is a connected graph of odd order $A+1 \ge 9$ with $t^+(G) \ge A^2/2 + 2$, then G contains a cut-vertex x having degree 2 such that $G - x$ consists of two components each of which contains $A/2$ vertices.*

Proof. Let T be a spanning tree of G. Then $t^+(T) \ge t^+(G)$, implying that $T \in S_A$ by Lemma 2.7. In particular, if G is a tree, then the result immediately follows. Thus, suppose that G is not a tree. We consider two cases according to whether T_A^* is a spanning tree of G or not.

Case 1. G contains a spanning tree $T \cong T_A^*$. Let T be a spanning tree of G constructed from two vertex-disjoint paths $(u_1, u_2, \ldots, u_{A/2})$ and $(w_1, w_2, \ldots, w_{A/2})$, both isomorphic to $P_{A/2}$, by adding a new vertex x and joining x to u_2 and w_1. Let $U = \{u_1, u_2, \ldots, u_{A/2}\}$ and $W = \{w_1, w_2, \ldots, w_{A/2}\}$.

We first show that $\deg_G x = 2$. If $u_1x \in E(G)$, then $T' = (T + u_1x) - u_1u_2$ is a spanning tree of G. However then, w_1x is the only edge whose component number equals $A/2$ and so $T' \notin S_A$, which is a contradiction. If $ux \in E(G)$ for some $u \in U - \{u_1, u_2\}$, then $T' = (T + ux) - u_2u_3$ is a spanning tree of G but $T' \notin S_A$ since $\text{cn}_{T'}(e) = A/2$ if and only if $e = w_1x$

and this is impossible. Similarly, if $wx \in E(G)$ for some $w \in W - \{w_1\}$, then it can be verified that $T' = (T + wx) - w_1w_2$ is a spanning tree of G not belonging to $\mathcal{S}_A$, which again cannot occur. Therefore, $\deg_G x = 2$ as claimed.

We next show that if $u \in U$ and $w \in W$, then $uw \notin E(G)$. Assume, to the contrary, that there exists an edge $e = uw$ for some $u \in U$ and $w \in W$. Consider

$$T' = \begin{cases} (T+e) - u_2x & \text{if } e = uw \text{ where } u \in \{u_1, u_{A/2}\} \\ & \text{and } w \in W - \{w_{A/2}\} \\ (T+e) - w_1w_2 & \text{if } e = u_1w_{A/2} \\ (T+e) - w_{A/2-1}w_{A/2} & \text{if } e = u_{A/2}w_{A/2} \\ (T+e) - w_1x & \text{if } e = uw \text{ where } u \in U - \{u_1, u_{A/2}\} \\ & \text{and } w \in W \end{cases}$$

and observe that T' is a spanning tree of G not belonging to $\mathcal{S}_A$, which is impossible.

Case 2. No spanning tree of G is isomorphic to T_A^*. Let T be a spanning tree of G. By Lemma 2.7 we may assume that there exist two vertex-disjoint trees T_1 and T_2 of order $A/2$ such that T is obtained from T_1 and T_2 by adding a new vertex x and joining x to u_1 and w_1, where u_1 is an end-vertex in T_1 and w_1 is an end-vertex in T_2. Let $V(T_1) = U = \{u_1, u_2, \ldots, u_{A/2}\}$ and $V(T_2) = W = \{w_1, w_2, \ldots, w_{A/2}\}$. Furthermore, let $N_T(u_1) = \{u_2, x\}$ and $N_T(w_1) = \{w_2, x\}$.

First we show that $\deg_G x = 2$. If not, then we may assume, without loss of generality, that $ux \in E(G)$ for some $u \in U - \{u_1\}$. However then $T' = (T + ux) - u_1u_2$ is a spanning tree of G, where $\text{cn}_{T'}(w_1x) = A/2$ and $\deg_{T'} x = 3$. Since $T' \not\cong T_A^*$ by assumption, $T' \notin \mathcal{S}_A$. This is a contradiction, concluding that $\deg_G x = 2$.

Next we show that if $u \in U$ and $w \in W$, then $uw \notin E(G)$. Assume, to the contrary, that $e = uw \in E(G)$ for some $u \in U$ and $w \in W$.

If at least one of u and w is not an end-vertex in T, say $\deg_T u \geq 2$, then let $T' = (T + e) - u_1x$. Then T' must be a spanning tree of G belonging to $\mathcal{S}_A - \{T_A^*\}$. However, $\text{cn}_{T'}(e) = A/2$ and $\deg_{T'} u \geq 3$, which contradicts Lemma 2.7.

Therefore, suppose that $\deg_T u = \deg_T w = 1$ and consider the graph $H = T + e \subseteq G$. Then H contains a cycle C containing the vertices in the set $\{u_1, u_2, u, w_1, w_2, w, x\}$. Since $t^+(G) \geq A^2/2+2$, it follows that C is not a Hamiltonian cycle of G, implying that there exists a vertex v belonging to C whose degree in H is at least 3. Without loss of generality suppose that $v \in U$. Also, let $d = d_T(u, x)$ and let

$$C = (v_0 = x, v_1 = u_1, v_2 = u_2, \ldots, v_d = u,$$
$$v_{d+1} = w, v_{d+2}, \ldots, v_{N-2} = w_2, v_{N-1} = w_1, v_N = x)$$

where N is the length of C. Then there exists a positive integer k such that $\deg_H v_i = 2$ for $0 \le i \le k$ and $\deg_H v_{k+1} \ge 3$. By assumption $1 \le k \le d-2 \le A/2-3$.

We claim that $\deg_H v_{d+i} = 2$ for $1 \le i \le k$. First observe that $\deg_H v_{d+1} = \deg_H w = 2$. If $k \ge 2$, then assume that $\deg_H v_{d+i} = 2$ for some i with $1 \le i \le k-1$ and consider the tree $T' = H - v_i v_{i+1}$, which must belong to $\mathcal{S}_A - \{T_A^*\}$. Then $\mathrm{cn}_{T'}(v_{d+i}v_{d+i+1}) = A/2$ and so $\deg_{T'} v_{d+i+1} = 2$ by Lemma 2.7, which in turn implies that $\deg_H v_{d+i+1} = 2$. Hence, $\deg_H v_{d+i} = 2$ for $1 \le i \le k$ by induction. (Also note that $v_{d+i} \in W$ for $1 \le i \le k$ since $k \le A/2-3$.)

Now for the tree $T'' = H - v_{d+k}v_{d+k+1}$ we have $\mathrm{cn}_{T''}(v_k v_{k+1}) = A/2$. However, this is a contradiction since $\deg_{T''} v_{k+1} = \deg_H v_{k+1} \ge 3$ and T'' is a spanning tree of G. ∎

We are now prepared to present the following result.

Proposition 2.9 *If G is a connected graph of odd order $A+1 \ge 9$ with $t^+(G) \ge A^2/2+2$, then $t^+(G)$ is odd. That is, if $A^2/2+2 \le B \le \lfloor (A+1)^2/2 \rfloor - 2$ and B is even, then (A,B) is a forbidden pair.*

Proof. Let x be the cut-vertex of G whose degree is 2 such that deleting x from G results in two components each of which contains $A/2$ vertices. Let $N(x) = \{u_1, u_2\}$ and $G - x = G_1 \cup G_2$, where G_1 and G_2 are the two components of $G - x$ and $u_i \in V(G_i)$ for $i = 1,2$. Furthermore, let T_i be the spanning tree of G_i such that $d_{T_i}(u_i, v) = d_{G_i}(u_i, v)$ for every $v \in V(G_i) - \{u_i\}$ for $i = 1,2$. Therefore, the tree T obtained from T_1 and T_2 by adding the vertex x and the edges u_1x and u_2x is a spanning tree of G and $d_G(u,v) = d_T(u,v)$ if $\{u,v\} \not\subseteq V(G_1)$ or $\{u,v\} \not\subseteq V(G_2)$. Also, there exists a linear ordering $s : v_1, v_2, \ldots, v_{A+1}$ of $V(T)$ such that $d_T(s) = t^+(T)$, where $v_1 = x$ and $v_i \in V(T_j)$ if and only if $i \equiv j \pmod 2$ for $2 \le i \le A+1$ and $j = 1,2$. Then $d_G(v_i, v_{i+1}) = d_T(v_i, v_{i+1})$ for $1 \le i \le A$ and so $t^+(G) \ge d_G(s) = d_T(s) = t^+(T)$, that is, $t^+(G) = t^+(T)$. Hence $t^+(G)$ must be odd.

2.2 Forbidden pairs (A,B) where A is odd

Next we consider those pairs (A,B) with $\lfloor A^2/2 \rfloor + 2 \le B \le \lfloor (A+1)^2/2 \rfloor - 2$, where $A \ge 7$ and is odd. We first present two observations.

Observation 2.10 *If T is a tree of order $A+1 = 8$ with $t^+(T) \ge \lfloor A^2/2 \rfloor + 3 = 27$, then T is isomorphic to one of the four trees shown in* Figure 3.

Observation 2.11 *If T is a tree of even order $A+1 \ge 10$ with $t^+(T) \ge \lfloor A^2/2 \rfloor + 3$, then there are four consecutive edges e_1, e_2, e_3, and e_4 such*

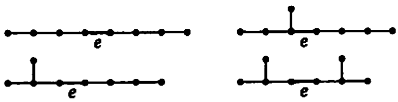

Figure 3: Four trees of order 8

that $\text{cn}(e_1) = \lfloor A/2 \rfloor - 1$, $\text{cn}(e_2) = \lfloor A/2 \rfloor$, $\text{cn}(e_3) = \lceil A/2 \rceil$, *and* $\text{cn}(e_4) \in \{\lfloor A/2 \rfloor - 1, \lfloor A/2 \rfloor\}$. *In particular, if* $\text{cn}(e_4) = \lfloor A/2 \rfloor - 1$, *then* T *is constructed from a path* $(v_1, v_2, \ldots, v_A)$ *of order* A *by adding a pendant edge at* $v_{\lfloor A/2 \rfloor}$. *We denote this tree by* T_A^*.

Observation 2.11 implies the following.

Lemma 2.12 *If* T *is a tree of even order* $A + 1 \geq 10$ *with* $t^+(T) \geq \lfloor A^2/2 \rfloor + 3$, *then either* (i) $T \cong T_A^*$ *or* (ii) T *is obtained from two vertex-disjoint trees* T_1 *and* T_2 *of order* $\lfloor A/2 \rfloor$ *and* $\lceil A/2 \rceil$, *respectively, by adding a new vertex* x *and joining* x *to* v_1 *and* v_2, *where* v_i *is an end-vertex of* T_i *for* $i = 1, 2$. *In particular, if* (ii) *occurs, then* $\text{cn}(v_2 x) = \lceil A/2 \rceil$ *and* $\deg x = \deg v_i = 2$ *for* $i = 1, 2$.

Let $\mathcal{S}_A$ be the set of trees of even order $A+1 \geq 10$ satisfying either (i) or (ii) described in Lemma 2.12. Also, let $\mathcal{S}_7$ be the set of four trees of order 8 in Figure 3. The following lemma is parallel to Lemma 2.8 presented in the previous subsection. We only present a proof for $A = 7$ since for $A \geq 9$ the argument is almost identical to that for $A = 7$ and Lemma 2.8.

Lemma 2.13 *If* G *is a connected graph of even order* $A + 1 \geq 8$ *with* $t^+(G) \geq \lfloor A^2/2 \rfloor + 2$, *then* G *contains a bridge* e *such that* $G - e$ *consists of two components each of which contains* $\lceil A/2 \rceil$ *vertices.*

Proof. We only verify the result for $A = 7$. If G is a graph of order 8 with $t^+(G) \geq 26$, then every spanning tree of G must belong to $\mathcal{S}_7$. In particular, the result is immediate if G is a tree.

If G is not a tree, then let $T \in \mathcal{S}_7$ be a spanning tree of G and assume, to the contrary, that $G - e$ is connected, where e is the edge in T shown in Figure 3. Then one can verify that either (i) G is Hamiltonian, (ii) G contains a spanning tree not belonging to $\mathcal{S}_7$, or (iii) G contains one of H_1 and H_2 in Figure 4 as a spanning subgraph. Since (i) and (ii) are clearly impossible, let us assume that (iii) occurs.

If $H_1 \subseteq G$, then $26 \leq t^+(G) \leq t^+(H_1)$. Observe that $d(u,v) = 5$ (= $\text{diam}(H_1)$) if and only if $\{u,v\} = \{v_1, v_4\}$ and $d(u,v) = 4$ if and only if $\{u,v\} \in \{\{v_1, v_3\}, \{v_2, v_4\}\}$, implying that $t^+(H_1) \leq 5 \cdot 1 + 4 \cdot 2 + 3 \cdot 4 = 25$, which is a contradiction. Thus, suppose that $H_2 \subseteq G$ and so $t^+(H_2) \geq 26$.

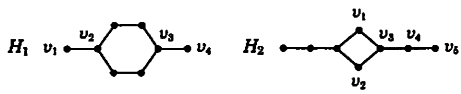

Figure 4: Graphs H_1 and H_2

Let s be a linear ordering of $V(H_2)$ such that $d_{H_2}(s) = t^+(H_2)$ and suppose that $v \in V(H_2) - \{v_1\}$ is a vertex that appears immediately before or after v_1 in s. If $v \in \{v_3, v_4, v_5\}$, then consider the tree $T = H_2 - v_1v_3$ and observe that $d_T(v_1, v) = d_{H_2}(v_1, v) + 2$. Therefore, $t^+(T) \geq d_T(s) \geq d_{H_2}(s) + 2 \geq 28$, which is a contradiction since $t^+(T) = 27$. Hence, we may assume that v_1 is the initial vertex in s followed by v_2. However, this in turn implies that $t^+(H_2) = d_{H_2}(s) \leq 2 + t^+(H_2 - v_1) = 25$. Therefore, (iii) is also impossible.

We are now prepared to state a result for connected graphs of even order $A+1 \geq 8$, which corresponds to Proposition 2.9 in the previous subsection. We omit its proof.

Proposition 2.14 *If G is a connected graph of even order $A+1 \geq 8$ with $t^+(G) \geq \lfloor A^2/2 \rfloor + 2$, then $t^+(G)$ is odd. That is, if $\lfloor A^2/2 \rfloor + 2 \leq B \leq \lfloor (A+1)^2/2 \rfloor - 2$ and B is even, then (A, B) is a forbidden pair.*

Combining Propositions 2.9 and 2.14 with (3), we now have the following.

Theorem 2.15 *Suppose that A and B are positive integers. If (A, B) is a realizable pair, then either* (I) *$A \leq B \leq \lfloor A^2/2 \rfloor$ or* (II) *$\lfloor A^2/2 \rfloor + 1 \leq B \leq \lfloor (A+1)^2/2 \rfloor - 1$ and B is odd.*

3 A realization result

We finally determine the set of realizable pairs by verifying that the converse of Theorem 2.15 also holds. Before doing this we present some additional definitions and notation.

For a fixed integer $k \geq 2$, define $f_k : \{1, 2, \ldots, k-1\} \to \mathbb{N}$ by $f_k(x) = kx - \binom{x+1}{2}$. Observe that f_k is strictly increasing on the domain. If $k \geq 3$, then for each integer p with $k \leq p \leq \binom{k}{2}$ let q be the unique integer $(1 \leq q \leq k-2)$ such that $f_k(q) < p \leq f_k(q+1)$ and let $r = p - f_k(q)$. Hence, $1 \leq r \leq k - q - 1$.

Definition 3.1 *Let k and p be integers with $k \geq 2$ and $k - 1 \leq p \leq \binom{k}{2}$.*

- *If $p = k-1$, then let $T_{k,p}$ be a star of order k rooted at the central vertex v_1 whose neighbors are $u_1, u_2, \ldots, u_{k-1}$. Let $G_{k,p}$ be the graph obtained from $T_{k,p}$ by joining u_i to u_{i+1} for $1 \le i \le k-2$ (if $k \ge 3$).*
- *If $3 \le k \le p \le \binom{k}{2}$, then let $T_{k,p}$ be a tree of order k, rooted at v_1, constructed from a path $(v_1, v_2, \ldots, v_q)$ and the vertices in the set $\{u_1, u_2, \ldots, u_{k-q-r}\} \cup \{w_1, w_2, \ldots, w_r\}$ by joining each u_i to v_q and each w_i to u_1. Let $G_{k,p}$ be the graph obtained from $T_{k,p}$ by joining u_i to u_{i+1} for $1 \le i \le k-q-r$ (if $k \ge q+r+2$) and w_i to w_{i+1} for $1 \le i \le r-1$ (if $r \ge 2$).*

In each case the vertex v_1 is called the root of $G_{k,p}$.

In Figure 5 the graphs $G_{5,p}$ $(4 \le p \le 10)$ are shown, where the root of each graph is indicated by a hollow vertex.

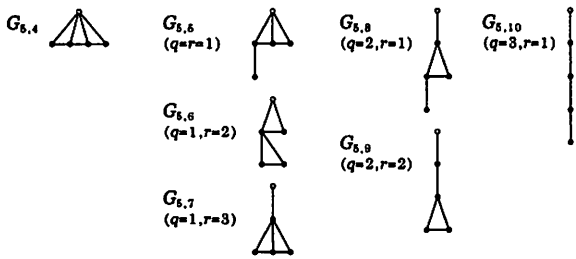

Figure 5: Graphs $G_{5,p}$ for $4 \le p \le 10$

We finally prove the main theorem in this paper.

Theorem 3.2 *For a pair A, B of positive integers, there exists a graph G such that $(t(G), t^+(G)) = (A, B)$ if and only if either* (I) *$A \le B \le \lfloor A^2/2 \rfloor$ or* (II) *$\lfloor A^2/2 \rfloor + 1 \le B \le \lfloor (A+1)^2/2 \rfloor - 1$ and B is odd.*

Proof. By Remark 2.4 the statement holds for $1 \le A \le 6$. Thus, we only prove the converse of Theorem 2.15 for $A \ge 7$ by showing that for each B satisfying either (I) or (II) there exists a traceable graph G_B of order $A+1$ having upper traceable number B.

If $A \le B \le 2A$, then let $P = (v_1, v_2, \ldots, v_{B-A+1})$ be a path and consider the graph G_B of order $A+1$ such that G_B consists of P and $2A-B$ additional isolated vertices. Then G_B is traceable and $t^+(G_B) \ge d(s_0) = B$, where s_0 is a linear ordering of $V(G_B)$ whose first $B-A+1$ terms are $v_1, v_2, \ldots, v_{B-A+1}$. Also, $\text{diam}(G_B) = 2$ and $d_{G_B}(u,v) = 2$ if and only if $uv \in E(P)$. Therefore, $d(s) \le 2 \cdot (B-A) + 1 \cdot (2A-B) = B$ for every linear ordering s of $V(G_B)$, implying that $t^+(G_B) = B$.

For $B \geq 2A+1$ we consider two cases according to the parity of A, beginning with assuming that A is odd.

Case 1. *A is odd.*

Subcase 1.1. *B is even.* If $2A+2 \leq B \leq 3A-5$, then $B = 2A+2\ell$ for some positive integer $\ell \leq \lfloor A/2 \rfloor - 2$. Let H_1, H_2, and H_3 be pairwise vertex-disjoint graphs where $H_1 \cong K_{\lfloor A/2 \rfloor - \ell, \lceil A/2 \rceil - \ell}$ whose partite sets are $\{v_1, v_2, \ldots, v_{\lfloor A/2 \rfloor - \ell}\}$ and $\{v_{\lceil A/2 \rceil - \ell}, v_{\lceil A/2 \rceil - \ell + 1}, \ldots, v_{A-2\ell}\}$, $H_2 \cong K_{\ell+1}$ with $V(H_2) = \{u_1, u_2, \ldots, u_{\ell+1}\}$, and $H_3 \cong K_\ell$ with $V(H_3) = \{w_1, w_2, \ldots, w_\ell\}$. Then G_B is constructed from H_1, H_2, and H_3 by joining (i) every vertex of H_2 to v_1 and $v_{A-2\ell}$ and (ii) every vertex of H_3 to $v_{\lfloor A/2 \rfloor - \ell}$ and $v_{\lceil A/2 \rceil - \ell}$. Hence, G_B is a traceable graph of order $A+1$ and $t^+(G_B) \geq d(s_0) = B$, where

$$
\begin{aligned}
s_0 :\ & v_1, v_2, \ldots, v_{\lfloor A/2 \rfloor - \ell}, u_1, w_1, u_2, w_2, \ldots, u_\ell, w_\ell, u_{\ell+1}, \\
& v_{\lceil A/2 \rceil - \ell}, v_{\lceil A/2 \rceil - \ell + 1}, \ldots, v_{A-2\ell}.
\end{aligned}
$$

Note that $d(u,v) = 3$ $(= \operatorname{diam}(G_B))$ if and only if $u \in V(H_2)$ and $v \in V(H_3)$. Therefore, $d(s) \leq 3 \cdot 2\ell + 2 \cdot (A - 2\ell) = B$ for every linear ordering s of $V(G_B)$, implying that $t^+(G_B) = B$.

If $B = 3A-3$, then for $1 \leq i \leq 6$ let

$$
V_i = \begin{cases} \{v_{i,1}\} & \text{if } i \neq 2, 5 \\ \{v_{2,1}, v_{2,2}, \ldots, v_{2,\lfloor A/2 \rfloor}\} & \text{if } i = 2 \\ \{v_{5,1}, v_{5,2}, \ldots, v_{5,\lfloor A/2 \rfloor - 2}\} & \text{if } i = 5 \end{cases}
$$

be pairwise disjoint sets of vertices such that for each i the vertices in V_i form a complete graph. Then G_B is the graph with $V(G_B) = \cup_{i=1}^{6} V_i$ obtained by joining every vertex in V_i to every vertex in V_{i+1} for $1 \leq i \leq 6$ (where $V_7 = V_1$). Hence G_B is a Hamiltonian graph of order $A+1$ and $t^+(G_B) \geq d(s_0) = B$, where

$$
\begin{aligned}
s_0 :\ & v_{2,1}, v_{5,1}, v_{2,2}, v_{5,2}, \ldots, v_{2,\lfloor A/2 \rfloor - 2}, v_{5,\lfloor A/2 \rfloor - 2}, v_{2,\lfloor A/2 \rfloor - 1}, \\
& v_{4,1}, v_{1,1}, v_{3,1}, v_{6,1}, v_{2,\lfloor A/2 \rfloor}.
\end{aligned}
$$

For every linear ordering s of $V(G_B)$, observe that $d(s)$ contains at most $A-3$ terms equal to 3 $(= \operatorname{diam}(G_B))$ and so $d(s) \leq 3 \cdot (A-3) + 2 \cdot 3 = B$. Thus, $t^+(G_B) = B$.

Next suppose that $B = 3A-1$. For $A = 7$ one can verify that the graph obtained from a 7-cycle $(v_1, v_2, \ldots, v_7, v_1)$ by adding a new vertex v_8 and joining v_8 to v_1 and v_4 is traceable and its upper traceable number equals 20. For $A \geq 9$ let G_B be the graph obtained from an $(A-1)$-cycle $(v_1, v_2, \ldots, v_{A-1}, v_1)$ by adding two new vertices v_A and v_{A+1} and joining v_A to v_{2i-1} and v_{A+1} to v_{2i} for $1 \leq i \leq \lfloor A/2 \rfloor$. Then G_B is a Hamiltonian graph of order $A+1$ and $t^+(G_B) \geq d(s_0) = B$, where

$$
s_0 :\ v_1, v_6, v_3, v_8, \ldots, v_{A-6}, v_{A-1}, v_{A-4}, v_2, v_{A-2}, v_4, v_A, v_{A+1}.
$$

Observe also that $d(s) < 3A = B+1$ for every linear ordering s of $V(G_B)$ since if $d(v_i, v_j) = 3$ (= diam(G_B)) and $\{i,j\} \cap \{A, A+1\} \neq \emptyset$, then $\{i,j\} = \{A, A+1\}$. Hence, $t^+(G_B) = B$.

If $3A+1 \leq B \leq 4A-6$, then $B = 3A+2\ell-1$ for some positive integer $\ell \leq \lceil A/2 \rceil - 3$. For $1 \leq i \leq 8$ let

$$V_i = \begin{cases} \{v_{i,1}\} & \text{if } i \text{ is even} \\ \{v_{i,1}, v_{i,2}, \ldots, v_{i,\ell}\} & \text{if } i = 1,5 \\ \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lfloor A/2 \rfloor - \ell - 1}\} & \text{if } i = 3,7 \end{cases}$$

be pairwise disjoint sets of vertices such that the vertices in V_i form a complete graph for each i. Construct G_B with $V(G_B) = \cup_{i=1}^{8} V_i$ by joining every vertex in V_i to every vertex in V_{i+1} for $1 \leq i \leq 8$ (where $V_9 = V_1$) as well as joining $v_{4,1}$ and $v_{6,1}$. Hence G_B is a Hamiltonian graph of order $A+1$ and $t^+(G_B) \geq d(s_0) = B$, where

$$s_0 : v_{4,1}, v_{8,1}, v_{3,1}, v_{7,1}, v_{3,2}, v_{7,2}, \ldots, v_{3,\lfloor A/2 \rfloor - \ell - 1}, v_{7,\lfloor A/2 \rfloor - \ell - 1},$$
$$v_{2,1}, v_{6,1}, v_{1,1}, v_{5,1}, v_{1,2}, v_{5,2}, \ldots, v_{1,\ell}, v_{5,\ell}.$$

On the other hand, $d(u,v) = 4$ (= diam(G_B)) if and only if $u \in V_1$ and $v \in V_5$. Therefore, $d(s) \leq 4 \cdot (2\ell - 1) + 3 \cdot (A - 2\ell + 1) = B$ for every linear ordering s of $V(G_B)$ and so $t^+(G_B) = B$.

Finally, suppose that $4A-4 \leq B \leq \lfloor A^2/2 \rfloor$. Let H_1 and H_2 be vertex-disjoint graphs such that

$$H_1 \cong \begin{cases} G_{\lceil A/2 \rceil, A-1} & \text{if } 4A-4 \leq B \leq (A^2+8A-9)/4 \\ G_{\lceil A/2 \rceil, (A^2-1)/8} & \text{if } (A^2+8A-9)/4 + 2 \leq B \leq \lfloor A^2/2 \rfloor \end{cases}$$

$$H_2 \cong \begin{cases} G_{\lceil A/2 \rceil, B/2-A+1} & \text{if } 4A-4 \leq B \leq (A^2+8A-9)/4 \\ G_{\lceil A/2 \rceil, B/2-(A^2-1)/8} & \text{if } (A^2+8A-9)/4 + 2 \leq B \leq \lfloor A^2/2 \rfloor \end{cases}$$

whose roots are u_1 and u_2, respectively. Observe that u_i is an end-vertex and so let u_i' be the neighbor of u_i for $i = 1,2$. Construct G_B from H_1 and H_2 by adding the three edges u_1u_2, u_1u_2', and $u_1'u_2$ and observe that G_B is a traceable graph of order $A+1$. Furthermore, $d(s) = B$ for every linear ordering $s : v_1, v_2, \ldots, v_{A+1}$ of $V(G_B)$ such that $v_i \in V(H_j)$ if and only if $i \equiv j \pmod 2$ for $1 \leq i \leq A+1$ and $j = 1,2$ with $v_1 = u_1$ and $v_{A+1} = u_2$. Hence, $t^+(G_B) \geq B$.

Let T_1 and T_2 be spanning trees of $H_1 \cong G_{k_1,p_1}$ and $H_2 \cong G_{k_2,p_2}$, respectively, such that $T_i \cong T_{k_i,p_i}$ for $i = 1,2$. Then the tree T obtained from T_1 and T_2 by adding the edges u_1u_2 and u_1u_2' and deleting the edge u_2u_2' is a spanning tree of G_B. One can verify that $\text{cn}(T) = B/2+1$ and so $t^+(G_B) \leq t^+(T) = B+1$. Assume, to the contrary, that $t^+(G_B) = B+1$ and let s_0 be a linear ordering of $V(G_B)$ with $d_{G_B}(s_0) = B+1$. Note that $d_{G_B}(u,v) \leq d_T(u,v)$ for every $u,v \in V(G_B)$. In particular, $d_{G_B}(u_2,v) = d_T(u_2,v)$ if and only if $v = u_1$. Therefore, we may assume that the initial term of s_0 is u_2 followed by u_1. Now consider the tree $T' = T - u_2$ and observe that $\text{cn}_{T'}(e) = \text{cn}_T(e)$ for every $e \in E(T')$ and so $\text{cn}(T') = \text{cn}(T) - 1$, that is, $t^+(T') = B - 1$. Furthermore, let s_0' be the linear ordering obtained by deleting u_2 from s_0 and observe that s_0' is a linear ordering of $V(T')$. However then, $B + 1 = d(s_0) = d(s_0') + 1 \leq t^+(T') + 1 \leq B$, a contradiction. Therefore, $t^+(G_B) = B$ as claimed.

Subcase 1.2. *B is odd.* Let B be an odd integer such that $2A+1 \leq B \leq (A+1)^2/2 - 1$. Let H_1 and H_2 be vertex-disjoint graphs such that

$$H_1 \cong \begin{cases} G_{\lceil B/2\rceil - A+1, \lceil B/2\rceil - A} & \text{if } 2A+1 \leq B \leq 3A-2 \\ G_{\lceil A/2\rceil, \lceil A/2\rceil} & \text{if } 3A \leq B \leq (A^2+8A+3)/4 \\ G_{\lceil A/2\rceil, (A^2-1)/8} & \text{if } (A^2+8A+3)/4+2 \leq B \leq \\ & (A+1)^2/2 - 1 \end{cases}$$

$$H_2 \cong \begin{cases} G_{2A - \lceil B/2\rceil, 2A - \lceil B/2\rceil - 1} & \text{if } 2A+1 \leq B \leq 3A-2 \\ G_{\lceil A/2\rceil, \lfloor B/2\rfloor - A} & \text{if } 3A \leq B \leq (A^2+8A+3)/4 \\ G_{\lceil A/2\rceil, \lceil B/2\rceil - (A^2+4A+3)/8} & \text{if } (A^2+8A+3)/4+2 \leq B \leq \\ & (A+1)^2/2 - 1 \end{cases}$$

whose roots are u_1 and u_2, respectively. Let G_B be the traceable graph of order $A+1$ obtained from H_1 and H_2 by joining the two roots. Furthermore, $t^+(G_B) \geq B$ since $d(s) = B$ for every linear ordering $s : v_1, v_2, \ldots, v_{A+1}$ of $V(G_B)$ such that $v_i \in V(H_j)$ if and only if $1 \leq i \leq A+1$ and $j = 1, 2$ with $v_1 = u_1$ and $v_{A+1} = u_2$.

On the other hand, let T_1 and T_2 be spanning tree of $H_1 \cong G_{k_1,p_1}$ and $H_2 \cong G_{k_2,p_2}$, respectively, such that $T_i \cong T_{k_i,p_i}$ for $i = 1, 2$. Then the tree T obtained from T_1 and T_2 by joining the two roots is a spanning tree of G_B. One can also verify that $\text{cn}(T) = \lceil B/2\rceil$ and so $t^+(T) = B$, implying that $t^+(G_B) = B$.

Case 2. *A is even.* We only describe how the graph G_B is constructed for each B since the proof verifying that $t^+(G_B) = B$ is almost identical to those presented in Case 1.

Subcase 2.1. *B is even.* If $2A+2 \leq B \leq 3A-4$, then $B = 2A + 2\ell$ for some positive integer $\ell \leq A/2 - 2$. Let H_1, H_2, and H_3 be pairwise vertex-disjoint graphs where $H_1 \cong K_{A/2-\ell, A/2-\ell}$, $H_2 \cong K_{\ell+1}$, and $H_3 \cong K_\ell$. Let $(v_1, v_2, v_3, v_4, v_1)$ be a 4-cycle in H_1 and obtain G_B from H_1, H_2, and H_3 by joining (i) every vertex of H_2 to v_1 and v_2 and (ii) every vertex of H_3 to v_3 and v_4.

If $B = 3A-2$, then let $V_1, V_2, \ldots, V_6$ be pairwise disjoint sets of vertices such that $|V_2| = A/2 - 1$, $|V_5| = A/2 - 2$, and $|V_i| = 1$ if $1 \leq i \leq 6$ and $i \neq 2, 5$. Also, suppose that the vertices in V_i form a complete graph for each i. Then G_B is the graph with $V(G_B) = \cup_{i=1}^6 V_i$ obtained by joining every vertex in V_i to every vertex in V_{i+1} for $1 \leq i \leq 6$ (where $V_7 = V_1$).

If $3A \leq B \leq 4A - 6$, then $B = 3A + 2\ell$ for some nonnegative integer $\ell \leq A/2 - 3$. Let $V_1, V_2, \ldots, V_7, V_6'$ be pairwise disjoint sets of vertices such that $|V_3| = A/2-2$, $|V_6| = A/2-\ell-2$, $|V_i| = 1$ if $1 \le i \le 7$ and $i \ne 3,6$, and $|V_6'| = \ell$. Furthermore, the vertices in each V_i ($1 \le i \le 7$) form a complete graph as well as the vertices in $V_6 \cup V_6'$ form a complete graph. Then G_B is the graph with $V(G_B) = (\cup_{i=1}^7 V_i) \cup V_6'$ obtained by joining every vertex in V_i to every vertex in V_{i+1} for $1 \le i \le 7$ (where $V_8 = V_1$).

If $4A-4 \le B \le A^2/2$, then let H_1 and H_2 be vertex-disjoint graphs such that

$$H_1 \cong \begin{cases} G_{A/2,A-2} & \text{if } 4A-4 \le B \le (A^2+10A-16)/4 \\ G_{A/2,B/2-(A^2+2A)/8} & \text{if } (A^2+10A-16)/4+2 \le B \le A^2/2 \end{cases}$$

$$H_2 \cong \begin{cases} G_{A/2+1,B/2-A+2} & \text{if } 4A-4 \le B \le (A^2+10A-16)/4 \\ G_{A/2+1,(A^2+2A)/8} & \text{if } (A^2+10A-16)/4+2 \le B \le A^2/2 \end{cases}$$

whose roots are u_1 and u_2, respectively. Since u_i is an end-vertex let u_i' be the neighbor of u_i for $i = 1,2$ and construct G_B from H_1 and H_2 by adding the three edges u_1u_2, u_1u_2', and $u_1'u_2$.

Subcase 2.2. *B is odd.* Let B be an odd integer such that $2A+1 \le B \le \lfloor (A+1)^2/2 \rfloor - 1$ and consider vertex-disjoint graphs H_1 and H_2 given by

$$H_1 \cong \begin{cases} G_{\lceil B/2\rceil-A+1,\lceil B/2\rceil-A} & \text{if } 2A+1 \le B \le 3A-3 \\ G_{A/2,\lfloor B/2\rfloor-A} & \text{if } 3A-1 \le B \le (A^2+6A+4)/4 \\ G_{A/2,(A^2-2A)/8} & \text{if } (A^2+6A+4)/4+2 \le B \le \\ & \lfloor (A+1)^2/2\rfloor - 1 \end{cases}$$

$$H_2 \cong \begin{cases} G_{2A-\lceil B/2\rceil,2A-\lceil B/2\rceil-1} & \text{if } 2A+1 \le B \le 3A-3 \\ G_{A/2+1,A/2+1} & \text{if } 3A-1 \le B \le (A^2+6A+4)/4 \\ G_{A/2+1,\lceil B/2\rceil-(A^2+2A)/8} & \text{if } (A^2+6A+4)/4+2 \le B \le \\ & \lfloor (A+1)^2/2\rfloor - 1. \end{cases}$$

Then the graph G_B is constructed from H_1 and H_2 by joining the two roots. ∎

Recall that for a pair n, A of positive integers, there exists a graph G such that $(n(G), t(G)) = (n, A)$ if and only if $(n, A) = (2, 1)$ or $2 \le n-1 \le A \le 2n-4$. Since the graphs G_B presented in the proof of Theorem 3.2 are traceable, now we have a parallel result for the order and upper traceable number of graphs, which we state as follows.

Corollary 3.3 *For a pair n, B of positive integers, there exists a graph G such that $(n(G), t^+(G)) = (n, B)$ if and only if either* (I) $1 \le n-1 \le B \le \lfloor (n-1)^2/2 \rfloor$ *or* (II) $\lfloor (n-1)^2/2 \rfloor + 1 \le B \le \lfloor n^2/2 \rfloor - 1$ *and B is odd.*

4 A related open question

We have seen that if G is a connected graph of order $n \ge 2$, then $n-1 \le t(G) \le t^+(G) \le \lfloor n^2/2 \rfloor - 1$. A triple (n, A, B) of positive integers is said to be *realizable* if there exists a graph G of order n with $t(G) = A$ and $t^+(G) = B$. Therefore, if (n, A, B) is a realizable triple, then $1 \le n-1 \le A \le B \le \lfloor n^2/2 \rfloor - 1$.

By the proof of Theorem 3.2, we see that a triple of the form $(n, n-1, B)$ is realizable if and only if $n \ge 2$ and either (i) $n-1 \le B \le \lfloor (n-1)^2/2 \rfloor$ or (ii) $\lfloor (n-1)^2/2 \rfloor + 1 \le B \le \lfloor n^2/2 \rfloor - 1$ and B is odd. Also, it is straightforward to determine the realizable triples for small values of n. If S_n denotes the set of realizable triples whose first entry equals $n \ge 2$, then

$$\begin{aligned} S_2 &= \{(2,1,1)\} \\ S_3 &= \{(3,2,2),(3,2,3)\} \\ S_4 &= \{(4,3,3),(4,3,4),(4,3,5),(4,3,7),(4,4,5)\} \\ S_5 &= \{(5,4,4),(5,4,5),(5,4,6),(5,4,7),(5,4,8),(5,4,9),(5,4,11), \\ &\qquad (5,5,7),(5,5,9),(5,6,7)\}. \end{aligned}$$

We conclude this paper with the following problem.

Problem 4.1 *For which triples (n, A, B) of positive integers with $5 \le n-1 \le A \le B \le \lfloor n^2/2 \rfloor - 1$ does there exist a graph of order n whose traceable number and upper traceable number are A and B, respectively?*

5 Acknowledgment

The author is grateful to the referee whose valuable suggestions resulted in an improved paper.

References

[1] T. Asano, T. Nishizeki, and T. Watanabe. An upper bound on the length of a hamiltonian walk of a maximal planar graph. *J. Graph Theory*, 4:315–336, 1980.

[2] T. Asano, T. Nishizeki, and T. Watanabe. An approximation algorithm for the hamiltonian walk problems on maximal planar graphs. *Discrete Appl. Math.*, 5:211–222, 1983.

[3] J. C. Bermond. On hamiltonian walks. *Congr. Numer.*, 15:41–51, 1976.

[4] G. Chartrand, L. Lesniak, and P. Zhang. *Graphs and Digraphs.* CRC Press, 4 edition, 2010.

[5] G. Chartrand, T. Thomas, V. Saenpholphat, and P. Zhang. A new look at hamiltonian walks. *Bull. Inst. Combin. Appl.*, 42:37–52, 2004.

[6] S. E. Goodman and S. T. Hedetniemi. On hamiltonian walks in graphs. *SIAM J. Comput.*, 3:214–221, 1974.

[7] L. Nebeský. A generalization of hamiltonian cycles for trees. *Czech. Math. J.*, 26:596–603, 1976.

[8] F. Okamoto, V. Saenpholphat, and P. Zhang. Measures of traceability in graphs. *Math. Bohem.*, 131:63–83, 2006.

[9] F. Okamoto, V. Saenpholphat, and P. Zhang. The upper traceable number of a graph. *Czech. Math. J.*, 58:271–287, 2008.

[10] F. Okamoto and P. Zhang. On upper traceable numbers of graphs. *Math. Bohem.*, 133:389–405, 2008.

[11] P. Vacek. On open hamiltonian walks in graphs. *Arch. Math.*, 27(1):105–111, 1991.

[12] P. Vacek. Bounds of lengths of open hamiltonain walks. *Arch. Math.*, 28(1):11–16, 1992.