Rainbow restrained domination numbers in graphs

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Abstract

A *2-rainbow dominating function* (2RDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u\in N(v)} f(u) = \{1,2\}$ is fulfilled, where $N(v)$ is the open neighborhood of v. A rainbow dominating function f is said to be a *rainbow restrained domination function* if the induced subgraph of G by the vertices with label $\emptyset$, has no isolated vertex. The weight of a rainbow restrained dominating function is the value $w(f) = \sum_{u\in V(G)} |f(u)|$. The minimum weight of a rainbow restrained dominating function of G is called the *rainbow restrained domination number* of G. In this paper we initiate the study of the rainbow restrained domination number and we present some bounds for this parameter.

Keywords: domination, rainbow dominating function, rainbow domination number, rainbow restrained dominating function, rainbow restrained domination number

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1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [12]. In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *k-regular* if $d(v) = k$ for each vertex v of G. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A *tree* is an acyclic connected graph. For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. If $A \subseteq V(G)$, then $G[A]$ is the subgraph induced by A. If $A, B \subseteq V(G)$, then $E(A, B)$ is the set of edges between A and B. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, P_n for a *path* of order n and C_n for a *cycle* of length n.

A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The *restrained domination number* $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of G. The restrained domination number was introduced by Domke et al. [7] and has been studied by several author (see for example [5, 6]).

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *k-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [1] and has been studied by several authors (see for example [2, 3, 4, 9, 10, 13, 14]).

A 2-rainbow dominating function $f : V \longrightarrow \mathcal{P}(\{1, 2\})$ can be represented by the ordered partition $(V_0, V_1, V_2, V_{1,2})$ (or $(V_0^f, V_1^f, V_2^f, V_{1,2}^f)$ to refer f) of V, where $V_0 = \{v \in V \mid f(v) = \emptyset\}$, $V_1 = \{v \in V \mid f(v) = \{1\}\}$, $V_2 = \{v \in V \mid f(v) = \{2\}\}$, $V_{1,2} = \{v \in V \mid f(v) = \{1, 2\}\}$. In this representation, its weight is $\omega(f) = |V_1| + |V_2| + 2|V_{1,2}|$.

A 2-rainbow dominating function $f = (V_0, V_1, V_2, V_{1,2})$ is called a *rainbow restrained dominating function* (RRDF) if the induced subgraph $G[V_0]$ has no isolated vertices. This definition is parallel to the definition of restrained dominating set of a graph defined in [7]. The *rainbow restrained* *domination number* of G, denoted by $\gamma_{rr}(G)$, is the minimum weight of an RRDF on G. A $\gamma_{rr}(G)$*-function* is an RRDF of G with $\omega(f) = \gamma_{rr}(G)$. If $f = (V_0, V_1, V_2, V_{1,2})$ is a $\gamma_{rr}(G)$-function, then since $V_1 \cup V_2 \cup V_{1,2}$ is a restrained dominating set, and since placing weight $\{1,2\}$ at the vertices of a restrained dominating set yields an RRDF, we observe that

$$\max\{\gamma_{r2}(G), \gamma_r(G)\} \leq \gamma_{rr}(G) \leq 2\gamma_r(G). \quad (1)$$

If $G_1, G_2, \cdots, G_s$ are the components of G, then $\gamma_{rr}(G) = \sum_{i=1}^{s} \gamma_{rr}(G_i)$. Hence, it is sufficient to study $\gamma_{rr}(G)$ for connected graphs.

Rainbow restrained domination number differs significantly from rainbow domination number. For example, for $n \geq 2$, $\gamma_{r2}(K_{1,n}) = 2$ and $\gamma_{rr}(K_{1,n}) = n$.

The purpose of this paper is to initiate the study of the rainbow restrained domination numbers in graphs. We first study basic properties of the rainbow restrained domination number and then we present some sharp bounds on the rainbow restrained domination number.

We make use of the following results in this paper.

Theorem A. ([7]) If G is a connected graph of order $n \geq 2$, then $\gamma_r(G) = n$ if and only if G is a star.

Observation 1. Let G be a graph of order n. Then
(a) $\gamma_{rr}(G) = 1$ if and only if $G = K_1$.

(b) $\gamma_{rr}(G) = 2$ if and only if $\Delta(G) = n-1$ and $\delta(G) \geq 2$ or $\Delta(G) = n-2$, $\delta(G) \geq 3$ and there exists two vertices u, v such that $N(u) \cap N(v) = V(G) - \{u, v\}$.

Observation 2. If H is a subgraph of G, then $\gamma_{rr}(G) \leq \gamma_{rr}(H) + |V(G)| - |V(H)|$.

Example 3. (a) $\gamma_{rr}(K_{1,n-1}) = n$ for $n \geq 2$.
(b) $\gamma_{rr}(P_n) = n$ for $1 \leq n \leq 6$ and $\gamma_{rr}(P_n) = \lceil \frac{2n+1}{3} \rceil + 1$ for $n \geq 7$.
(c) $\gamma_{rr}(C_n) = 2\lceil \frac{n}{3} \rceil$ when $n \not\equiv 2 \pmod 3$ and $\gamma_{rr}(C_n) = 2\lceil \frac{n}{3} \rceil + 1$ otherwise.
(d) $\gamma_{rr}(K_{p,q}) = 4$ for $2 \leq p \leq q$.

The next theorem shows that for every pair a, b of integers, with $1 \leq a \leq b \leq 2a$, there exists a simple connected graph G such that $\gamma_r(G) = a$ and $\gamma_{rr}(G) = b$.

Theorem 4. For every two positive integers a and b, with $1 \leq a \leq b$ and $b \leq 2a$, there exists a simple connected graph G such that $\gamma_r(G) = a$ and $\gamma_{rr}(G) = b$.

Proof. If $a = b = 1$, then let $G = K_1$ and if $a = 1, b = 2$, then let $G = K_n$ for $n \geq 3$.

If $a = b = 2$, let $G = K_2$, if $a = 2$ and $b = 4$ then let $G = P_4$, and if $a = 2$ and $b = 3$ then let G be the graph obtained from K_n ($n \geq 3$) by adding a new vertex and joining it to one of the vertices of K_n.

If $a = b = 3$ then let $G = P_3$, if $a = 3$ and $b = 5$ then let $G = P_5$, if $a = 3$ and $b = 6$ then let $G = P_7$, if $a = 3$ and $b = 4$ then let G be the graph obtained from K_n ($n \geq 3$) with $V(K_n) = \{v_1, v_2, \dots, v_n\}$ by adding two new vertices u_1, u_2 and joining u_i to v_i for $i = 1, 2$.

Finally, let $a \geq 4$. If $b = 2a$, then let $G = P_{3a-2}$, and if $b = 2a - 1$, then let $G = P_{3a-4}$. Thus we may assume that $b \leq 2a - 2$. Then let G be the graph obtained from the double star $S(p, q)$ with $p \geq q \geq 2$ and $p + q = a$, by subdividing $b - a$ pendant edges such that at least one pendant edge at each central vertex is not subdivided. □

2 Basic properties of the rainbow restrained domination number

Proposition 5. Let G be a connected graph of order n. Then $\gamma_r(G) = \gamma_{rr}(G)$ if and only if G is a star or G has a $\gamma_r(G)$-set S that partitions into two nonempty subsets S_1 and S_2 such that $N(S_1) = V(G) - (S_1 \cup S_2)$ and $N(S_2) = V(G) - (S_1 \cup S_2)$.

Proof. Assume that $\gamma_r(G) = \gamma_{rr}(G)$ and let $f = (V_0, V_1, V_2, V_{1,2})$ be a γ_{rr}-function of G. If $\gamma_r(G) = \gamma_{rr}(G) = n$, then G is a star by Theorem A. Let $\gamma_r(G) = \gamma_{rr}(G) < n$. Then we have $\gamma_r(G) \leq |V_1| + |V_2| + |V_{1,2}| \leq |V_1| + |V_2| + 2|V_{1,2}| = \gamma_{rr}(G)$. This implies that $|V_{1,2}| = 0$ and hence each vertex in V_0 has at least one neighbor in V_1 and one neighbor in V_2. Therefore, $V(G) - (V_1 \cup V_2) \subseteq N(V_1)$ and $V(G) - (V_1 \cup V_2) \subseteq N(V_2)$.

We claim that $V_1 \cup V_2$ is independent. Assume to the contrary that $uv \in G[V_1 \cup V_2]$ and let H be the component of $G[V_1 \cup V_2]$ containing uv. Since G is connected, H has a vertex x with a neighbor in V_0. Then $V_1 \cup V_2 - \{x\}$ is a restrained dominating set of G of weight less than $\gamma_r(G)$, a contradiction. Thus $V_1 \cup V_2$ is independent. This implies that $V(G) - (V_1 \cup V_2) = N(V_1)$ and $V(G) - (V_1 \cup V_2) = N(V_2)$.

Conversely, assume that G is a star or G has a minimum restrained dominating set S that partitions into two nonempty subsets S_1 and S_2 such that $N(S_1) = V(G) - (S_1 \cup S_2)$ and $N(S_2) = V(G) - (S_1 \cup S_2)$. If G is a star then clearly $\gamma_r(G) = \gamma_{rr}(G) = n$. Assume that G has a minimum restrained dominating set S that partitions into two nonempty subsets S_1 and S_2 such that $N(S_1) = V(G) - (S_1 \cup S_2)$ and $N(S_2) = V(G) - (S_1 \cup S_2)$. It is straightforward to verify that the function $(V(G) - (S_1 \cup S_2), S_1, S_2, \emptyset)$

is a rainbow restrained dominating function of G of weight $\gamma_r(G)$ and hence $\gamma_r(G) = \gamma_{rr}(G)$. This completes the proof. □

Proposition 6. Let G be a connected graph of order n and clique number $\omega(G) \geq 3$. Then $\gamma_{rr}(G) \leq n-\omega(G)+2$. This bound is sharp for the corona graph $K_m o K_1$ when $m \geq 3$.

Proof. Let S be a maximum clique of G and let $v \in S$. Then $f = (S - \{v\}, V(G)-S, \emptyset, \{v\})$ is an RRDF of G and hence $\gamma_{rr}(G) \leq n-\omega(G)+2$. □

Next we characterize the graphs G with $\gamma_{rr}(G) = n$. We start with the following lemma.

Lemma 7. Let T be a tree of order n with $\text{diam}(T) \geq 3$. If $T \neq P_n$, then $\gamma_{rr}(T) < n$.

Proof. Let v be a vertex of maximum degree of the tree T, and let $N(v) = \{v_1, v_2, \ldots, v_k\}$. Since $T \neq P_n$, we have $k \geq 3$. Root T in v. Since $\text{diam}(T) \geq 3$, we may assume, without loss of generality, that $\deg(v_k) \geq 2$. Let $u \in N(v_k) - \{v\}$. Then the function

$$f = (\{v_k, v\}, V(T) - \{u, v, v_k, v_2\}, \{v_2\}, \{u\})$$

is a rainbow restrained dominating function of T of weight $n-1$. Thus $\gamma_{rr}(T) \leq \omega(f) < n$ and the proof is complete. □

Theorem 8. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{rr}(G) = n$ if and only if $G \simeq K_{1,n-1}, C_4, C_5$ or $G = P_n$ for $n = 2, 3, 4, 5, 6$.

Proof. If $G \simeq K_{1,n-1}, C_4, C_5$ or $G = P_n$ for $n = 2, 3, 4, 5, 6$, then $\gamma_{rr}(G) = n$ by Example 3.

Conversely, let $\gamma_{rr}(G) = n$. If G is a tree, then it follows from Lemma 7 that $\text{diam}(G) \leq 2$ or $\Delta(T) \leq 2$. This implies that T is star or $T = P_n$. By Example 3, $T = K_{1,n-1}$ or $T = P_n$ for $n = 2, 3, 4, 5, 6$. So let G have a cycle. If G has a triangle $(v_1v_2v_3)$, then the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(v_1) = f(v_2) = \emptyset, f(v_3) = \{1, 2\}$ and $f(x) = \{1\}$ otherwise, is an RRDF of G with weight less than n which is a contradiction. Thus G is triangle-free. It follows from Example 3 and Observation 2 that the length of a longest cycle in G is at most 5. Let $C = (v_1v_2 \ldots v_r)$ be a longest cycle in G. First let $r = 4$. If $n \geq 5$, then we may assume, without loss of generality, that there is a vertex w such that $v_1w \in E(G)$. Then the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(v_1) = f(v_2) = f(v_4) = \emptyset, f(w) = f(v_3) = \{1, 2\}$ and $f(x) = \{1\}$ otherwise, is an RRDF of G with weight less than n which is a contradiction. Hence $n = 4$ and $G = C_4$. Now let $r = 5$. If $n \geq 6$, then we may assume, without loss of generality, that there is a vertex w such that $v_1w \in E(G)$. Then the function $f : V(G) \to$

$\mathcal{P}(\{1,2\})$ defined by $f(v_1) = f(v_2) = \emptyset, f(v_5) = \{2\}, f(v_3) = \{1,2\}$ and $f(x) = \{1\}$ otherwise, is an RRDF of G of weight less than n which is a contradiction. Hence $n = 5$ implying that $G = C_5$. This completes the proof. □

Next we establish a lower bound for the rainbow restrained domination number of a graph G in terms of order and size of G.

Observation 9. If for each $\gamma_{rr}(G)$-function $f = (V_0, V_1, V_2, V_{1,2})$ of a connected graph G, $V_0 = \emptyset$, then $G \simeq K_1, K_2$ or $K_{1,s}$, $s \geq 2$.

Theorem 10. Let G be a connected graph of order $n \geq 2$ and size m. Then

$$\gamma_{rr}(G) \geq \frac{3n}{2} - m$$

with equality if and only if $G = K_2$ or $G = S(2,2)$.

Proof. The result is immediate for $n = 2,3,4$ with equality if and only if $G = K_2$. Let now $n \geq 5$. If we have $V_0 = \emptyset$ for each $\gamma_{rr}(G)$-function $f = (V_0, V_1, V_2, V_{1,2})$ of G, then by Observation 9, $G \simeq K_{1,s}$, $s \geq 4$. Then $\gamma_{rr}(G) = n > \frac{3n}{2} - m$.

Now let $f = (V_0, V_1, V_2, V_{1,2})$ be a $\gamma_{rr}(G)$-function so that $V_0 \neq \emptyset$. Let $m_i = |E(G[V_i])|$ for $i = 0,1,2$, $m_3 = |E(V_0, V_1)|, m_4 = |E(V_0, V_2)|, m_5 = |E(V_0, V_{1,2})|, m_6 = |E(V_1, V_2)|, m_7 = |E(V_1, V_{1,2})|$, $m_8 = |E(V_2, V_{1,2})|$ and $m_9 = |E(G[V_{1,2}])|$. Since $G[V_0]$ has no isolated vertex, we have $2m_0 = \sum_{v\in V_0} \deg_{G[V_0]}(v) \geq |V_0| = n - |V_1| - |V_2| - |V_{1,2}|$. Since G is connected, we must have $m_1 + m_3 + m_6 + m_7 \geq |V_1|$ and $m_2 + m_4 + m_6 + m_8 \geq |V_2|$. On the other hand, since V_0 is dominated by $V_1 \cup V_{1,2}$ and $V_2 \cup V_{1,2}$ we have $m_3 + m_5 \geq |V_0|$ and $m_4 + m_5 \geq |V_0|$. Thus

$$\begin{aligned}
2m &= 2m_0 + 2m_1 + 2m_2 + 2m_3 + 2m_4 + 2m_5 + 2m_6 \\
&+ 2m_7 + 2m_8 + 2m_9 \\
&\geq 3(n - |V_1| - |V_2| - |V_{1,2}|) + |V_1| + |V_2| + m_1 + m_2 \\
&+ m_7 + m_8 + 2m_9 \\
&= 3n - 2|V_1| - 2|V_2| - 3|V_{1,2}| + m_1 + m_2 + m_7 + m_8 + 2m_9 \\
&= 3n - (2|V_1| + 2|V_2| + 4|V_{1,2}|) + |V_{1,2}| + m_1 + m_2 \\
&+ m_7 + m_8 + 2m_9 \\
&= 3n - 2\gamma_{rr}(G) + |V_{1,2}| + m_1 + m_2 + m_7 + m_8 + 2m_9 \\
&\geq 3n - 2\gamma_{rr}(G).
\end{aligned}$$

This implies that $\gamma_{rr}(G) \geq \frac{3n}{2} - m$.

If $G = K_2$ or $G = S(2,2)$, then obviously $\gamma_{rr}(G) = 2 = \frac{3n}{2} - m$ or $\gamma_{rr}(G) = 4 = \frac{3n}{2} - m$.

Conversely, let $\gamma_{rr}(G) = \frac{3n}{2} - m$ and let f be a $\gamma_{rr}(G)$-function. If $2 \leq n \leq 4$, then it is easy to see that $G = K_2$. Suppose next that $n \geq 5$. Since G is connected, we have $\gamma_{rr}(G) < n$ and hence $V_0 \neq \emptyset$. Then all inequalities occurring in the proof become equalities. In particular, $|V_{1,2}| = m_1 = m_2 = m_7 = m_8 = m_9 = 0$, $2m_0 = |V_0|$, $m_3 + m_6 = |V_1|$, $m_4 + m_6 = |V_2|$, $m_3 + m_5 = |V_0|$ and $m_4 + m_5 = |V_0|$. It follows from $|V_{1,2}| = 0$ that $m_5 = 0$ and hence $m_3 = |V_0|$ and $m_4 = |V_0|$. This implies that each vertex in V_0 is adjacent to exactly one vertex in V_1 and one vertex in V_2. Hence

(a) $V_{1,2} = \emptyset$,

(b) for $i = 1, 2$, V_i is an independent dominating set of $G[V_0 \cup V_i]$;

(c) $G[V_0]$ is a 1-regular graph;

(d) every vertex in V_0 is adjacent to exactly one vertex in V_1 and one vertex in V_2.

We claim that $m_6 = 0$. Assume to the contrary that $m_6 \geq 1$ and $uv \in E(G)$, where $u \in V_1$ and $v \in V_2$.

If $G - uv$ is connected, then f is also a rainbow restrained dominating function on $G - uv$, and we obtain the contradiction

$$\frac{3n}{2} - m = \gamma_{rr}(G) = \omega(f) \geq \gamma_{rr}(G - uv) \geq \frac{3n}{2} - (m - 1).$$

If $G - uv$ is disconnected, then let G_1 and G_2 be the components of $G - uv$. Obviously, the function f restricted to G_i is a rainbow restrained dominating function on G_i for $i = 1, 2$. If $|V(G_i)| \geq 2$ for $i = 1, 2$, then we have

$$\begin{aligned}
\gamma_{rr}(G) &= \omega(f) \\
&= \omega(f|_{G_1}) + \omega(f|_{G_2}) \\
&\geq \tfrac{3|V(G_1)|}{2} - |E(G_1)| + \tfrac{3|V(G_2)|}{2} - |E(G_2)| \\
&= \tfrac{3n}{2} - (m - 1) > \tfrac{3n}{2} - m
\end{aligned}$$

which is a contradiction. If $|V(G_i)| = 1$ for some i, say $i = 1$, then

$$\begin{aligned}
\gamma_{rr}(G) &= \omega(f|_{G_1}) + \omega(f|_{G_2}) \\
&\geq 1 + \tfrac{3(n-1)}{2} - (m - 1) \\
&= \tfrac{3n}{2} - m + \tfrac{1}{2} > \tfrac{3n}{2} - m
\end{aligned}$$

a contradiction again. Thus $m_6 = 0$ implying that $|V_0| = |V_1| = |V_2|$ and $V_1 \cup V_2$ is independent. Furthermore, by connectedness of G we deduce that each vertex of V_1 and V_2 has a neighbor in V_0. It follows from (c) and connectedness of G that $G = S(2, 2)$. □

Corollary 11. Let T be a tree of order $n \geq 2$. Then

$$\gamma_{rr}(T) \geq \frac{n}{2} + 1$$

with equality if and only if $T = P_2$ or $T = S(2, 2)$.

3 Nordhaus-Gaddum type results

Many problems in extremal graph theory seek the extreme values of graph parameters on families of graphs. Results of *Nordhaus–Gaddum type* study the extreme values of the sum (or product) of a parameter on a graph and its complement, following the classic paper of Nordhaus and Gaddum [11] solving these problems for the chromatic number on n-vertex graphs. In this section, we study such problems for the rainbow restrained domination number.

The next result is an immediate consequence of Theorem 8.

Corollary 12. If G is a connected graph of order $n \geq 4$, different from C_5, with $\gamma_{rr}(G) = n$, then $\gamma_{rr}(\overline{G}) \leq 4$.

The Dutch-windmill graph, $K_3^{(m)}$, is a graph which consists of m copies of K_3 with a vertex in common. Obviously, $\overline{K_3^{(m)}} = K_1 \cup K_{\underbrace{2,\ldots,2}_{m \text{ times}}}$. It is easy to see that $\gamma_{rr}(K_3^{(m)}) = 2$ and $\gamma_{rr}(\overline{K_3^{(m)}}) = 3$ for $m \geq 3$.

Theorem 13. For any connected graph G of order $n \geq 4$,

$$\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \geq 5.$$

The bound is sharp for $K_3^{(m)}$, where $m \geq 3$.

Proof. Without loss of generality, we may assume that $\gamma_{rr}(G) \leq \gamma_{rr}(\overline{G})$. If $\gamma_{rr}(G) \geq 3$, then $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \geq 6$. Let $\gamma_{rr}(G) = 2$. It follows from Observation 1 that $\Delta(G) = n-1$ and $\delta(G) \geq 2$ or $\Delta(G) = n-2$, $\delta(G) \geq 3$ and G has two vertices u, v such that $N(u) \cap N(v) = V(G) - \{u, v\}$.

First let $\Delta(G) = n-1$ and $\delta(G) \geq 2$. Then $G = K_1 + H$, where H is a graph without isolated vertices. Let $v \in V(G)$ be a vertex of maximum degree $\Delta(G) = n-1$. Then $\overline{G} = \{v\} \cup \overline{H}$ and so $\gamma_{rr}(\overline{G}) = 1 + \gamma_{rr}(\overline{H}) \geq 3$. Hence $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \geq 5$.

Now let $\Delta(G) = n-2$, $\delta(G) \geq 3$ and G has two vertices u, v such that $N(u) \cap N(v) = V(G) - \{u, v\}$. Then $G = \overline{K_2} + H$ where H is a graph without isolated vertices. Then $\overline{G} = K_2 \cup \overline{H}$ and so $\gamma_{rr}(\overline{G}) = 2 + \gamma_{rr}(\overline{H}) \geq 4$ and thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \geq 6$. Since $\gamma_{rr}(G) = 1$ is not possible, the proof is complete. □

Lemma 14. If G is a connected graph with $\text{diam}(G) \geq 5$, then $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$.

Proof. Let $P = v_1 v_2 \dots v_k$ be a diametral path in G. Define $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ by $f(v_1) = f(v_k) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_k\}$. It is easy to see that f is an RRDF of $\overline{G}$ of weight 4 and so $\gamma_{rr}(\overline{G}) \leq 4$. Thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$. □

Lemma 15. If G is a connected graph of order n with $\mathrm{diam}(G) = 4$, then $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$.

Proof. Let $P = v_1 v_2 v_3 v_4 v_5$ be a diametral path in G. If $n = 5$, then $G = P_5$ and clearly $\gamma_{rr}(\overline{P_5}) = 4$ and hence $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$. Let $n \geq 6$. It follows from Theorem 8 that $\gamma_{rr}(G) \leq n - 1$. Define $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ by $f(v_1) = f(v_4) = \{1,2\}, f(v_3) = \{1\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_3, v_4\}$. It is easy to see that f is an RRDF of $\overline{G}$ of weight 5. Thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$. □

Lemma 16. If G is a connected graph of order n with $\mathrm{diam}(G) = 3$, then $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$.

Proof. Let $P = v_1 v_2 v_3 v_4$ be a diametral path in G. If $n = 4$, then clearly $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) = n + 4$. Let $n \geq 5$. If $\overline{G}[V - \{v_1, v_4\}]$ has no isolated vertex, then the function $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ defined by $f(v_1) = f(v_4) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(\overline{G}) - \{v_1, v_4\}$, is an RRDF of $\overline{G}$ of weight 4 and so $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$.

Let $\overline{G}[V - \{v_1, v_4\}]$ have an isolated vertex, say w. If $G[V - \{v_1, v_4, w\}]$ has no isolated vertex, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_1) = f(v_4) = \{1\}, f(w) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_4, w\}$, is an RRDF of G of weight 4 and again $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$. Suppose that $G[V - \{v_1, v_4, w\}]$ has an isolated vertex. Then the function $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ defined by $f(v_1) = f(v_4) = \{1,2\}, f(w) = \{1\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_4, w\}$, is an RRDF of $\overline{G}$ of weight 5. It follows from Theorem 8 that $\gamma_{rr}(G) \leq n - 1$ and hence $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$. This completes the proof. □

Theorem 17. For any connected graph G of order $n \geq 4$ and different from C_5,

$$\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4.$$

The bound is sharp for C_4, P_4 and P_5.

Proof. We may assume that G is connected because the complement of a disconnected graph is connected. Suppose that $u, v \in V(G)$ are adjacent vertices in G such that $|N(u) \cap N(v)|$ is maximum. For convenience, let $N = N(u) \cap N(v)$. Let $X = V(G) - (N \cup \{u, v\})$ and let $I = \{w \mid w$ is an isolated vertex in $\overline{G}[X]\}$. We consider three cases.

Case 1. $|I| \geq 2$.
Assume $w_1, w_2 \in I$. Then w_i is adjacent to all vertices of $X - \{w_i\}$ in G

for $i = 1, 2$. If $N = \emptyset$, then by the choice of u and v we must have $|I| = 2$ and so $n = 4$. Then clearly $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \le n + 4$. Assume $N \ne \emptyset$. Since G is connected, we may choose an edge xy such that $x \in N \cup \{u, v\}$ and $y \in X$. Assume that $v_1 \in \{u, v\} - \{x\}$ and $v_2 \in \{w_1, w_2\} - \{y\}$. Then the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(v_1) = f(v_2) = \{1, 2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_2\}$, is an RRDF of G of weight 4. It follows that $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \le n + 4$.

Case 2. $|I| \le 1$ and $N = \emptyset$.
If $I = \emptyset$, then the function $f : V(\overline{G}) \to \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1, 2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{u, v\}$, is an RRDF of $\overline{G}$ of weight 4 and the result follows. Now let $|I| = 1$ and suppose that $I = \{w\}$. Then the function $f : V(\overline{G}) \to \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1, 2\}, f(w) = \{1\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{u, v, w\}$, is an RRDF of $\overline{G}$ of weight 5. It follows from Corollary 12 and the fact that $G \ne C_5$ that $\gamma_{rr}(\overline{G}) \le 4$ or $\gamma_{rr}(G) \le n - 1$. Hence $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \le n + 4$.

Case 3. $|I| \le 1$ and $N \ne \emptyset$.
First let $I = \emptyset$. It is easy to see that the function $f : V(G) \to \mathcal{P}(\{1, 2\})$ defined by $f(u) = \{1, 2\}$ and $f(x) = \{1\}$ for $x \in X$ and $f(x) = \emptyset$ otherwise is an RRDF of G of weight $n - |N|$ and the function $g : V(\overline{G}) \to \mathcal{P}(\{1, 2\})$ defined by $f(u) = f(v) = \{1, 2\}$ and $f(x) = \{1\}$ for $x \in N$ and $f(x) = \emptyset$ otherwise is an RRDF of $\overline{G}$ of weight $|N| + 4$. Thus

$$\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \le (n - |N|) + (4 + |N|) = n + 4.$$

Now let $|I| = 1$ and suppose $I = \{w\}$. By Lemmas 14, 15 and 16, we may assume that $\operatorname{diam}(G) \le 2$. Since $X \ne \emptyset$, we deduce that $\operatorname{diam}(G) = 2$. If $\gamma_{rr}(G[X]) < |X|$, then let g be a $\gamma_{rr}(G[X])$-function and define $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(u) = \{1, 2\}$ and $f(x) = g(x)$ for $x \in X$ and $f(x) = \emptyset$ otherwise. It is clear that f is an RRDF of G of weight at most $n - |N| - 1$. On the other hand, $(X - \{w\}, N \cup \{w\}, \emptyset, \{u, v\})$ is an RRDF of $\overline{G}$ and hence

$$\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \le (n - |N| - 1) + (5 + |N|) = n + 4.$$

Now let $\gamma_{rr}(G[X]) = |X|$. It follows from Theorem 8 and the fact $|I| = 1$ that $G[X] = K_{1,|X|-1}$ with the central vertex w. Since $w \notin N$, we may assume that v and w are not adjacent in G. Consider two subcases.

Subcase 3.1. $|X| \ge 3$.
If $N(w) \cap (N \cup \{v\}) \ne \emptyset$ (the case $N(w) \cap (N \cup \{u\}) \ne \emptyset$ is similar), then let $w_1 \in X - \{w\}$ and define $f : V(G) \to \mathcal{P}(\{1, 2\})$ by $f(u) = \{1, 2\}, f(w_1) = \{1\}$, and $f(x) = \{2\}$ for $x \in X - \{w, w_1\}$ and $f(x) = \emptyset$ otherwise. It is clear that f is an RRDF of G of weight $n - |N| - 1$ and the result follows as above. Thus we may assume $N(w) \cap (N \cup \{u, v\}) = \emptyset$. Since $\operatorname{diam}(G) = 2$,

u must have a neighbor in $X - \{w\}$ such as w'. Let $w'' \in X - \{w, w'\}$ and define $f : V(G) \to \mathcal{P}(\{1,2\})$ by $f(u) = \{1,2\}, f(w'') = \{1,2\}$, and $f(x) = \{1\}$ for $x \in X - \{w, w', w''\}$ and $f(x) = \emptyset$ otherwise. Obviously, f is an RRDF of G of weight $n - |N| - 1$ and the result follows as above again.

Subcase 3.2. $|X| \leq 2$.
Then the function $(N \cup \{v\}, X, \emptyset, \{u\})$ is an RRDF of G and hence $\gamma_{rr}(G) \leq 4$. Thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq 4 + n$.

Hence we have $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 4$ in all cases, and the proof is complete. □

If the graph G has order $n \geq 6$, then we can improve Lemmas 14, 15 and 16 and thus Theorem 17 when $\mathrm{diam}(G) \geq 3$.

Theorem 18. If G is a connected graph of order $n \geq 6$ with $\mathrm{diam}(G) \geq 3$, then

$$\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 3.$$

The bound is sharp for P_6.

Proof. Let $P = v_1 v_2 \ldots v_k$ be a diametral path in G.

Assume first that $k \geq 6$. If $n = 6$, then $G = P_6$. Define $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ by $f(v_1) = \{1,2\}, f(v_2) = \{1\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_2\}$. It is easy to see that f is an RRDF of $\overline{G}$ of weight 3. Thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 3$.

If $n \geq 7$, then it follows from Theorem 8 that $\gamma_{rr}(G) \leq n - 1$. Define $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ by $f(v_1) = f(v_k) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_k\}$. It is easy to see that f is an RRDF of $\overline{G}$ of weight 4. Thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 3$.

Now assume that $k = 5$. Since $n \geq 6$, it follows from Theorem 8 that $\gamma_{rr}(G) \leq n - 1$. Define $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ by $f(v_1) = f(v_4) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_4\}$. It is easy to see that f is an RRDF of $\overline{G}$ of weight 4. Thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 3$.

Finally, assume that $k = 4$.

Case 1. Assume that $\overline{G}[V - \{v_1, v_4\}]$ has no isolated vertex. Then the function $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ defined by $f(v_1) = f(v_4) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(\overline{G}) - \{v_1, v_4\}$, is an RRDF of $\overline{G}$ of weight 4. It follows from Theorem 8 that $\gamma_{rr}(G) \leq n - 1$ and so $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n + 3$.

Case 2. Assume that $\overline{G}[V - \{v_1, v_4\}]$ has an isolated vertex, say w. Let, without loss of generality, $wv_1 \in E(\overline{G})$.

Subcase 2.1. Assume that $G[V - \{v_1, v_4, w\}]$ has no isolated vertex. If $wv_4 \in E(G)$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_1) = \{1\}, f(w) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, w\}$, is an RRDF of G of weight 3, and this leads to the desired bound. If $wv_4 \in E(\overline{G})$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_1) = f(v_4) = \{1\}, f(w) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(G) - \{v_1, v_4, w\}$, is an RRDF of G of weight 4. Since $\overline{G}$ is connected and different from the star $K_{1,n-1}$, it follows from Theorem 8 that $\gamma_{rr}(\overline{G}) \leq n-1$ and thus $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n+3$.

Subcase 2.2. Assume that $G[V - \{v_1, v_4, w\}]$ has an isolated vertex, say z.

If $wv_4 \in E(G)$, then the function $f : V(G) \to \mathcal{P}(\{1,2\})$ defined by $f(v_2) = f(v_3) = f(v_4) = \emptyset$, $f(w) = \{1,2\}$ and $f(x) = 1$ for $x \in V(G) - \{v_2, v_3, v_4, w\}$, is an RRDF of G of weight $n-2$, and therefore $\gamma_{rr}(G) \leq n-2$. In addition, the function $g : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ defined by $g(v_1) = g(v_4) = \{1,2\}, g(w) = \{1\}$ and $g(x) = \emptyset$ for $x \in V(G) - \{v_1, v_4, w\}$, is an RRDF of $\overline{G}$ of weight 5. Consequently, $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n+3$.

Finally, assume that $wv_4 \in E(\overline{G})$. If $G[V - \{v_1, v_4, w\}]$ has an edge different from v_2v_3, then we obtain $\gamma_{rr}(G) \leq n-2$ and hence $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n+3$ as above. If $G[V - \{v_1, v_4, w\}]$ only contains the edge v_2v_3, then the function $f : V(\overline{G}) \to \mathcal{P}(\{1,2\})$ defined by $f(w) = f(z) = \{1,2\}$ and $f(x) = \emptyset$ for $x \in V(\overline{G}) - \{w, z\}$, is an RRDF of $\overline{G}$ of weight 4. It follows from Theorem 8 that $\gamma_{rr}(G) + \gamma_{rr}(\overline{G}) \leq n+3$. □

4 Ladders

In this section we find the rainbow restrained domination number of ladders. Throughout this section we assume the vertices of the i-th copy of P_2 in ladders $P_2 \times P_n$ are u_i, v_i for $i = 1, 2, \ldots, n$.

Theorem 19. For $n \geq 1$,

$$\gamma_{rr}(P_2 \times P_n) = \begin{cases} n+2 & \text{if} \quad n \text{ is even} \\ n+1 & \text{if} \quad n \text{ is odd.} \end{cases}$$

Proof. First let n be even. Define $f : V(P_2 \times P_n) \to \mathcal{P}(\{1,2\})$ by $f(u_n) = f(v_n) = \{1\}$, $f(u_{4i+1}) = f(v_{4i+1}) = \{1\}$ for $0 \leq i \leq \lceil \frac{n}{4} \rceil - 1$, $f(u_{4i+3}) = f(v_{4i+3}) = \{2\}$ for $0 \leq i \leq \lfloor \frac{n}{4} \rfloor - 1$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is an RRDF of $P_2 \times P_n$ of weight $n+2$ and hence $\gamma_{rr}(P_2 \times P_n) \leq n+2$.

To prove $\gamma_{rr}(P_2 \times P_n) \geq n+2$, we proceed by induction on n. If $n = 2$, then $P_2 \times P_2 = C_4$ and we have $\gamma_{rr}(P_2 \times P_2) = 4$. Let $n \geq 4$ and $P_2 \times P_{n-2} = P_2 \times P_n - \{u_n, u_{n-1}, v_n, v_{n-1}\}$. Suppose f is a $\gamma_{rr}(P_2 \times P_n)$-function. We consider the following cases.

Case 1. $f(u_n) = \emptyset$ (the case $f(v_n) = \emptyset$ is similar).
Since f is an RRDF of $P_2 \times P_n$, we must have $f(u_{n-1}) = \emptyset$ or $f(v_n) = \emptyset$. Consider two subcases.

Subcase 1.1 $f(v_n) = \emptyset$.
To rainbowly dominate u_n and v_n, we must have $f(u_{n-1}) = f(v_{n-1}) = \{1,2\}$. If $f(u_{n-2}) \neq \emptyset$ and $f(v_{n-2}) \neq \emptyset$, then the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF and by the induction hypothesis we have $\gamma_{rr}(P_2 \times P_n) = \omega(f) \geq 4 + \gamma_{rr}(P_2 \times P_{n-2}) \geq 4 + n$. We assume, without loss of generality, that $f(u_{n-2}) = \emptyset$. Since f is an RRDF of $P_2 \times P_n$, we must have $f(u_{n-3}) = \emptyset$ or $f(v_{n-2}) = \emptyset$.

If either $f(u_{n-3}) = f(v_{n-2}) = f(v_{n-3}) = \emptyset$ or $f(u_{n-3}) = f(v_{n-2}) = f(u_{n-4}) = \emptyset$ or $f(v_{n-3}) = f(v_{n-2}) = f(v_{n-4}) = \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = g(v_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and by the induction hypothesis we have $\gamma_{rr}(P_2 \times P_n) = \omega(f) = 2 + \omega(g) \geq 2 + \gamma_{rr}(P_2 \times P_{n-2}) \geq n + 2$.

If $f(u_{n-3}) = \emptyset$ and $f(v_{n-2}) \neq \emptyset$ or $f(u_{n-3}) = f(v_{n-2}) = \emptyset$ and $f(v_{n-3}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(v_{n-2}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{v_{n-2}\}$, is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the results follows by the induction hypothesis.

If $f(v_{n-3}) = f(v_{n-2}) = \emptyset$ and $f(u_{n-3}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$, is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the results follows by the induction hypothesis.

If $f(u_{n-3}) \neq \emptyset, f(v_{n-3}) \neq \emptyset$ and $f(v_{n-2}) = \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = g(v_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}\}$, is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the results follows as above.

Subcase 1.2 $f(u_{n-1}) = \emptyset$.
To rainbowly dominate u_n, this condition leads to $f(v_n) = \{1,2\}$. If $f(u_{n-2}) = f(u_{n-3}) = \emptyset$, then to rainbowly dominate u_{n-1}, u_{n-2}, we must have $f(v_{n-1}) = f(v_{n-2}) = \{1,2\}$ and clearly the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF of $P_2 \times P_{n-2}$. And by the induction hypothesis we obtain $\gamma_{rr}(P_2 \times P_n) = \omega(f) = 4 + \omega(f|_{P_2 \times P_{n-2}}) \geq 4 + \gamma_{rr}(P_2 \times P_{n-2}) \geq n+4$.

If $f(u_{n-2}) = f(v_{n-2}) = f(v_{n-3}) = \emptyset$, then to rainbowly dominate u_{n-1}, u_{n-2}, we must have $f(v_{n-1}) = f(u_{n-3}) = \{1,2\}$. Then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and by the induction hypothesis we have $\gamma_{rr}(P_2 \times P_n) = \omega(f) = 2 + \omega(g) \geq 2 + \gamma_{rr}(P_2 \times P_{n-2}) \geq n + 2$.

If $f(u_{n-2}) = f(v_{n-2}) = \emptyset$ and $f(v_{n-3}) \neq \emptyset$, then to rainbowly dominate u_{n-1}, u_{n-2}, we must have $f(v_{n-1}) = f(u_{n-3}) = \{1,2\}$. Then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(v_{n-3}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{v_{n-3}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most

$\omega(f) - 3$ and the result follows as above.

If $f(u_{n-2}) = f(v_{n-3}) = \emptyset$ and $f(v_{n-2}) \neq \emptyset$, then to rainbowly dominate u_{n-1}, we must have $f(v_{n-1}) = \{1,2\}$. Then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 3$ and the result follows as above.

Thus we may assume $f(u_{n-2}) \neq \emptyset$.

If $f(u_{n-2}) \neq \emptyset$ and $f(v_{n-1}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = f(u_{n-2}) \cup f(v_{n-1})$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the result follows by the induction hypothesis.

If $f(u_{n-2}) \neq \emptyset$ and $f(v_{n-1}) = f(v_{n-2}) = f(v_{n-3}) = \emptyset$, then $f(u_{n-2}) = \{1,2\}$ and the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF and by the induction hypothesis the result follows.

If $f(u_{n-2}) \neq \emptyset$ and $f(v_{n-2}) \neq \emptyset$, then the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF and by the induction hypothesis the result follows.

Hence we may assume $f(v_{n-1}) = f(v_{n-2}) = \emptyset$ and $f(v_{n-3}) \neq \emptyset$. Then to rainbowly dominate u_{n-1}, we must have $f(u_{n-2}) = \{1,2\}$.

If $f(u_{n-3}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1\}, g(v_{n-2}) = \{2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the result follows by the induction hypothesis.

Let $f(u_{n-3}) = \emptyset$. Since f is an RRDF of $P_2 \times P_n$, we must have $f(u_{n-4}) = \emptyset$.

If $f(v_{n-4}) \neq \emptyset$ then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1\}, g(v_{n-2}) = g(v_{n-3}) = \{2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}, v_{n-3}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the result follows by the induction hypothesis.

If $f(v_{n-4}) = \emptyset$ and $f(v_{n-3}) = \{1,2\}$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = g(v_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the result follows by the induction hypothesis.

Finally, if $f(v_{n-4}) = \emptyset$ and $|f(v_{n-3})| = 1$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1,2\} - f(v_{n-3}), g(v_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the result follows by the induction hypothesis.

Case 2. $f(u_n) \neq \emptyset$ and $f(v_n) \neq \emptyset$.
Consider the following subcases.

Subcase 2.1. $f(u_{n-1}) = f(v_{n-1}) = \emptyset$.
If $f(u_{n-2}) = f(v_{n-2}) = \emptyset$ or $f(u_{n-2}) = f(u_{n-3}) = \emptyset$ or $f(v_{n-2}) = f(v_{n-3}) = \emptyset$ or $f(u_{n-2}) \neq \emptyset$ and $f(v_{n-2}) \neq \emptyset$, then the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF and it follows from the induction hy-

pothesis that $\gamma_{rr}(P_2 \times P_n) = \omega(f) \geq \gamma_{rr}(P_2 \times P_{n-2}) + 2 \geq n + 2$.

If $f(u_{n-2}) = \emptyset, f(u_{n-3}) \neq \emptyset$ and $f(v_{n-2}) \neq \emptyset$ (the case $f(v_{n-2}) = \emptyset, f(v_{n-3}) \neq \emptyset$ and $f(u_{n-2}) \neq \emptyset$ is similar), then $f(u_n) = \{1,2\}$ and the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight $\omega(f) - 2$ and the result follows by the induction hypothesis.

Subcase 2.2. $f(u_{n-1}) = \emptyset$ and $f(v_{n-1}) \neq \emptyset$ (the case $f(u_{n-1}) \neq \emptyset$ and $f(v_{n-1}) = \emptyset$ is similar).
Since f is an RRDF, we must have $f(u_{n-2}) = \emptyset$. If $f(u_{n-3}) = \emptyset$, then the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF and the result follows by the induction hypothesis. Let $f(u_{n-3}) \neq \emptyset$. If $f(v_{n-2}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the result follows as above.

If $f(v_{n-2}) = \emptyset$ and $f(v_{n-3}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(v_{n-3}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{v_{n-3}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the result follows by the induction hypothesis.

If $f(v_{n-2}) = f(v_{n-3}) = \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the result follows by the induction hypothesis.

Subcase 2.3. $f(u_{n-1}) \neq \emptyset$ and $f(v_{n-1}) \neq \emptyset$.
If $f(u_{n-2}) \neq \emptyset$ and $f(v_{n-2}) \neq \emptyset$, then the function f, restricted to $P_2 \times P_{n-2}$ is an RRDF of weight at most $\omega(f) - 4$ and the result follows by the induction hypothesis. Thus we may assume, without loss of generality, that $f(u_{n-2}) = \emptyset$. If $f(u_{n-3}) = f(v_{n-2}) = f(v_{n-3}) = \emptyset$ or $f(v_{n-2}) = \emptyset, f(u_{n-3}) \neq \emptyset, f(v_{n-3}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = f(u_{n-1}), g(v_{n-2}) = f(v_{n-1})$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}, v_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the result follows by the induction hypothesis.

If $f(u_{n-3}) = \emptyset$ and $f(v_{n-2}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(v_{n-2}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{v_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 3$ and the result follows as above.

If $f(v_{n-3}) = f(v_{n-2}) = \emptyset$ and $f(u_{n-3}) \neq \emptyset$, then the function $g : V(P_2 \times P_{n-2}) \to \mathcal{P}(\{1,2\})$ defined by $g(u_{n-2}) = \{1,2\}$ and $g(x) = f(x)$ for $x \in V(P_2 \times P_{n-2}) - \{u_{n-2}\}$ is an RRDF of $P_2 \times P_{n-2}$ of weight at most $\omega(f) - 2$ and the result follows as above.

Since we discuss all possible cases, we have $\gamma_{rr}(P_2 \times P_n) \geq n + 2$ when n is even. Thus $\gamma_{rr}(P_2 \times P_n) = n + 2$ when n is even.

Now let n be odd. Define $f : V(P_2 \times P_n) \to \mathcal{P}(\{1,2\})$ by $f(u_{4i+1}) =$

$f(v_{4i+1}) = \{1\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$ and $f(u_{4i+3}) = f(v_{4i+3}) = \{2\}$ for $0 \le i \le \lceil \frac{n}{4} \rceil - 1$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is an RRDF of $P_2 \times P_n$ of weight $n+1$ and hence $\gamma_{rr}(P_2 \times P_n) \le n+1$. Using an argument similar to that described above we can see that $\gamma_{rr}(P_2 \times P_n) \ge n+1$. Thus $\gamma_{rr}(P_2 \times P_n) = n+1$ when n is odd and the proof is complete. □

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