The smallest Merrifield-Simmons index of trees with exactly six leaves *

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Abstract

The Merrifield-Simmons index $\sigma(G)$ of a graph G is defined as the number of subsets of the vertex set, in which any two vertices are non-adjacent, *i.e.*, the number of independent vertex sets of G. A tree is called r-leave tree if it contains r vertices with degree one. In this paper, we obtain the smallest Merrifield-Simmons index among all trees with n vertices and exactly six leaves, and characterize the corresponding extremal graph.

Key Words: Merrifield-Simmons index; Trees with exactly six leaves; Fibonacci number.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the number of vertices of G by $n(G)$. For any vertex $u \in V(G)$, we denote the neighborhood and the degree of u of G by $N_G(u)$ and $d_G(u)$, respectively. When no confusion occurs, we will denote $N_G(u)$ and $d_G(u)$ by $N(u)$ and $d(u)$, respectively. Denote the path and the star with n vertices by P_n and S_n, respectively. Denote the maximum degree of G by $\Delta(G)$. All graphs considered here are finite and simple. Undefined notations and terminology will conform to those in [1].

The Merrifield-Simmons index or σ-index $\sigma(G)$ of a graph G, is defined as the number of subsets of $V(G)$, in which no two vertices are adjacent, *i.e.*,

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the total number of the independent vertex sets of the graph G, including the empty set. For example, for the cycle $C_4 = v_1v_2v_3v_4$, all this kind of subsets of $V(C_4)$ are as follow: $\emptyset$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_1, v_3\}$, $\{v_2, v_4\}$, and then $\sigma(C_4) = 7$. As for the path P_n, $\sigma(P_n)$ is exactly equal to the Fibonacci number f_{n+2}. This is perhaps why some researchers call the σ-index the "Fibonacci number" of the graph. The concept of the Merrifield-Simmons index was introduced by Prodinger and Tichy in 1982 [8], this index is one of the most important topological index in chemistry, which was extensively studied in a monograph [6]. Now there are many results about the Merrifield-Simmons indices of graphs. In [8] the authors proved that the path P_n has the smallest σ-index and the star S_n has the largest σ-index among all trees with n vertices. In [7] Pedersen and Vestergaard obtained the smallest σ-index among all unicyclic graphs with n vertices. In [10] B. Wang et al. obtained the first, second and third smallest σ-index among all unicyclic graphs with n vertices and girth k. In [14] Yu and Lv characterized the largest σ-index among all trees with n vertices and k pendent vertices. In [5] Zhao and Li characterized the second and third smallest σ-index among all trees with n vertices. In [9] Wanger showed that a tree T with $\sigma(T) < 18f_{n-5} + 21f_{n-6}$ has at most three leaves. In [11] M.L.Wang et al. obtained the first and second largest σ-index among all trees with n vertices and k pendent vertices. In [12] Yan investigated the σ-index of a special class of tree with four leaves, and obtained the σ-index orderings of this class of trees. In [2] Gao and Wei obtained the smallest σ-index among all trees with n vertices and five leaves, and characterize the extremal graph. In [13] Ye characterized trees with the second and third minimal Merrifield-Simmons index in the set of 5-leaf-trees of order n. In this paper, we obtained the smallest σ-index among all trees with n vertices and six leaves, and characterize the corresponding extremal graph.

For a graph G, a leaf is a vertex of degree one of G, it is also called pendent vertex. The distance between u and v denote by $d(u, v)$. We denote the simple path with two end-vertices u and v by P_{uv}. If $W \subseteq V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, if $E' \subseteq E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E'. If $W = \{v\}$ and $E' = \{xy\}$, we write $G - \{v\}$ and $G - \{xy\}$, respectively. Let (G_1, v_1) and (G_2, v_2) be two graphs rooted at v_1 and v_2, respectively, then $G = (G_1, v_1) \bullet (G_2, v_2)$ denote the graph obtained by identifying v_1 with v_2 as one common vertex.

Let f_n and l_n denote the n-th Fibonacci number and n-th Lucas number, respectively. It is well known that f_n and l_n satisfy the following recursive relations:

$f_n = f_{n-1} + f_{n-2}, f_1 = f_2 = 1, n \geq 3$, where $f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi)^{-n})$

and $\phi = \frac{1+\sqrt{5}}{2}$. And $l_n = l_{n-1} + l_{n-2}, l_1 = 1, l_2 = 3, n \geq 3$, where $l_n = (\phi^n + (-\phi)^{-n})$. So from the definition we can conclude that

$$f_n f_m = \frac{1}{5}(l_{n+m} - (-1)^n l_{m-n}), m \geq n. \quad (1)$$

It is easy to see that $\sigma(P_n) = f_{n+2}$ and $\sigma(S_n) = 2^{n-1} + 1$, for $n \geq 1$.

2 Preliminaries

In this section, we introduce some known lemmas and definitions, which will be helpful to the proofs of our main results.

Lemma 1. *([3, 4, 8]) For any graph G with any $u \in V(G)$, we have*

$$\sigma(G) = \sigma(G-u) + \sigma(G-[u]), \textit{where } [u] = N_G(u) \cup \{u\}.$$

Lemma 2. *([3, 4, 8]) Let G be a graph with m components $G_1, G_2, \ldots, G_m$. Then*

$$\sigma(G) = \prod_{i=1}^{m} \sigma(G_i).$$

Lemma 3. *([3, 4, 8]) Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. If $V(G_1) = V(G_2)$ and $E(G_1) \subset E(G_2)$, then $\sigma(G_1) > \sigma(G_2)$.*

Lemma 4. *([9]) Let $G \ncong P_1$ be a connected graph and choose $v \in V(G)$. Let $P(n,k,G,v)$ denote the graph obtained by identifying v with the vertex v_k of a simple path $v_1, \cdots, v_n$. Let $n = 4m + i, i \in 1,2,3,4, m \geq 0$. Then $\sigma(P(n,2,G,v)) > \sigma(P(n,4,G,v)) > \cdots > \sigma(P(n,2m+2l,G,v)) > \sigma(P(n,2m+1,G,v)) > \cdots > \sigma(P(n,3,G,v)) > \sigma(P(n,1,G, v))$, where $l = \lfloor \frac{i-1}{2} \rfloor$.*

Definition 5. *([9]) We call a tree with only one vertex v of degree $d(v) > 2$ a d-pode. In particular, we use the term tripode of 3-podes. v is called the center. To each partition $(c_1, \cdots, c_d)$ of $n-1$, there is exactly one corresponding d-pode, which we denote by $R(c_1, \cdots, c_d)$. Here, c_i is the length of the i-th "ray" going out from the center.*

Lemma 6. *([9]) For all positive integers c_i we have*

$$\sigma(R(c_1, c_2, \cdots, c_d)) = \prod_{i=1}^{d} f_{c_i+2} + \prod_{i=1}^{d} f_{c_i+1}.$$

Definition 7. *([9]) Let a, b_{ij} be positive integers with $a + b_{11} + b_{12} + b_{21} + b_{22} = n$. Then, the n-vertex tree that is shown in Figure 1 (a) is denoted by $H(a; b_{11}, b_{12}; b_{21}, b_{22})$, where $d(v_1, u_2) = b_{11}$, $d(v_1, u_4) = b_{12}$, $d(v_a, w_2) = b_{21}$ and $d(v_a, w_4) = b_{22}$. Note that $d(v_1, v_a) = a - 1$. Here, $a = 1$ means that v_1 and v_a coincide.*

Lemma 8. *([9]) For all positive integers a, b_{ij}, we have*
$$\sigma(H(a; b_{11}, b_{12}; b_{21}, b_{22})) = f_a \prod_{1\le i,j\le 2} f_{b_{ij}+2} + f_{a-1}\Big(\prod_{1\le i,j\le 2} f_{b_{ij}+i} + \prod_{1\le i,j\le 2} f_{b_{ij}+3-i}\Big) + f_{a-2} \prod_{1\le i,j\le 2} f_{b_{ij}+1}.$$

Lemma 9. *([9]) Let T be a tree, $v \in V(T)$, let S be one of the subtrees at v in T that contains more than one leaf. S can be replaced in such a way that the resulting tree T' has exactly one leaf less than T and the tree T' preserves the number of vertices of the tree T. Then the σ-index of the tree T' is smaller.*

Lemma 10. *([9]) For a given number n of vertices and given maximal degree d, the tree T with minimal σ-index is*

$$\begin{cases} R(\underbrace{2,\cdots,2}_{n-1-d},\underbrace{1,\cdots,1}_{2d-n+1}) & \text{if } d \ge \frac{n-1}{2}, \\ R(\underbrace{2,\cdots,2}_{d-1}, n-2d+1) & \text{if } d \le \frac{n-1}{2}. \end{cases}$$

The σ-index of these trees is $(\frac{3}{2})^{n-1}(\frac{4}{3})^d + 2^{n-d-1}$ and $3^{d-1}f_{n-2d+3} + 2^{d-1}f_{n-2d+2}$, respectively.

Lemma 11. *([2]) Let G and H be two connected graphs, $u \in V(H)$, let $P_{H,m}(n,n,G,v)$ denote the graph obtained by joining the vertex v_m of the simple path $v_1, v_2 \cdots, v_n$ of the graph $P(n,n,G,v)$ and u with a new edge uv_m. If $n \ge 4$, $2 \le m < n-1$, then*

$$\sigma(P_{H,m}(n,n,G,v)) \ge \sigma(P_{H,3}(n,n,G,v)).$$

Lemma 12. *([2]) Let G be a connected graph, $H \cong P_5$, let $P'_{H,m}(n,n,G,v)$ denote the graph obtained by identifying v_m of the simple path $v_1, v_2 \cdots, v_n$ of the graph $P(n,n,G,v)$ with the center u of $H = P_5$. If $n > m \ge 4$ then*

$$\sigma(P'_{H,m}(n,n,G,v)) \ge \sigma(P'_{H,3}(n,n,G,v)).$$

Corollary 13. *([2]) Let G be a connected tree. Let $H = (G,u)\bullet(R(i,j,k),v)$ be the graph obtained by identifying any vertex u of G with any pendent vertex v of the tree $R(i,j,k)$. If $i+j+k=5$, then*

$$\sigma(H) \geq \sigma(H'),$$

where $H' = (G,u)\bullet(R(1,2,2),v)$, v is the pendent vertex of the tree $R(1,2,2)$ and v is adjacent to the center of the tree $R(1,2,2)$.

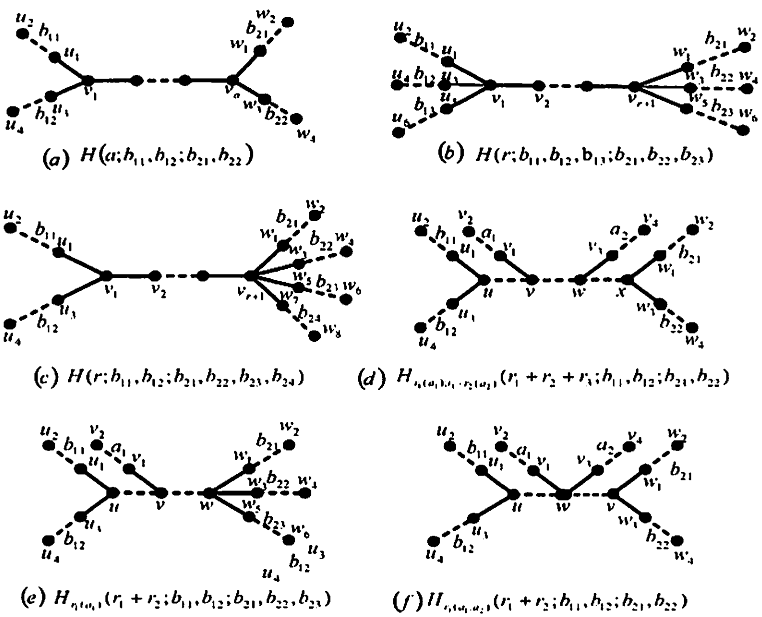

Figure 1

Definition 14. *Let b_{ij}, r be positive integers with $b_{11}+b_{12}+b_{13}+b_{21}+b_{22}+b_{23}+r+1=n$. Then, the n-vertex tree that is shown in Figure 1 (b) is denoted by $H(r;b_{11},b_{12},b_{13};b_{21},b_{22},b_{23})$, where $d(v_1,u_2)=b_{11}$, $d(v_1,u_4)=b_{12}$, $d(v_1,u_6)=b_{13}$, $d(v_{r+1},w_2)=b_{21}$, $d(v_{r+1},w_4)=b_{22}$ and $d(v_{r+1},w_6)=b_{23}$. Note that $d(v_1,v_{r+1})=r$.*

Definition 15. *Let b_{ij}, r be positive integers with $b_{11}+b_{12}+b_{21}+b_{22}+b_{23}+b_{24}+r+1=n$. Then, the n-vertex tree that is shown in Figure 1 (c) is denoted by $H(r;b_{11},b_{12};b_{21},b_{22},b_{23},b_{24})$, where $d(v_1,u_2)=b_{11}$, $d(v_1,u_4)=b_{12}$, $d(v_{r+1},w_2)=b_{21}$, $d(v_{r+1},w_4)=b_{22}$, $d(v_{r+1},w_6)=b_{23}$ and $d(v_{r+1},w_8)=b_{24}$. Note that $d(v_1,v_{r+1})=r$.*

Definition 16. *Let a_i, b_{ij}, r_i be positive integers with $a_1+a_2+b_{11}+b_{12}+b_{21}+b_{22}+r_1+r_2+r_3+1=n$. Then the n-vertex tree that is shown in Figure 1 (d) is denoted by $H_{r_1(a_1);r_1+r_2(a_2)}(r_1+r_2+r_3;b_{11},b_{12};b_{21},b_{22})$, where $d(u,u_2)=b_{11}$, $d(u,u_4)=b_{12}$, $d(x,w_2)=b_{21}$, $d(x,w_4)=b_{22}$, $d(v,v_2)=a_1$ and $d(w,v_4)=a_2$. Note that $d(u,v)=r_1$, $d(v,w)=r_2$, $d(w,x)=r_3$.*

Definition 17. *Let a_i, b_{ij}, r_i be positive integers with $a_1+b_{11}+b_{12}+b_{21}+b_{22}+b_{23}+r_1+r_2+1=n$. Then the n-vertex tree that is shown in Figure 1 (e) is denoted by $H_{r_1(a_1)}(r_1+r_2;b_{11},b_{12};b_{21},b_{22},b_{23})$, where $d(u,u_2)=b_{11}$, $d(u,u_4)=b_{12}$, $d(w,w_2)=b_{21}$, $d(w,w_4)=b_{22}$, $d(w,w_6)=b_{23}$ and $d(v,v_2)=a_1$. Note that $d(u,v)=r_1$, $d(v,w)=r_2$.*

Definition 18. *Let a_i, b_{ij}, r_i be positive integers with $a_1+a_2+b_{11}+b_{12}+b_{21}+b_{22}+r_1+r_2+1=n$. Then the n-vertex tree that is shown in Figure 1 (f) is denoted by $H_{r_1(a_1,a_2)}(r_1+r_2;b_{11},\ b_{12};b_{21},b_{22})$, where $d(u,u_2)=b_{11}$, $d(u,u_4)=b_{12}$, $d(v,w_2)=b_{21}$, $d(v,w_4)=b_{22}$, $d(w,v_2)=a_1$ and $d(w,v_4)=a_2$. Note that $d(u,w)=r_1$, $d(w,v)=r_2$.*

3 Main results

In this section, we investigate the smallest Merrifield-Simmons index (or σ-index) among all trees with n vertices and exactly six leaves, and characterize the extremal graph.

Lemma 19. *Let G be a given connected graph, H be a d-pode tree with n vertices and let u and v be the center of H and any vertex of G, respectively. Then the graph $G'=(G,v)\bullet(H,u)$ attains the minimal σ-index only if the form of H is one of the following two cases.*

$$\begin{cases} R(\underbrace{2,\cdots,2}_{n-1-d},\underbrace{1,\cdots,1}_{2d-n+1}) & \text{if } d \geq \frac{n-1}{2}, \\ R(\underbrace{2,\cdots,2}_{d-1},n-2d+1) & \text{if } d \leq \frac{n-1}{2}. \end{cases}$$

Proof. We prove the result by induction on n. Because the d-pode tree has only one vertex u satisfying $d(u)>2$, we will complete the proof by distinguishing the two cases.

Case 1. $d \geq \frac{n-1}{2}$

When $n=5$ and $d=3$, H must be the tree $R(1,1,2)$. Hence the result holds.

Assume that the result holds for any d-pode tree with $n-1$ vertices, that is, when $n(H)=n-1$ and $H \cong R(\underbrace{2,\cdots,2}_{n-2-d},\underbrace{1,\cdots,1}_{2d-n+2})$, then the graph $G' = (G,v)\bullet(H,u)$ attains the minimal σ-index.

Suppose that $n(H)=n$, then H must be one of the three trees: $H \cong R(3,\underbrace{2,\cdots,2}_{n-3-d},\underbrace{1,\cdots,1}_{2d-n+2})$ $(=H_1)$, $H \cong R(\underbrace{2,\cdots,2}_{n-1-d},\underbrace{1,\cdots,1}_{2d-n+1})(=H_2)$ or $H \cong R(\underbrace{2,\cdots,2}_{n-2-d},\underbrace{1,\cdots,1}_{2d-n+3})(=H_3)$.

Let $\sigma(G-v)=A$, $\sigma(G-[v])=B$. Then $A>B$.

By Lemmas 1 and 6, we get that

$$\sigma((G,v)\bullet(H_1,u)) = 5\times 3^{n-d-3}\times 2^{2d-n+2}\times A + 3\times 2^{n-d-3}\times B,$$

$$\sigma((G,v)\bullet(H_2,u)) = 3^{n-d-1}\times 2^{2d-n+1}\times A + 2^{n-d-1}\times B$$

and $\sigma((G,v)\bullet(H_3,u)) = 3^{n-d-2}\times 2^{2d-n+3}\times A + 2^{n-d-2}\times B$.

Hence, we have that

$\sigma((G,v)\bullet(H_2,u)) - \sigma((G,v)\bullet(H_1,u)) = A\times(3^{n-d-1}\times 2^{2d-n+1} - 5\times 3^{n-d-3}\times 2^{2d-n+2}) + B\times(2^{n-d-1} - 3\times 2^{n-d-3})$.

$\sigma((G,v)\bullet(H_3,u)) - \sigma((G,v)\bullet(H_2,u)) = A\times(3^{n-d-2}\times 2^{2d-n+3} - 3^{n-d-1}\times 2^{2d-n+1}) + B\times(2^{n-d-2} - 2^{n-d-1})$.

Because

$$(3^{n-d-1}\times 2^{2d-n+1} - 5\times 3^{n-d-3}\times 2^{2d-n+2}) = -(3^{n-d-3}\times 2^{2d-n+1}),$$

and $2^{n-d-1} - 3\times 2^{n-d-3} = 2^{n-d-3}$. Therefore $\sigma((G,v)\bullet(H_2,u)) < \sigma((G,v)\bullet(H_1,u))$.

Similarly, we have that $\sigma((G,v)\bullet(H_2,u)) < \sigma((G,v)\bullet(H_3,u))$.

Hence, when $n(H)=n$ and $H \cong R(\underbrace{2,\cdots,2}_{n-d-1},\underbrace{1,\cdots,1}_{2d-n+1})$, then the graph $G' = (G,v)\bullet(H,u)$ attains the minimal σ-index.

Case 2. $3 \le d \le \frac{n-1}{2}$

In this case, the smallest number of vertices of H must be $n(H)=7$ and let $\Delta(H)=d(v)=3$. Then H must be one of the three possible trees: $H\cong R(1,1,4)(=H_1), H\cong R(1,2,3)(=H_2)$ or $H\cong R(2,2,2)(=H_3)$. Let $\sigma(G-v)=A$, $\sigma(G-[v])=B$. Then $A>B$.

By Lemmas 1 and 6, we get that

$$\sigma((G,v)\bullet(H_1,u)) = 32A+5B,$$

$$\sigma((G,v)\bullet(H_2,u)) = 30A+6B$$

and $\sigma((G,v)\bullet(H_3,u)) = 27A+8B$. Hence

$$\sigma((G,v)\bullet(H_3,u)) - \sigma((G,v)\bullet(H_2,u)) = 2B-3A<0$$

and

$$\sigma((G,v)\bullet(H_3,u))-\sigma((G,v)\bullet(H_1,u))=3B-5A<0.$$

Therefore, when $H\cong R(2,2,2)$, then $G'=(G,v)\bullet(H,u)$ attains the minimal σ-index. Hence, the result holds.

Assume that the result holds for any d-pode tree with $n-1$ vertices, that is, when $n(H)=n-1$ and $H\cong R(\underbrace{2,\cdots,2}_{d-1},n-2d)$, then the graph $G'=(G,v)\bullet(H,u)$ attains the minimal σ-index.

Suppose that $n(H)=n$, then H will be one of the following three trees: $H\cong R(\underbrace{2,\cdots,2}_{d-1},n-2d+1)(=H_1)$, $H\cong R(\underbrace{2,\cdots,2}_{d},n-2d-1)(=H_2)$ and $H\cong R(3,\underbrace{2,\cdots,2}_{d-2},n-2d)(=H_3)$.

By Lemmas 1 and 6, we get that

$$\sigma((G,v)\bullet(H_1,u))=3^{d-1}f_{n-2d+3}A+2^{d-1}f_{n-2d+2}B,$$

$$\sigma((G,v)\bullet(H_2,u))=3^{d}f_{n-2d+1}A+2^{d}f_{n-2d}B$$

and $\sigma((G,v)\bullet(H_3,u))=5\times 3^{d-2}f_{n-2d+2}A+3\times 2^{d-2}f_{n-2d+1}B$.

Hence, $\sigma((G,v)\bullet(H_1,u))-\sigma((G,v)\bullet(H_3,u))=A(3^{d-1}f_{n-2d+3}-5\times 3^{d-2}f_{n-2d+2})+B(2^{d-1}f_{n-2d+2}-3\times 2^{d-2}f_{n-2d+1})<0$. So $\sigma((G,v)\bullet(H_1,u))<\sigma((G,v)\bullet(H_3,u))$.

Similarly we have $\sigma((G,v)\bullet(H_1,u))<\sigma((G,v)\bullet(H_2,u))$.

Thus when $n(H)=n$ and $H\cong R(\underbrace{2,\cdots,2}_{d-1},n-2d+1)$, then the graph $G'=(G,v)\bullet(H,u)$ attains the minimal σ-index.

This completes the proof. □

Theorem 20. *If T is a tree with $n\geq 19$ vertices and exactly six leaves, then*

$$\sigma(T)\geq 291f_{n-11}+372f_{n-14},$$

where the equality holds if and only if the tree $T\cong H_{1(2)}(4;2,n-16;2,2,2)$.

Proof. Because T is a tree with $n\geq 19$ vertices and exactly six leaves, the tree T has the following five possible cases: There is only one vertex with degree 6 and all other vertices have degree less than 3; There are two vertices with degree 4 and all other vertices have degree less than 3; There are one vertex with degree 3, one vertex with degree 5 and all other vertices have degree less than 3; There are two vertices with degree 3, one vertex with degree 4 and all other vertices have degree less than 3; There are four vertices with degree 3 and all other vertices have degree less than 3. Next we will discuss these cases respectively.

Case 1. When the tree T has only one vertex with degree 6 and all other vertices have degree less than 3. From Lemma 10, we get that $\sigma(T) \geq \sigma(T_1)$, where $T_1 \cong R(2,2,2,2,2,n-11)$. By Lemma 6, we get that $\sigma(T_1) = 243f_{n-9} + 32f_{n-10}$.

Case 2. When the tree T has two vertices with degree 4 and all other vertices have degree less than 3. Let $u, v \in V(T)$, $d(u) = 4, d(v) = 4$. Then T can be regarded as the tree obtained by identifying two end-vertices u and v of a simple path P_{uv} with two centers of two 3-pode trees respectively, the corresponding two 3-pode trees are denoted by T_u and T_v. Suppose that $n(T_u) \geq 4, n(T_v) \geq 4$, we will discuss the following four cases.

Case 2.1. If $n(T_u) + n(P_{uv}) = 6$, then $d(u,v) = 1$. From Lemma 19 we get that $\sigma(T) \geq \sigma(T_2)$, $T_2 = (G, w_1) \bullet (H, w_2) \cong H(1;1,1,1;2,2,n-9)$, where $G = S_5$, $H = R(2,2,n-9)$ and w_1, w_2 are an end-vertex of $G = S_5$ and the center of $R(2,2,n-9)$ respectively. By Lemma 1, we get that $\sigma(T_2) = 81f_{n-7} + 32f_{n-8}$.

Case 2.2. If $n(T_u) + n(P_{uv}) = 7$, then $1 \leq d(u,v) \leq 2$. If $d(u,v) = 1$, then $T_u \cong R(1,1,2)$. From Lemma 19, we know that there exists the tree $T_3^1 \cong H(1;1,1,2;2,2,n-10)$ such that $\sigma(T) \geq \sigma(T_3^1)$. If $d(u,v) = 2$ then $T_u \cong S_4$. From Lemma 19, we know that there exists the tree $T_3^2 \cong H(2;1,1,1;2,2,n-10)$ such that $\sigma(T) \geq \sigma(T_3^2)$. By Lemma 1, we get that $\sigma(T_3^1) = 126f_{n-8} + 48f_{n-9}$ and $\sigma(T_3^2) = 153f_{n-8} + 36f_{n-9}$, Thus $\sigma(T_3^1) < \sigma(T_3^2)$. Let $T_3 = T_3^1 = H(1;1,1,1;2,2,n-10)$. Hence we get that $\sigma(T) \geq \sigma(T_3)$ and $\sigma(T_3) = 126f_{n-8} + 48f_{n-9}$.

Case 2.3. If $n(T_u) + n(P_{uv}) = 8$, then $1 \leq d(u,v) \leq 3$. If $d(u,v) = 1$, then $T_u \cong R(1,2,2)$. From Lemma 19, we know that there exists the tree $T_4^1 \cong H(1;1,2,2;2,2,n-11)$ such that $\sigma(T) \geq \sigma(T_4^1)$. If $d(u,v) = 2$, then $T_u \cong R(1,1,2)$. From Lemma 19, we know that there exists the tree $T_4^2 \cong H(2;1,1,2;2,2,n-11)$ such that $\sigma(T) \geq \sigma(T_4^2)$. If $d(u,v) = 3$, then $T_u \cong S_4$. From Lemma 19, we know that there exists the tree $T_4^3 \cong H(3;1,1,1;2,2,n-11)$ such that $\sigma(T) \geq \sigma(T_4^3)$. By Lemma 1, we get that $\sigma(T_4^1) = 198f_{n-9} + 72f_{n-10}$, $\sigma(T_4^2) = 234f_{n-9} + 56f_{n-10}$ and $\sigma(T_4^3) = 234f_{n-9} + 68f_{n-10}$. Thus $\sigma(T_4^1) < \sigma(T_4^2) < \sigma(T_4^3)$. Let $T_4 = T_4^1$. Hence $\sigma(T) \geq \sigma(T_4)$ and $\sigma(T_4) = 198f_{n-9} + 72f_{n-10}$.

Case 2.4. If $n(T_u) + n(P_{uv}) > 8$, because $n(T) \geq 19$, thus $n(T_v) \geq 10$. Then we discuss the following two cases.

Case 2.4.1. If $n(T_v) = 10$, by Lemmas 10 and 19, the σ-index of T is minimal when $T_v \cong R(2,2,5)$.

Case 2.4.1.1. If $d(u,v) = 1$, then T_u is a tree with $n-10$ vertices and maximum degree 3. By Lemmas 10 and 19, we know that the tree T attains the minimal σ-index only if $T_u \cong R(2,2,n-15)$. Hence we have $\sigma(T) \geq \sigma(T_5)$, where $T_5 \cong H(1;2,2,n-15;2,2,5)$. By Lemmas 1 and 6, we get $\sigma(T_5) = 1341f_{n-13} + 468f_{n-14}$.

Case 2.4.1.2. If $d(u,v) > 1$, then by Lemmas 12 and 19, we have

$\sigma(T) \geq \sigma(T_6)$, where $T_6 \cong H(n-16;2,2,2;2,2,5)$. By Lemma 1, we get that $\sigma(T_6) = 3159f_{n-15} + 1800f_{n-16} + 256f_{n-17}$.

Case 2.4.2. If $n(T_v) = m(m > 10)$, by Lemma 10 and 19, then we get that the tree T attains the minimal σ-index only if $T_v \cong R(2,2,r)$, where $r \geq 6$, $m = r+5$.

Case 2.4.2.1. If $d(u,v) = 1$ and $n-r-5 \geq 7$, then by Lemmas 10 and 19, the tree T attains the minimal σ-index only if $T_u \cong R(2,2,n-r-10)$. Hence we have $\sigma(T) \geq \sigma(T_7)$, where $T_7 \cong H(1;2,2,n-r-10;2,2,r)$. By Lemma 1, we get $\sigma(T_7) = 81f_{r+2}f_{n-r-8} + 36f_{r+2}f_{n-r-9} + 36f_{r+1}f_{n-r-8}$.

If $r \leq \frac{n-11}{2}$, by the formula (1.1), then

$$\sigma(T_7) = \frac{1}{5}(81l_{n-6}+72l_{n-7}) - \frac{1}{5}(-1)^r(81l_{n-2r-10}+36l_{n-2r-11}-36l_{n-2r-9}).$$

Since $81l_{n-2r-10} + 36l_{n-2r-11} - 36l_{n-2r-9} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 6$, let $T = H(1;2,2,n-16;2,2,6) = T_7^1$, the tree T attains the minimal σ-index and $\sigma(T_7^1) = 2169f_{n-14} + 756f_{n-15}$.

If $r \geq \frac{n-9}{2}$, by the formula (1.1), then
$\sigma(T_7) = \frac{1}{5}(81l_{n-6} + 72l_{n-7}) - \frac{1}{5}(-1)^{n-r}(81l_{2r-n+10} - 36l_{2r-n+11} + 36l_{2r-n+9})$.

Since $81l_{2r-n+10} - 36l_{2r-n+11} + 36l_{2r-n+9} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r = 12$, let $T = H(1;2,2,2;2,2,n-12) = T_7^2$, the tree T attains the minimal σ-index and $\sigma(T_7^2) = 315f_{n-10} + 108f_{n-11}$.

Case 2.4.2.2. If $d(u,v) = 1$ and $n-r-5 \leq 6$, then the tree T will have following three possible cases that $H(1;2,2,n-11;2,2,1)$, $H(1;2,2,n-10;2,1,1)$ and $H(1;2,2,n-9;1,1,1)$. By Lemma 1, we get that $\sigma(H(1;2,2,n-11;2,2,1)) = 198f_{n-9}+72f_{n-10}$, $\sigma(H(1;2,2,n-10;2,1,1)) = 48f_{n-9}+126f_{n-10}$ and $\sigma(H(1;2,2,n-9;1,1,1)) = 81f_{n-7} + 32f_{n-8}$.

Comparing all the σ-indices of the these trees in Case 2.4.2.1 and Case 2.4.2.2, we conclude that the minimal σ-index among the three trees in Case 2.4.2.2 is larger than the minimal σ-index of the tree T in Case 2.4.2.1. Therefore, when $d(u,v) = 1$, the tree T attains the minimal σ-index only if $n-r-5 \geq 7$ but not $n-r-5 \leq 6$.

Case 2.4.2.3. If $d(u,v) > 1$ and $n-r-5 \geq 9$. From Lemma 12, we know that there exists the tree $T_8 \cong H(n-r-11;2,2,2;2,2,r)$ such that $\sigma(T) \geq \sigma(T_8)$. By Lemma 1, we get $\sigma(T_8) = 243f_{r+2}f_{n-r-10} + 108f_{r+1}f_{n-r-11} + 72f_{r+2}f_{n-r-11} + 32f_{r+1}f_{n-r-12}$.

If $r \leq \frac{n-13}{2}$, by the formula (1.1), we have
$\sigma(T_8) = \frac{1}{5}(243l_{n-8}+108l_{n-10}+72l_{n-9}+32l_{n-11}) - \frac{1}{5}(-1)^r(243\,l_{n-2r-12} - 108l_{n-2r-12} + 72l_{n-2r-13} - 32l_{n-2r-13})$.

Since $243l_{n-2r-12} - 108l_{n-2r-12} + 72l_{n-2r-13} - 32l_{n-2r-13} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 6$, let

$T = H(n-17; 2,2,2; 2,2,6) = T_8^1$, the tree T attains the minimal σ-index and $\sigma(T_8^1) = 5103f_{n-16} + 2916f_{n-17} + 416f_{n-18}$.

If $r \geq \frac{n-12}{2}$, by the formula (1.1), we have $\sigma(T_8) = \frac{1}{5}(243l_{n-8} + 108l_{n-10} + 72\, l_{n-9} + 32l_{n-11}) - \frac{1}{5}(-1)^{n-r}(243l_{2r-n+12} - 108l_{2r-n+12} - 72l_{2r-n+13} + 32l_{2r-n+13})$. Since $243l_{2r-n+12} - 108l_{2r-n+12} - 72l_{2r-n+13} + 32l_{2r-n+13} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r = 14$, let $T = H(3; 2,2,2; 2,2,n-14) = T_8^2$, the tree T attains the minimal σ-index and $\sigma(T_8^2) = 873f_{n-12} + 248f_{n-13}$.

Case 2.4.2.4. If $d(u,v) > 1$ and $n-r-5 \leq 8$, then the tree T will have following three possible cases: $H(2; 2,2,1; 2,2,n-12)$, $H(3; 2,1,1; 2,2,n-12)$ and $H(4; 1,1,1; 2,2,n-12)$. By Lemma 1, we get $\sigma(H(2; 2,2,1; 2,2,n-12)) = 360f_{n-10} + 88f_{n-11}$, $\sigma(H(3; 2,1,1; 2,2,n-12)) = 360f_{n-10} + 104f_{n-11}$ and $\sigma(H(4; 1,1,1; 2,2,n-12)) = 387f_{n-10} + 104f_{n-11}$.

Comparing all the σ-indices of these trees in Case 2.4.2.3. and Case 2.4.2.4., we conclude that the minimal σ-index among the three trees in Case 2.4.2.4. is larger than the minimal σ-index of the tree T in Case 2.4.2.3. Therefore when $d(u,v) > 1$, the tree T attains the minimal σ-index only if $n-r-5 \geq 9$ but not $n-r-5 \leq 8$.

Hence we can conclude that the minimal σ-index of the tree T in Case 2.4 must be attained in the case $d \leq \frac{n'-1}{2}$ but not in the case $d \geq \frac{n'-1}{2}$, where $n' = n - n(T_v)$.

Therefore, the minimal σ-index of T in Case 2 is $\sigma(T) = 315f_{n-10} + 108f_{n-11}$, where $T \cong H(1; 2,2,2; 2,\ 2,\ n-12) = T_7^2$.

Case 3. When the tree T has one vertex with degree 3 and one vertex with degree 5 and all other vertices have degree less than 3. Let $u, v \in V(T)$, $d(u) = 3, d(v) = 5$. Then T can be regarded as the tree obtained by identifying two end-vertices u and v of a simple path P_{uv} with any vertex of degree 2 of a path P' and the center of a 4-pode tree respectively. The corresponding path P' and the 4-pode tree are denoted by T_u and T_v, respectively.

Case 3.1. If $n(T_u) + n(P_{uv}) = 5$, then $d(u,v) = 1$. From Lemma 19, we know that there exists the tree $T_9 \cong H(1; 1,1; 2,2,2,n-10)$ such that $\sigma(T) \geq \sigma(T_9)$. By Lemma 1, we get that $\sigma(T_9) = 135f_{n-8} + 32f_{n-9}$.

Case 3.2. If $n(T_u) + n(P_{uv}) = 6$, then $1 \leq d(u,v) \leq 2$.

If $d(u,v) = 1$, then $T_u \cong P_4$. From Lemma 19, we have that there exist the tree $T_{10}^1 \cong H(1; 1,2; 2,2,2,n-11)$ such that $\sigma(T) \geq \sigma(T_{10}^1)$.

If $d(u,v) = 2$, then $T_u \cong P_3$. Similarly we have $\sigma(T) \geq \sigma(T_{10}^2)$, where $T_{10}^2 \cong H(2; 1,1; 2,2,\ 2,n-11)$. By Lemma 1, we get that $\sigma(T_{10}^1) = 216f_{n-9} + 48f_{n-10}$ and $\sigma(T_{10}^2) = 243f_{n-9} + 40f_{n-10}$. Thus $\sigma(T_{10}^1) < \sigma(T_{10}^2)$.

Let $T_{10}^1 = T_{10}$. Hence we have $\sigma(T) \geq \sigma(T_{10})$ and $\sigma(T_{10}) = 216f_{n-9} + 48f_{n-10}$.

Case 3.3. If $n(T_u) + n(P_{uv}) > 6$, because $n(T) \geq 19$, thus $n(T_v) \geq 12$. Then we will discuss the distinguishing two cases.

Case 3.3.1. If $n(T_v) = 12$, by Lemmas 10 and 19, then the σ-index of the tree T is minimal when $T_v \cong R(2,2,2,5)$.

If $d(u,v) = 1$, by Lemma 4, then we have $\sigma(T) \geq \sigma(T_{11})$, where $T_{11} \cong H(1;2,n-15;2,2,2,5)$. By Lemma 1, we get that $\sigma(T_{11}) = 1245f_{n-13} + 702f_{n-14}$.

If $d(u,v) > 1$, by Corollary 13 and Lemma 19, then we have $\sigma(T) \geq \sigma(T_{12})$, where $T_{12} \cong H(n-16;2,2;2,2,2,5)$ and $\sigma(T_{12}) = 3159f_{n-15} + 1980f_{n-16} + 256f_{n-17}$.

Case 3.3.2. If $n(T_v) > 12$, by Lemmas 10 and 19, then the σ-index of the tree T is minimal when $T_v \cong R(2,2,2,r)$, where $r \geq 6$.

Similarly to Case 2.4.2, we only discuss the tree T in the case $d \geq \frac{n'-1}{2}$, where $n' = n - n(T_v)$.

Case 3.3.2.1. If $d(u,v) = 1$, then by Lemmas 4 and 19, we have $\sigma(T) \geq \sigma(T_{13})$, where $T_{13} \cong H(1;2,n-r-10;2,2,2,r)$ and $\sigma(T_{13}) = 27f_{n-r-5}f_{r+2} + 24f_{r+1}f_{n-r-8}$.

If $r \leq \frac{n-9}{2}$, by the formula (1.1), then

$$\sigma(T_{13}) = \frac{1}{5}(27l_{n-3} + 24l_{n-7}) - \frac{1}{5}(-1)^r(27l_{n-2r-7} - 24l_{n-2r-9}).$$

Since $27l_{n-2r-7} - 24l_{n-2r-9} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 6$, let $T = H(1;2,n-16;2,2,2,6) = T_{13}^1$, the tree T attains the minimal σ-index and $\sigma(T_{13}^1) = 567f_{n-11} + 312f_{n-14}$.

If $r \geq \frac{n-7}{2}$, by the formula (1.1), then

$$\sigma(T_{13}) = \frac{1}{5}(27l_{n-3} + 24l_{n-7}) - \frac{1}{5}(-1)^{n-r}(-27l_{2r-n+7} + 24l_{2r-n+9}).$$

Since $-27l_{2r-n+7} + 24l_{2r-n+9} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r = 14$, let $T = H(1;2,4;2,2,2,n-14) = T_{13}^2$, the tree T attains the minimal σ-index and $\sigma(T_{13}^2) = 918f_{n-12} + 192f_{n-13}$.

Case 3.3.2.2. If $d(u,v) > 1$, by Corollary 13 and Lemma 19, we have $\sigma(T) \geq \sigma(T_{14})$, where $T_{14} \cong H(n-r-11;2,2;2,2,2,r)$ and $\sigma(T_{14}) = 243f_{r+2}f_{n-r-10} + 108f_{r+2}f_{n-r-11} + 72f_{r+1}f_{n-r-11} + 32f_{r+1}f_{n-r-12}$.

If $r \leq \frac{n-13}{2}$, by the formula (1.1), then $\sigma(T_{14}) = \frac{1}{5}(243l_{n-8} + 72l_{n-10} + 108l_{n-9} + 32l_{n-11}) - \frac{1}{5}(-1)^r(243l_{n-2r-12} - 72l_{n-2r-12} + 108l_{n-2r-13} - 32l_{n-2r-13})$. Since $243l_{n-2r-12} - 72l_{n-2r-12} + 108l_{n-2r-13} - 32l_{n-2r-13} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 6$, let $T = H(n-17;2,2;2,2,2,6) = T_{14}^1$, the tree T attains the minimal σ-index and $\sigma(T_{14}^1) = 5103f_{n-16} + 3204f_{n-17} + 416f_{n-18}$.

If $r \geq \frac{n-12}{2}$, by the formula (1.1), then $\sigma(T_{14}) = \frac{1}{5}(243l_{n-8} + 72l_{n-10} + 108l_{n-9} + 32l_{n-11}) - \frac{1}{5}(-1)^{n-r}(243\, l_{2r-n+12} - 72l_{2r-n+12} - 108l_{2r-n+13} +

$32l_{2r-n+13}$). Since $243 \cdot l_{2r-n+12} - 72l_{2r-n+12} - 108l_{2r-n+13} + 32l_{2r-n+13} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r=14$, let $T = H(3;2,2;2,2,2,n-14) = T_{14}^2$, the tree T attains the minimal σ-index and $\sigma(T_{14}^2) = 945f_{n-12} + 176f_{n-13}$.

As the above discussion in Case 3, we get the minimal σ-index of T is $\sigma(T) = 567f_{n-11} + 312f_{n-14}$, where $T = H(1;2,n-16;2,2,2,6) = T_{13}^1$.

Case 4. When the tree T has four vertices with degree 3 and all other vertices have degree less than 3. According to Lemmas 4, 12 and Corollary 13, we can get that the tree T attains the minimal σ-index when $T \cong H_{r(2);r+p(2)}(n-13;2,2;2,2)$. Let $u,v,w,x \in V(T)$ and $d(u)=d(v)=d(w)=d(x)=3$. The tree is shown in Figure 1 (d). Let $d(u,v)=r_1=r$, $d(v,w)=r_2=p$.

Case 4.1. Let p be a fixed positive integer, we will discuss for r in the following two cases.

Case 4.1.1. If $r \geq 1, p=1$ and $n-r-14 \geq 1$, by the formula (1.1), then $\sigma(T_{15}) = 243f_{r+1}f_{n-r-10} + 108f_{r+1}f_{n-r-11} + 108f_rf_{n-r-10} + 48f_rf_{n-r-11} + 486 \cdot f_rf_{n-r-13} + 216f_rf_{n-r-14} + 216f_{r-1}f_{n-r-13} + 96f_{r-1}f_{n-r-14}$.

If $r \leq \frac{n-14}{2}$, by the formula (1.1), since $-243l_{n-2r-11} - 108l_{n-2r-12} + 108l_{n-2r-10} + 48l_{n-2r-11} + 486l_{n-2r-13} + 216l_{n-2r-14} - 216l_{n-2r-12} - 96l_{n-2r-13} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $r=1$, let $T = H_{1(2);2(2)}(n-13;2,2;2,2) = T_{15}^1$, the tree T attains the minimal σ-index and $\sigma(T_{15}^1) = 351f_{n-11} + 156f_{n-12} + 486f_{n-14} + 216f_{n-15}$.

If $r \geq \frac{n-10}{2}$, by the formula (1.1), since $243l_{2r-n+11} - 108l_{2r-n+12} + 108l_{2r-n+10} - 48l_{2r-n+11} - 486l_{2r-n+13} + 216l_{2r-n+14} - 216l_{2r-n+12} + 96 \cdot l_{2r-n+13} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r=15$, let $T = H_{n-15(2);n-14(2)}(n-13;2,2;2,2) = T_{15}^2$, the tree attains the minimal σ-index and $\sigma(T_{15}^2) = 351f_{n-11} + 156f_{n-12} + 486f_{n-14} + 216f_{n-15}$.

Therefore, when $r \geq 1, p=1$, the minimal σ-index of the tree T in Case 4.1 is $\sigma(T) = 351f_{n-11} + 156f_{n-12} + 486f_{n-14} + 216f_{n-15} = \sigma(T_{15}^1) = \sigma(T_{15}^2)$.

Case 4.1.2. If $r \geq 1$, $p > 1$, $n-r-p-13 \geq 1$, by Lemmas 1 and 8, we have $\sigma(H_{r(2);r+p(2)}(n-13;2,2;2,2) = f_4\sigma(R(2,2,r-1))\sigma(H(n-r-p-13;2,p-1;2,2)) + f_3\sigma(R(2,2,r-2))\sigma(H(n-r-p-13;2,p-2;2,2))$.

We get that $\sigma(H_{r+1(2);r+p+1(2)}(n-13;2,2;2,2) = f_4\sigma(R(2,2,r))\sigma(H(n-r-p-14;2,p-1;2,2)) + f_3\sigma(R(2,2,r-1))\sigma(H(n-r-p-14;2,p-2;2,2))$, and

$$\begin{cases} \sigma(H_{r(2);r+p(2)}(n-13;2,2;2,2)) < \sigma(H_{r+1(2);r+p+1(2)}(n-13;2,2;2,2)), \\ \text{if } r \text{ is odd}, \\ \sigma(H_{r(2);r+p(2)}(n-13;2,2;2,2)) > \sigma(H_{r+1(2);r+p+1(2)}(n-13;2,2;2,2)), \\ \text{if } r \text{ is even}. \end{cases}$$

and

$$\begin{cases}\sigma(H_{r+2(2);r+p+2(2)}(n-13;2,2;2,2)) > \sigma(H_{r(2);r+p(2)}(n-13;2,2;2,2)),\\ \text{if } r \text{ is odd,}\\ \sigma(H_{r+2(2);r+p+2(2)}(n-13;2,2;2,2)) < \sigma(H_{r(2);r+p(2)}(n-13;2,2;2,2)),\\ \text{if } r \text{ is even.}\end{cases}$$

Therefore, when $d(v,w) = r_2 = p$, $d(u,v) = r_1 = 1$, the tree T attains the minimal σ-index.

Because the tree T has the symmetrical property in Case 4, the tree T attains the minimal σ-index only if $d(w,x) = 1$. Therefore, when $d(v,w) = p$, let $T = H_{1(2);n-14(2)}(n-13;2,2;2,2) = T_{16}$, the tree T attains the minimal σ-index, and $\sigma(T) = 2223f_{n-15} + 2547f_{n-16} + 702f_{n-17}$.

Case 4.2. Let $d(v,w) = p$ $(p > 1)$, we will discuss for p. By Lemmas 10 and 19, we have $\sigma(T) \geq \sigma(T_{17})$, where $T_{17} \cong H_{1(2);p+1(2)}(n-13;2,2;2,2)$ and $\sigma(T_{17}) = 81f_{p+3}f_{n-p-10} + 36f_{p+3}f_{n-p-11} + 108f_pf_{n-p-10} + 48f_pf_{n-p-11} + 243f_{p+2}f_{n-p-13} + 108f_{p+2}f_{n-p-14} + 324f_{p-1} \cdot f_{n-p-13} + 144f_{p-1}f_{n-p-14}$.

If $p \leq \frac{n-16}{2}$, since $-225l_{n-2p-13} - 324l_{n-2p-12} - 36l_{n-2p-14} + 108l_{n-2p-10} + 48l_{n-2p-11} + 243l_{n-2p-15}$
$+108l_{n-2p-16} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $p = 3$, let $T = H_{1(2);4(2)}(n-13;2,2;2,2) = T_{17}^1$, the tree T attains the minimal σ-index and $\sigma(T_{17}^1) = 864f_{n-13} + 384f_{n-14} + 1539f_{n-16} + 684f_{n-17}$.

If $p \geq \frac{n-10}{2}$, since $225l_{2p-n+13} - 36l_{2p-n+14} + 108l_{2p-n+10} - 48l_{2p-n+11} - 243l_{2p-n+15} + 108l_{2p-n+16} - 324l_{2p-n+12} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - p = 15$, let $T = H_{1(2);n-14(2)}(n-13;2,2;2,2) = T_{17}^2$, the tree T attains the minimal σ-index and $\sigma(T_{17}^2) = 513f_{n-12} + 351f_{n-13} + 684f_{n-15} + 468f_{n-16}$.

Therefore the minimal σ-index of the tree T in Case 4.2 is $\sigma(T) = \sigma(T_{17}^2) = 513f_{n-12} + 351f_{n-13} + 684f_{n-15} + 468f_{n-16}$.

Case 4.3. The tree T ia shown in Figure 1 (d). Let $d(v,v_2) = a_1 = q$. By Lemma 19, we have $\sigma(T) \geq T_{18}$, where $T_{18} \cong H_{1(q);n-q-12(2)}(n-q-11;2,2;2,2)$ and $\sigma(T_{18}) = 741f_{q+2}f_{n-q-13} + 513f_{q+1}f_{n-q-14} + 507 \cdot f_{q+2}f_{n-q-14} + 351f_{q+1}f_{n-q-15}$.

If $q \leq \frac{n-16}{2}$, since $741l_{n-2q-15} - 513l_{n-2q-15} + 507l_{n-2q-16} - 351l_{n-2q-16} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $q = 2$, let $T = H_{1(2);n-14(2)}(n-13;2,2;2,2) = T_{18}^1$, the tree T attains the minimal σ-index and $\sigma(T_{18}^1) = 2223f_{n-15} + 2547f_{n-16} + 702f_{n-17}$.

If $q \geq \frac{n-15}{2}$, since $-741l_{2q-n+15} + 513l_{2q-n+15} + 507l_{2q-n+16} - 351l_{2q-n+16} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - q = 16$, let $T = H_{1(n-16);4(2)}(5;2,2;2,2) = T_{18}^2$, the tree T attains the minimal σ-index and $\sigma(T_{18}^2) = 1984f_{n-14} + 864f_{n-15}$.

Because the tree T has the symmetrical property in Case 4, all possible cases of the tree T have been considered. Therefore the minimal σ-index of T in Case 4 is $\sigma(T) = 2223f_{n-15} + 2547f_{n-16} + 702f_{n-17}$, where $T \cong H_{1(2);n-14(2)}(n-13; 2, 2; 2, 2) = T_{18}^1$.

Case 5. When the tree T has two vertices with degree 3 and one vertex with degree 4 and all other vertices have degree less than 3. Let $u, v, w \in V(T)$, and let $d(u) = d(v) = 3$, $d(w) = 4$. Then there are two cases for the tree T: 1) v is between u and w, T is shown in Figure 1 (e); 2) w is between u and v, T is shown in Figure 1 (f).

Case 5.1. Let v be between u and w. The tree T can be regarded as the tree obtained by identifying two end-vertices u, v and any vertex w of degree 2 of a simple (u, v, w)-path P_{uw} with any vertex of degree 2 of a path P', the center of a 3-pode tree and one end-vertex of a path P'' respectively, the corresponding path P', 3-pode tree and the path P'' are denoted by T_u, T_w and T_v, respectively. The tree T is shown in Figure 1 (e). Let $d(v, v_2) = a_1 = r$.

Case 5.1.1. $n(T_u) + n(P_{uv}) = 5$.

Case 5.1.1.1. If $n(T_u)+n(P_{uv}) = 5$, $d(v, w) = 1$ and $n-r-4 \geq 7$. By Lemma 19, we have $\sigma(T) \geq \sigma(T_{20})$, where $T_{20} \cong H_{1(r)}(2; 1, 1; 2, 2, n-r-9)$ and $\sigma(T_{20}) = 45f_{r+2}f_{n-r-7} + 20f_{r+2}f_{n-r-8} + 36f_{r+1}f_{n-r-7}$.

If $r \leq \frac{n-10}{2}$, since $9l_{n-2r-9} + 4l_{n-2r-10} - \frac{36}{5}l_{n-2r-8} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 1$, let $T = H_{1(1)}(2; 1, 1; 2, 2, n-10) = T_{20}^1$, the tree T attains the minimal σ-index and $\sigma(T_{20}^1) = 126f_{n-8} + 40f_{n-9}$.

If $\frac{n-8}{2} \leq r \leq n-11$, since $-9l_{2r-n+9} + 4l_{2r-n+10} - \frac{36}{5}l_{2r-n+8} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 11$, let $T = H_{1(n-11)}(2; 1, 1; 2, 2, 2) = T_{20}^2$, the tree T attains the minimal σ-index and $\sigma(T_{20}^2) = 175f_{n-9} + 108f_{n-10}$.

When $d(v, w) = 1$ and $n - r - 7 \leq 6$, the tree will be one of the three possible cases: $H_{1(n-10)}^1(2; 1, 1; 1, 2, 2)$, $H_{1(n-9)}^2(2; 1, 1; 1, 1, 2)$ and $H_{1(n-8)}^3(2; 1, 1; 1, 1, 1)$. We can get the minima σ-index of the three trees is $\sigma(H_{1(n-10)}^1(2; 1, 1; 1, 2, 2)) = 72f_{n-7} + 38f_{n-8}$. So the minimal σ-index of the tree T in Case 5.1.1.1. is $\sigma(T) = \sigma(T_{20}^2) = 175f_{n-9} + 108f_{n-10}$, where $T_{20}^2 = H_{1(n-11)}(2; 1, 1; 2, 2, 2)$.

That means the minimal σ-index of T in Case 5.1.1.1. will occur in the case $n - r - 4 \geq 7$. Hence, similarly to Case 5.1.1.1., we only discuss in the case $n - r - 4 \geq 9$ in the following Case 5.1.1.2.

Case 5.1.1.2. If $d(v, w) > 1$ and $n-r-4 \geq 9$. By Lemmas 10 and 19, then $\sigma(T) \geq \sigma(T_{21})$, where $T_{21} \cong H_{1(r)}(n-r-9; 1, 1; 2, 2, 2)$ and $\sigma(T_{21}) = 135f_{r+2}f_{n-r-9} + 40f_{r+2}f_{n-r-10} + 108f_{r+1}f_{n-r-10} + 32f_{r+1}f_{n-r-11}$.

If $r \leq \frac{n-12}{2}$, since $8l_{n-2r-12} + 27l_{n-2r-11} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 2$, let $T = H_{1(2)}(n -$

$11; 1, 1; 2, 2, 2) = T_{21}^1$, the tree T attains the minimal σ-index and $\sigma(T_{21}^1) = 405f_{n-11} + 336f_{n-12} + 64f_{n-13}$.

If $r \geq \frac{n-11}{2}$, since $-27l_{2r-n+11} + 5l_{2r-n+12} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 13$, let $T = H_{1(n-13)}(4; 1, 1; 2, 2, 2) = T_{21}^2$, the tree T attains the minimal σ-index and $\sigma(T_{21}^2) = 485f_{n-11} + 248f_{n-12}$.

Therefore the minimal σ-index of T in Case 5.1.1.2. is $\sigma(T_{21}) = 405f_{n-11} + 336f_{n-12} + 64f_{n-13}$, where $T \cong H_{1(2)}(n-11; 1, 1; 2, 2, 2)$.

Case 5.1.2. If $n(T_u) + n(P_{uv}) = 6$, then $1 \leq d(u, v) \leq 2$. By Lemma 19, when $d(u, v) = 1$, the σ-index of the tree T is smaller.

Case 5.1.2.1. If $d(v, w) = 1$ and $n - r - 5 \geq 7$, by Lemma 19, then we have $\sigma(T) \geq \sigma(T_{22})$, where $T_{22} \cong H_{1(r)}(2; 1, 2; 2, 2, n-r-10)$ and $\sigma(T_{22}) = 72f_{r+2}f_{n-r-8} + 32f_{r+2}f_{n-r-9} + 54f_{r+1}f_{n-r-8}$.

If $r \leq \frac{n-11}{2}$, since $72l_{n-2r-10} + 32l_{n-2r-11} - 54l_{n-2r-9} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 2$, let $T = H_{1(2)}(2; 1, 2; 2, 2, n-12) = T_{22}^1$, the tree T attains the minimal σ-index and $\sigma(T_{22}^1) = 324f_{n-10} + 96f_{n-11}$.

If $\frac{n-9}{2} \leq r \leq n - 12$, since $72l_{2r-n+10} - 32l_{2r-n+11} + 54l_{2r-n+9} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 12$, let $T = H_{1(n-12)}(2; 1, 2; 2, 2, 2) = T_{22}^2$, the tree T attains the minimal σ-index and $\sigma(T_{22}^2) = 280f_{n-10} + 162f_{n-11}$.

Hence the minimal σ-index of the tree T in Case 5.1.2.1. is $\sigma(T) = 280f_{n-10} + 162f_{n-11} = \sigma(T_{22}^2)$.

Case 5.1.2.2. If $d(v, w) = 1$ and $n-r-5 \leq 6$, when T attains the minimal σ-index, T will have two possible cases: $T = T_{23}^{'} \cong H_{1(r)}(2; 1, 2; 1, 1, n-r-8)$ and $T = T_{23}^{''} \cong H_{1(r)}(2; 1, 2; 1, 2, n-r-9)$, respectively.

By Lemma 1, we have $\sigma(T_{23}^{'}) = 32f_{r+2}f_{n-r-6} + 8f_{r+2}f_{n-r-7} + 24f_{r+1} \cdot f_{n-r-6}$. If $r \leq \frac{n-9}{2}$, since $32l_{n-2r-8} + 8l_{n-2r-9} - 24l_{n-2r-7} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 1$, let $T = H_{1(1)}(2; 1, 2; 1, 1, n-9) = T_{23}^{'1}$, the tree T attains the minimal σ-index and $\sigma(T_{23}^{'1}) = 88f_{n-7} + 16f_{n-8}$.

If $\frac{n-7}{2} \leq r \leq n - 11$, since $32l_{2r-n+8} - 8l_{2r-n+9} + 24l_{2r-n+7} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 10$, let $T = H_{1(n-10)}(2; 1, 2; 1, 1, 2) = T_{23}^{'2}$, the tree T attains the minimal σ-index and $\sigma(T_{23}^{'2}) = 112f_{n-8} + 72f_{n-9}$. So the minimal σ-index of tree T is $\sigma(T_{23}^{'2}) = 112f_{n-8} + 72f_{n-9}$.

Because $\sigma(T_{23}^{'}) > 280f_{n-10} + 162f_{n-11}$, so the minimal σ-index of T will be found in the case $n - r - 5 \geq 7$.

Similarly we can get that $\sigma(T_{23}^{''}) > 280f_{n-10} + 162f_{n-11}$.

Case 5.1.2.3. If $d(v, w) > 1$ and $n - r - 5 \geq 9$, by Lemmas 12 and 19, then we have $\sigma(T) \geq \sigma(T_{24})$, where $T_{24} \cong H_{1(r)}(n-r-10; 1, 2; 2, 2, 2)$ and $\sigma(T_{24}) = 216f_{r+2}f_{n-r-10} + 64f_{r+2}f_{n-r-11} + 162f_{r+1}f_{n-r-11} + 48f_{r+1}f_{n-r-12}$.

If $r \le \frac{n-13}{2}$, since $\frac{54}{5}l_{n-2r-12} + \frac{16}{5}l_{n-2r-13} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 2$, let $T = H_{1(2)}(n-12; 1,2; 2,2,2) = T_{24}^1$, the tree T attains the minimal σ-index and $\sigma(T_{24}^1) = 648f_{n-12} + 516f_{n-13} + 96f_{n-14}$.

If $r \ge \frac{n-12}{2}$, since $54l_{2r-n+12} - 16l_{2r-n+11} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 14$, let $T = H_{1(n-14)}(4; 1,2; 2,2,2) = T_{24}^2$, the tree T attains the minimal σ-index and $\sigma(T_{24}^2) = 776f_{n-11} + 372f_{n-12}$.

Therefore the minimal σ-index of T in Case 5.1.2.3. is $\sigma(T) = 776f_{n-12} + 372f_{n-13} = \sigma(T_{24}^2)$.

Case 5.1.2.4. If $d(v,w) > 1$ and $n - r - 5 \le 8$, then $\sigma(T) \ge \sigma(T_{25})$, where $T_{25} \cong H_{1(r)}(n-r-8; 1,2; 1,1,2)$ and $\sigma(T_{25}) = 96f_{r+2}f_{n-r-8} + 16f_{r+2}f_{n-r-9} + 72f_{r+1}f_{n-r-9} + 12f_{r+1}f_{n-r-10}$.

If $r \le \frac{n-11}{2}$, since $24l_{n-2r-10} + 4l_{n-2r-11} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 2$, let $T = H_{1(2)}(n-10; 1,2; 1,1,2) = T_{25}^1$, the tree T attains the minimal σ-index and $\sigma(T_{25}^1) = 288f_{n-10} + 192f_{n-11} + 24f_{n-12}$.

If $r \ge \frac{n-10}{2}$, since $96l_{2r-n+10} - 16l_{2r-n+11} - 72l_{2r-n+10} + 12l_{2r-n+11} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 12$, let $T = H_{1(n-12)}(4; 1,2; 1,1,2) = T_{25}^2$, the tree T attains the minimal σ-index and $\sigma(T_{25}^2) = 320f_{n-10} + 156f_{n-11}$.

Comparing all the σ-index of these tree, the smaller σ-index of T will be found in the case $n - r - 5 \ge 7$ and $d(v,w) = 1$ or $n - r - 5 \ge 9$ and $d(v,w) > 1$, Therefore the minimal σ-index of T in Case 5.1.2. is $\sigma(T) = 280f_{n-10} + 162f_{n-11}$, where $T = H_{1(n-12)}(2; 1,2; 2,2,2) = T_{22}^2$.

Case 5.1.3. $n(T_u) + n(P_{uv}) = 7$, by Corollary 13, Lemma 19, we know when $d(u,v) = 1$ and u is the center of $T_u = P_5$, then the σ-index of T is smaller.

Case 5.1.3.1. If $d(v,w)=1$ and $n-r-6 \ge 7$ then $T_v \cong R(2,2,n-r-11)$, so $\sigma(T) \ge \sigma(T_{26})$, where $T_{26} \cong H_{1(r)}(2; 2,2; 2,2,n-r-11)$ and $\sigma(T_{26}) = 117f_{r+2}f_{n-r-9} + 52f_{r+2}f_{n-r-10} + 81f_{r+1}f_{n-r-9}$.

If $r \le \frac{n-12}{2}$, since $117l_{n-2r-11} + 52l_{n-2r-12} - 81l_{n-2r-10} > 0$, and l_n is monotonically increasing on the natural number n, hence when $r = 2$, let $T = H_{1(2)}(2; 2,2; 2,2,n-13) = T_{26}^1$, the tree T attains the minimal σ-index and $\sigma(T_{26}^1) = 513f_{n-11} + 156f_{n-12}$.

If $\frac{n-10}{2} \le r \le n-13$, since $-117l_{2r-n+11} + 52l_{2r-n+12} - 81l_{2r-n+10} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 13$, let $T = H_{1(n-13)}(2; 2,2; 2,2,2) = T_{26}^2$, the tree T attains the minimal σ-index and $\sigma(T_{26}^2) = 455f_{n-11} + 243f_{n-12}$.

Case 5.1.3.2. If $d(v,w) > 1$ and $n - r - 6 \ge 9$, by the Corollary 13 and Lemma 19, we have $\sigma(T) \ge \sigma(T_{27})$, where $T_{27} \cong H_{1(r)}(n-r-11; 2,2; 2,2,2)$, and $\sigma(T_{27}) = 351f_{r+2}f_{n-r-11} + 104f_{r+2}f_{n-r-12} + 243f_{r+1}f_{n-r-12} + 72f_{r+1}f_{n-r-13}$.

If $r \le \frac{n-14}{2}$, since $351l_{n-2r-13}+104l_{n-2r-14}-243l_{n-2r-13}-72l_{n-2r-14} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r=2$, let $T = H_{1(2)}(n-13; 2,2; 2,2,2) = T_{27}^1$, the tree T attains the minimal σ-index and $\sigma(T_{27}^1) = 1053f_{n-13} + 798f_{n-14} + 144f_{n-15}$.

If $\frac{n-13}{2} \le r \le n-15$, since $-351l_{2r-n+13}+104l_{2r-n+14}+243l_{2r-n+13}-72l_{2r-n+14} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r=15$, let $T = H_{1(n-15)}(4; 2,2; 2,2,2) = T_{27}^2$, the tree T attains the minimal σ-index and $\sigma(T_{27}^2) = 1261f_{n-13} + 558f_{n-14}$.

Hence the minimal σ-index of T in Case 5.1.3. is $\sigma(T) = 455f_{n-11} + 243f_{n-12}$, where $T \cong H_{1(n-13)}(2; 2, 2; 2,2,2) = T_{26}^2$.

We know the minimal σ-index of T in the case $n-r-6 < 7$ is larger than the minimal σ-index of T in the case $n-r-6 \ge 7$. Therefore we only discuss the tree T in the case $n-r-6 \ge 7$ but not in the case $n-r-6 < 7$.

Case 5.1.4. If $n(T_u) + n(P_{uv}) = m (m > 8)$, then $1 \le d(u,v) \le m-4$, because $d(u) = 3$, by Lemma 7 and the path is the minimal σ-index of trees with m vertices. Hence, when $d(u,v) = 1$, $n-m-r \ge 7$, and $d(v,w) = 1$, we have $\sigma(T) \ge \sigma(T_{28})$, where $T_{28} \cong H_{1(r)}(2; 2, m-5; 2, 2, n-m-r-4)$ and $\sigma(T_{28}) = 9f_{r+2}f_m f_{n-m-r-2} + 4f_{r+2}f_m f_{n-m-r-3} + 27f_{r+1}f_{m-3}f_{n-m-r-2}$.

Next, we will discuss for the positive integer r.

Case 5.1.4.1. If $r = 1$ then $\sigma(T_{28}^1) = 18f_m f_{n-m-3} + 8f_m f_{n-m-4} + 27f_{m-3}f_{n-m-3}$.

If $m \le \frac{n-6}{2}$, since $18l_{n-2m-3} + 8l_{n-2m-4} - 27l_{n-2m} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $m = 9$, let $T = H_{1(1)}(2; 2, 4; 2, 2, n-14) = T_{28}^{11}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{11}) = 828f_{n-12} + 272f_{n-13}$.

If $m \ge \frac{n}{2}$, since $-18l_{2m-n+3} + 8l_{2m-n+4} - 27l_{2m-n} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-m = 9$, let $T = H_{1(1)}(2; 2, n-14; 2, 2, 4) = T_{28}^{12}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{12}) = 184f_{n-9} + 216f_{n-12}$.

Case 5.1.4.2. If $r = 2$ then $\sigma(T_{28}^2) = 27f_m f_{n-m-4} + 12f_m f_{n-m-5} + 54f_{m-3}f_{n-m-4}$.

If $m \le \frac{n-5}{2}$, since $27l_{n-2m-4} + 12l_{n-2m-5} - 54l_{n-2m-1} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $m = 9$, let $T = H_{1(2)}(2; 2, 4; 2, 2, n-15) = T_{28}^{21}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{21}) = 1350f_{n-13} + 408f_{n-14}$.

If $m \ge \frac{n-1}{2}$, since $27l_{2m-n+4} - 12l_{2m-n+5} + 54l_{2m-n+1} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-m = 10$, let $T = H_{1(2)}(2; 2, n-15; 2, 2, 4) = T_{28}^{22}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{22}) = 276f_{n-10} + 432f_{n-13}$.

Case 5.1.4.3. If $r \ge 3$ then $\sigma(T_{28}^3) = \sigma(P_{r-3})\sigma(T_{28}^2) + \sigma(P_{r-4})\sigma(T_{28}^1)$.

If $m \le \frac{n-6}{2}$, when $m = 9$, both $\sigma(T_{28}^{11})$ and $\sigma(T_{28}^{21})$ attain the minimal σ-index at the same time. So the minimal σ-index of T is $\sigma(T_{28}^3) = 306f_{r+2}f_{n-r-11} + 136f_{r+2}f_{n-r-12} + 216f_{r+1}f_{n-r-11}$.

If $r \leq \frac{n-14}{2}$, then $\sigma(T_{28}^3) = \frac{1}{5}(306l_{n-9} + 136l_{n-10} + 216l_{n-10}) - \frac{1}{5}(-1)^r \cdot (306l_{n-2r-13} + 136l_{n-2r-14} - 216l_{n-2r-12})$. Since $306l_{n-2r-13} + 136l_{n-2r-14} - 216l_{n-2r-12} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 4$, let $T = H_{1(4)}(2; 2, 4; 2, 2, n-17) = T_{28}^{31}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{31}) = 3528f_{n-15} + 1088f_{n-16}$.

If $r \geq \frac{n-12}{2}$, then $\sigma(T_{28}^3) = \frac{1}{5}(306l_{n-9} + 136l_{n-10} + 216l_{n-10}) - \frac{1}{5}(-1)^{n-r} \cdot (-306l_{2r-n+13} + 136l_{2r-n+14} - 216l_{2r-n+12})$. Since $-306l_{2r-n+13} + 136 \cdot l_{2r-n+14} - 216l_{2r-n+12} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 15$, let $T = H_{1(n-15)}(2; 2, 4; 2, 2, 2) = T_{28}^{32}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{32}) = 1190f_{n-13} + 648f_{n-14}$.

If $m \geq \frac{n}{2}$, when both $\sigma(T_{28}^{12})$ and $\sigma(T_{28}^{22})$ attain the minimal σ-index at the same time, then the minimal σ-index of T is $\sigma(T) = \sigma(T_{28}^3) = 92f_{r+2}f_{n-r-8} + 216f_{r+1}f_{n-r-11}$.

If $r \leq \frac{n-12}{2}$, then $\sigma(T_{28}^3) = \frac{1}{5}(92l_{n-6} + 216l_{n-10}) - \frac{1}{5}(-1)^r(92l_{n-2r-10} - 216l_{n-2r-12})$. Since $92l_{n-2r-10} - 216l_{n-2r-12} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 4$, let $T = H_{1(4)}(2; 2, n-13; 2, 2, 4) = T_{28}^{33}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{33}) = 736f_{n-12} + 1080f_{n-15}$.

If $r \geq \frac{n-10}{2}$, then $\sigma(T_{28}^{34}) = \frac{1}{5}(92l_{n-6} + 216l_{n-10}) - \frac{1}{5}(-1)^{n-r}(92l_{2r-n+10} - 216l_{2r-n+12})$. Since $92 \cdot l_{2r-n+10} - 216l_{2r-n+12} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 17$, let $T = H_{n-17}(2; 2, n-13; 2, 2, 4) = T_{28}^{34}$, the tree T attains the minimal σ-index and $\sigma(T_{28}^{34}) = 3128f_{n-15} + 1728f_{n-16}$.

We compare all the σ-indices of these trees, we get that the minimal σ-index of the tree T is $\sigma(T) = 1190f_{n-13} + 648f_{n-14} = \sigma(T_{28}^{32})$.

If $d(w, v) > 1$ and $n - m - r \geq 8$, from Lemma 19, we have $\sigma(T) \geq \sigma(T_{29})$, where $T_{29} \cong H_{1(r)}(n - m - r - 5; 2, m - 5; 2, 2, 2)$ and $\sigma(T_{29}) = 27f_{r+2}f_m f_{n-m-r-5} + 8f_{r+2}f_m f_{n-m-r-6} + 81f_{r+1}f_{m-3}f_{n-m-r-6} + 24f_{r+1} \cdot f_{m-3}f_{n-m-r-7}$.

Next, we will discuss for the positive integer r.

Case 5.1.4.4. If $r = 1$ then $\sigma(T_{29}^1) = 54f_m f_{n-m-6} + 16f_m f_{n-m-7} + 81f_{m-3}f_{n-m-7} + 24f_{m-3}f_{n-m-8}$.

If $m \leq \frac{n-7}{2}$, since $54l_{n-2m-6} + 16l_{n-2m-7} - 81l_{n-2m-4} - 24l_{n-2m-5} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $m = 9$, let $T = H_{1(1)}(n - 15; 2, 4; 2, 2, 2) = T_{29}^{11}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{11}) = 1836f_{n-15} + 1192f_{n-16} + 192f_{n-17}$.

If $m \geq \frac{n-4}{2}$, since $54l_{2m-n+6} - 16l_{2m-n+7} - 81l_{2m-n+4} + 24l_{2m-n+5} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - m = 10$, let $T = H_{1(1)}(4; 2, n - 15; 2, 2, 2) = T_{29}^{12}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{12}) = 194f_{n-10} + 186f_{n-13}$.

Case 5.1.4.5. If $r = 2$ then $\sigma(T_{29}^2) = 81f_m f_{n-m-7} + 24f_m f_{n-m-8} + 162f_{m-3}f_{n-m-8} + 48f_{m-3}f_{n-m-9}$.

If $m \le \frac{n-8}{2}$, since $81l_{n-2m-7}+24l_{n-2m-8}-162l_{n-2m-5}-48l_{n-2m-6} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $m = 9$, let $T = H_{1(2)}(n-16; 2,4; 2,2,2) = T_{29}^{21}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{21}) = 2754f_{n-16} + 2112f_{n-17} + 384f_{n-18}$.

If $m \ge \frac{n-5}{2}$, since $-81l_{2m-n+7}+24l_{2m-n+8}+162l_{2m-n+5}-48l_{2m-n+6} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-m = 11$, let $T = H_{1(2)}(4; 2, n-16; 2,2,2) = T_{29}^{22}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{22}) = 291f_{n-11} + 372f_{n-14}$.

Case 5.1.4.6. If $r \ge 3$ then $\sigma(T_{29}^{3}) = \sigma(P_{r-3})\sigma(T_{29}^{2}) + \sigma(P_{r-4})\sigma(T_{29}^{1})$.

If $m \le \frac{n-8}{2}$, when $m = 9$, both $\sigma(T_{29}^{11})$ and $\sigma(T_{29}^{21})$ attain the minimal σ-index at the same time. Then the minimal σ-index of the tree T is $\sigma(T_{29}^{3}) = 918f_{r+2}f_{n-r-14} + 272f_{r+2}f_{n-r-15} + 648f_{r+1}f_{n-r-15} + 192f_{r+1}f_{n-r-16}$.

If $r \le \frac{n-17}{2}$, then $\sigma(T_{29}^{3}) = \frac{1}{5}(918l_{n-12}+272l_{n-13}+648l_{n-14}+192l_{n-15}) - \frac{1}{5}(-1)^r(918\, l_{n-2r-16} + 272l_{n-2r-17} - 648l_{n-2r-16} - 192l_{n-2r-17})$. Since $918l_{n-2r-16} + 272l_{n-2r-17} - 648l_{n-2r-16} - 192l_{n-2r-17} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 4$, let $T = H_{1(4)}(n-18; 2,4; 2,2,2) = T_{29}^{31}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{31}) = 7344f_{n-18} + 5416f_{n-19} + 960f_{n-20}$.

If $r \ge \frac{n-17}{2}$, then $\sigma(T_{29}^{3}) = \frac{1}{5}\ (918l_{n-12} + 272l_{n-13} + 648l_{n-14} + 192 \cdot l_{n-15}) - \frac{1}{5}(-1)^{n-r}(918\, l_{2r-n+16} - 272l_{2r-n+17} - 648l_{2r-n+16} + 192\, l_{2r-n+17})$. Since $918l_{2r-n+16} - 272l_{2r-n+17} - 648l_{2r-n+16} + 192l_{2r-n+17} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r = 18$, let $T = H_{1(4)}(4; 2,4; 2,2,2) = T_{29}^{32}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{32}) = 3298f_{n-16} + 1488f_{n-17}$.

If $m \ge \frac{n-5}{2}$, when both $\sigma(T_{29}^{12})$ and $\sigma(T_{29}^{22})$ attain the minimal σ-index at the same time, so the minimal σ-index of T is $\sigma(T_{29}^{3}) = 97f_{r+2}f_{n-8-r} + 186f_{r+1}f_{n-r-11}$, where $T = T_{29}^{3} = H_{1(r)}(4; 2, n-r-13; 2,2,2)$.

If $r \le \frac{n-12}{2}$, then $\sigma(T_{29}^{3}) = \frac{1}{5}(97l_{n-6}+186l_{n-10}) - \frac{1}{5}(-1)^r(97l_{n-2r-10} - 186l_{n-2r-12})$. Since $97l_{n-2r-10} - 186l_{n-2r-12} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 4$, let $T = H_{1(4)}(4; 2, n-17; 2,2,2) = T_{29}^{33}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{33}) = 776f_{n-12} + 930f_{n-15}$.

If $r \ge \frac{n-10}{2}$, then $\sigma(T_{29}^{3}) = \frac{1}{5}(97l_{n-6}+186l_{n-10}) - \frac{1}{5}(-1)^{n-r}(97l_{2r-n+10} - 186l_{2r-n+12})$. Since $97l_{2r-n+10} - 186l_{2r-n+12} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n-r = 17$, let $T = H_{1(4)}(4; 2,4; 2,2,2) = T_{29}^{34}$, the tree T attains the minimal σ-index and $\sigma(T_{29}^{34}) = 3298f_{n-15} + 1488f_{n-16}$.

Hence the minimal σ-index of T in Case 5.1 is $\sigma(T) = 291f_{n-11} + 372f_{n-14}$, where $T = H_{1(2)}(4; 2, n-16; 2,2,2) = T_{29}^{22}$.

Case 5.2. Let $u, v, w \in V(T)$, let $d(u) = d(v) = 3$ and $d(w) = 4$, and all other vertices have degree less than 3. Let w be between u and v. Then T can be regarded as the tree obtained by identifying two end-vertices

u, v and any vertex w of degree 2 of a simple (u, w, v)-path P_{uv} with any three vertices of degree 2 of three path P', P'' and P''' respectively, the corresponding three path P', P'' and P''' are denoted by T_u, T_v and T_w, respectively. The tree T is shown in Figure 1 (f).

Case 5.2.1. If $n(T_u) + n(P_{uw}) = 5$, by Lemmas 12 and 19, when $d(w, v) = 1$, then $\sigma(T) \geq \sigma(T_{30})$, where $T_{30} \cong H_{1(2,2)}(2; 1, 1; 2, n-11)$ and $\sigma(T_{30}) = 45f_{n-6} + 48f_{n-9}$.

If $d(w, v) > 1$, by Lemma 19, then $\sigma(T) \geq \sigma(T_{31})$, where $T_{31} \cong H_{1(2,2)}(n-11; 1, 1; 2, 2)$, and $\sigma(T_{31}) = 405f_{n-11} + 324f_{n-12} + 64f_{n-13}$.

Case 5.2.2. If $n(T_u) + n(P_{uw}) = 6$ then $1 \leq d(u, w) \leq 2$. It easily find that the σ-index of T when $d(u, w) = 1$ is smaller than the σ-index of T when $d(u, w) = 2$. If $d(w, v) = 1$, by Lemma 4, then there exits the tree $T_{32} \cong H_{1(2,2)}(2; 1, 2; 2, n-12)$ such that $\sigma(T) \geq \sigma(T_{32})$ and $\sigma(T_{32}) = 72f_{n-7} + 72f_{n-10}$. If $d(w, v) > 1$, by Lemma 12 and Corollary 13, we have $\sigma(T) \geq \sigma(T_{33})$, where $T_{33} \cong H_{1(2,2)}(n-12; 1, 2; 2, 2)$ and $\sigma(T_{33}) = 648f_{n-12} + 504f_{n-13} + 96f_{n-14}$.

Case 5.2.3. If $n(T_u)+n(P_{uw}) = 7$, similarly to Case 5.2.2., the σ-index of T when $d(u, w) = 1$ is minimal. If $d(w, v) = 1$ we have $\sigma(T) \geq \sigma(T_{34})$, where $T_{34} \cong H_{1(2,r)}(2; 2, 2; 2, n-r-11)$ and $\sigma(T_{34}) = 39f_{r+2}f_{n-r-6} + 54f_{r+1}f_{n-r-9}$.

If $r \leq \frac{n-10}{2}$, then $\sigma(T_{34}) = \frac{1}{5}(39l_{n-4} + 54l_{n-8}) - \frac{1}{5}(-1)^r(39l_{n-2r-8} - 54l_{n-2r-10})$. Since $39l_{n-2r-8} - 54l_{n-2r-10} > 0$, and l_n is monotonically increasing on the natural number n, hence, when $r = 2$, let $T = H_{1(2,2)}(2; 2, 2; 2, n-13) = T^1_{34}$, the tree T attains the minimal σ-index and $\sigma(T^1_{34}) = 117f_{n-8} + 108f_{n-11}$.

If $r \geq \frac{n-8}{2}$, then $\sigma(T_{34}) = \frac{1}{5}(39l_{n-4} + 54l_{n-8}) - \frac{1}{5}(-1)^{n-r}(39l_{2r-n+8} - 54l_{2r-n+10})$. Since $39l_{2r-n+8} - 54l_{2r-n+10} < 0$, and l_n is monotonically increasing on the natural number n, hence, when $n - r = 13$, let $T = H_{1(2,n-13)}(2; 2, 2; 2, 2) = T^2_{34}$, the tree T attains the minimal σ-index and $\sigma(T^2_{34}) = 507f_{n-11} + 162f_{n-12}$.

If $d(w, v) > 1$, by Lemma 4 and Corollary 13, we have $\sigma(T) \geq \sigma(T_{35})$, where $T_{35} \cong H_{1(2,2)}(n-13; 2, 2; 2, 2)$ and $\sigma(T_{35}) = 1053f_{n-13} + 792f_{n-14} + 144f_{n-15}$.

Case 5.2.4. If $n(T_u)+n(P_{u,w}) = m(m > 7)$, without loss of generality, let $n(T_u) + n(P_{uw}) > n(T_v) + n(P_{wv})$. If $d(u, w) = 1$ and $d(w, v) = 1$, then $\sigma(T) \geq \sigma(T_{36})$, where $T_{36} \cong H_{1(2,2)}(2; 2, m-5; 2, n-m-6)$ and $\sigma(T_{36}) = 9f_m f_{n-m-1} + 36f_{m-3}f_{n-m-4}$.

If $m \leq \frac{n-1}{2}$ then $\sigma(T_{36}) = \frac{1}{5}(9l_{n-1} + 36l_{n-7}) - \frac{1}{5}(-1)^m(-27l_{n-2m-1})$, and l_n is monotonically increasing on the natural number n, hence, when $m = 9$, let $T = H_{1(2,2)}(2; 2, 4; 2, n-15) = T^1_{36}$, the tree T attains the minimal σ-index and $\sigma(T^1_{36}) = 306f_{n-10} + 288f_{n-13}$.

If $m \geq \frac{n-1}{2}$, then $\sigma(T_{36}) = \frac{1}{5}(9l_{n-1} + 36l_{n-7}) - \frac{1}{5}(-1)^{n-m}27l_{2m-n+1}$, and l_n is monotonically increasing on the natural number n, hence, when

$n-m=8$, let $T=H_{1(2,2)}(2;2,n-13;2,2)=T_{36}^2$, the tree T attains the minimal σ-index and $\sigma(T_{36}^2)=117f_{n-8}+108f_{n-11}$.

If $d(u,w)>1, d(w,v)=1$, by Lemma 12, then $\sigma(T)\geq\sigma(T_{37})$, where $T_{37}\cong H_{m-6(2,2)}(m-5;2,2;2,n-m-6)$ and $\sigma(T_{37})=81f_{m-5}f_{n-m-1}+36f_{m-6}f_{n-m-1}+108f_{m-6}f_{n-m-4}+48f_{m-7}f_{n-m-4}$.

If $m\leq\frac{n+2}{2}$, then $\sigma(T_{37})=\frac{1}{5}(81l_{n-6}+36l_{n-7}+108l_{n-10}+48l_{n-11})-\frac{1}{5}(-1)^m(-81\cdot l_{n-2m+4}+36l_{n-2m+5}+108l_{n-2m+2}-48l_{n-2m+3})$, since $-81\cdot l_{n-2m+4}+36l_{n-2m+5}+108l_{n-2m+2}-48l_{n-2m+3}<0$, and l_n is monotonically increasing on the natural number n, hence, when $m=8$, let $T=H_{2(2,2)}(3;2,2;2,n-14)=T_{37}^1$, the tree T attains the minimal σ-index and $\sigma(T_{37}^1)=315f_{n-10}+264f_{n-13}$.

If $m\geq\frac{n+5}{2}$, then $\sigma(T_{37})=\frac{1}{5}(81l_{n-5}+36l_{n-6}+108l_{n-9}+48l_{n-10})-\frac{1}{5}(-1)^{n-m}(-81\ l_{2m-n-4}-36l_{2m-n-5}+108l_{2m-n-2}+48l_{2m-n-3})$, since $-81l_{2m-n-4}-36l_{2m-n-5}+108\cdot l_{2m-n-2}+48l_{2m-n-3}>0$, and l_n is monotonically increasing on the natural number n, hence, when $n-m=8$, let $T=H_{n-14(2,2)}(n-13;2,2;2,2)=T_{37}^2$, the tree T attains the minimal σ-index and $\sigma(T_{37}^2)=1053f_{n-13}+792f_{n-14}+144f_{n-15}$.

If $d(u,w)>1, d(w,v)>1$, by Lemma 11, then we have $\sigma(T)\geq\sigma(T_{38})$, where $T_{38}\cong H_{m-6(2,2)}(n-13;2,2;2,2)$ and $\sigma(T_{38})=729f_{m-5}f_{n-m-6}+324f_{m-5}f_{n-m-7}+324f_{m-6}f_{n-m-6}+468f_{m-6}\ f_{n-m-7}+144f_{m-6}f_{n-m-8}+144f_{m-7}f_{n-m-7}+64f_{m-7}f_{n-m-8}$.

If $m\leq\frac{n-2}{2}$, since $-729l_{n-2m-1}-324l_{n-2m-2}+324l_{n-2m}+468l_{n-2m-1}-144l_{n-2m}+144l_{n-2m-2}-64l_{n-2m-1}<0$, and l_n is monotonically increasing on the natural number n, hence, when $m=9$, let $T=H_{3(2,2)}(n-13;2,2;2,2)=T_{38}^1$, the tree T attains the minimal σ-index and $\sigma(T_{38}^1)=2835f_{n-15}+2052f_{n-16}+352f_{n-17}$.

If $m\geq\frac{n}{2}$, since $729l_{2m-n+1}-324l_{2m-n+2}+324l_{2m-n}-468l_{2m-n+1}-144l_{2m-n}+144l_{2m-n+2}+64l_{2m-n+1}>0$, so by the monotonically of the Lucas number, hence, when $n-m=10$, let $T=H_{n-16(2,2)}(n-13;2,2;2,2)=T_{38}^2$, the tree T attains the minimal σ-index and $\sigma(T_{38}^2)=2835f_{n-15}+2052f_{n-16}+352f_{n-17}$.

Therefore the minimal σ-index of T in Case 5.2. is $\sigma(T)=117f_{n-8}+108f_{n-11}$, where $T\cong H_{1(2,2)}(2;2,n-13;2,2)=T_{36}^2$.

Thus we have considered all the cases for all trees with n (≥ 19) vertices and exactly six leaves, by comparing all the minimal σ-indices of the trees in all the cases, we obtain the smallest σ-index of T is $\sigma(T)=291f_{n-11}+372f_{n-14}$ and $T\cong H_{1(2)}(4;2,n-15;2,2,2)$. □

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