On the existence of a $(2,3)$-spread in $V(7,2)$

Olof Heden and Papa A. Sissokho*

January 23, 2011

Abstract

An (s,t)-spread in a finite vector space $V = V(n,q)$ is a collection $\mathcal{F}$ of t-dimensional subspaces of V with the property that every s-dimensional subspace of V is contained in exactly one member of $\mathcal{F}$. It is remarkable that no (s,t)-spreads has been found yet, except in the case $s = 1$.

In this note, the concept α-point to a $(2,3)$-spread $\mathcal{F}$ in $V = V(7,2)$ is introduced. A classical result of Thomas, applied to the vector space V, states that all points of V cannot be α-points to a given $(2,3)$-spread $\mathcal{F}$ in V. In this note, we strengthened this result by proving that every 6-dimensional subspace of V must contain at least one point that is not an α-point to a given $(2,3)$-spread of V.

1 Introduction

An (s,t)-*spread* in the finite vector space $V = V(n,q)$ over $\mathrm{GF}(q)$ is a collection $\mathcal{F}$ of t-dimensional subspaces of V with the property that every s-dimensional subspace of V is contained in exactly one member of $\mathcal{F}$. So far no (s,t)-spread, with $s > 1$, has been found, and it was conjectured by Metsch that none exists, see [1] for a survey.

If there exists an (s,t)-spread $\mathcal{F}$ in V then for any point P in V, the members of $\mathcal{F}$ that contain P induce an $(s-1,t-1)$-spread $\mathcal{F}_P$ in the quotient space V/P. A $(1,t)$-spread, or for short *spread*, $\mathcal{S}$ of V is called *geometric* if for any three members S_1, S_2 and S_3 of $\mathcal{S}$ such that $S_3 \cap \langle S_1 \cup S_2\rangle \neq \{0\}$, we have $S_3 \subseteq \langle S_1 \cup S_2\rangle$.

Thomas [2] proved the following theorem.

Theorem 1 *Given a $(2,t)$-spread $\mathcal{F}$ of $V = V(n,q)$, there exists a point P in V such that the derived $(1,t-1)$-spread $\mathcal{F}_P$ is not geometric.*

*Second author supported by grant KAW 2005.0098 from the Knut and Alice Wallenberg Foundation.

It must be remarked that geometric spreads are the spreads that are most natural and "easiest" to construct, although most of the spreads are not geometric.

The existence of $(2,3)$-spreads in $V(7,2)$ is the "first" open case for this conjecture. In this note, we give a property of $(2,3)$-spreads in $V(7,2)$, which, in this particular case, yields the result of Thomas as a corollary.

Assume that $\mathcal{F}$ is a $(2,3)$-spread in $V = V(7,2)$. As every spread in a 6-dimensional subspace U of V is of size 21, we get that every 1-dimensional subspace P, or *point*, of V is contained in 21 members of $\mathcal{F}$. As each of these 21 members of $\mathcal{F}$ contains 7 points, of which three belongs to U, it follows that U contains 45 members of $\mathcal{F}$. Similarly, we may derive that every point P in U is contained in exactly 5 of these 45 members of $\mathcal{F}$ and that every 5-dimensional subspace T of U contains exactly five members of $\mathcal{F}$.

We will say that a point P is an *α-point* to $\mathcal{F}$ if every 5-dimensional subspace T of V that contains two of the members of $\mathcal{F}$ that meet at P, has the property that all its five members from $\mathcal{F}$ will meet at the point P. From the definition of a geometric spread, it follows that in the case of $(2,3)$-spreads in $V = V(7,2)$, Theorem 1 of Thomas states that at least one point of V is not an α-point to $\mathcal{F}$.

We will show the following Theorem.

Theorem 2 *Assume that $\mathcal{F}$ is a $(2,3)$-spread in $V = V(7,2)$. Every 6-dimensional subspace of V contains at least one point which is not an α-point to $\mathcal{F}$.*

2 Proof of Theorem 2

Assume that $\mathcal{F}$ is a $(2,3)$-spread in $V = V(7,2)$. Let U be any 6-dimensional subspace of V. Assume that all points in U are α-points to $\mathcal{F}$. Then every 5-dimensional subspace T of U will contain a point P where all its five members of $\mathcal{F}$ meet. This point P will be called the *α-point of* T. Moreover, each point P of U is contained in exactly five of the members of $\mathcal{F}$ that belong to U, and hence these five members of $\mathcal{F}$ that meet the point P will all belong to the same 5-dimensional subspace T of U.

We claim that there is a 4-dimensional subspace W of U that does not contain any member of $\mathcal{F}$. To see this, just observe that every 3-dimensional subspace of a 5-dimensional subspace T of U is contained in exactly three 4-dimensional subspaces of T, and as T contains exactly five members of $\mathcal{F}$, there will be at least 16 subspaces W of dimension 4 of T that do not contain any member of $\mathcal{F}$. Such a 4-dimensional subspace W of U will be called a *poor space*.

There are three 5-dimensional subspaces T_1, T_2 and T_3 of U such that

$$W = T_1 \cap T_2 = T_1 \cap T_3 = T_2 \cap T_3 , \qquad \text{and} \qquad U = T_1 \cup T_2 \cup T_3 . \quad (1)$$

For $1 \leq i \leq 3$, let P_i be the α-point in the space T_i.

We first note that none of the points P_1, P_2, or P_3 belongs to W.

To prove this fact, assume for instance that P_1 belongs to W. Since W is a poor 4-dimensional space, each of the five members of $\mathcal{F}$ that belongs to U and contains the point P_1 meet W in two points, besides the point P_1. This leads to a contradiction since W contains 15 points and every point $Q \neq P_1$ in T_1 (and thus in W) belongs to exactly one of the five members of $\mathcal{F}$ in U that meet the point P_1.

Since $\mathcal{F}$ is a (2, 3)-spread and since the points P_i, $1 \leq i \leq 3$, do not belong to W and they are the α-points of the respective spaces T_i, we can conclude that the members of $\mathcal{F}$ that are subspaces of T_i will intersect W in a spread $\mathcal{S}_i$. Furthermore, since $\mathcal{F}$ is a (2, 3)-spread, these three spreads are mutually disjoint.

Now, let Q be any point of W. Let T_Q denote the unique 5-dimensional subspace of U, that contains the two members of $\mathcal{F}$ that meet the point Q and belong to T_1 and T_2, respectively. We note from Equation (1) that $P_1 \notin T_2 \cup T_3$ and $P_2 \notin T_1 \cup T_3$. Hence, T_Q cannot be one of the spaces T_i, $1 \leq i \leq 3$. As these are the only 5-dimensional subspaces of U that contain W, it follows that

$$\dim(T_Q \cap W) \leq 3 .$$

Moreover, since all 5-dimensional subspaces of U have a unique point where all its members of $\mathcal{F}$ meet, and as there are two members of $\mathcal{F}$ in T_Q meeting Q, we conclude that Q is the α-point of the space T_Q. This implies that the member of $\mathcal{F}$ that is a subspace of T_3 and meets the point Q must also belong to T_Q. This space will be denoted by $Z_{Q,3}$; and we define $Z_{Q,1}$ and $Z_{Q,2}$ similarly. For $1 \leq i \leq 3$, the intersection of $Z_{Q,i}$ with W is a 2-dimensional subspace which we denote by $L_{Q,i}$.

Now, the space $Z_{Q,3}$ is completely contained in T_Q and intersects W in the 2-dimensional space $L_{Q,3}$, which thus also must be a subspace of T_Q, so,

$$L_{Q,3} \subseteq T_Q \cap W = \langle L_{Q,1}, L_{Q,2} \rangle . \quad (2)$$

The last step in our proof is to show that there is at least one point Q in W, for which the above relation does not hold.

Let us assume for a moment that

$$\mathcal{S}_1 = \{ L_1, L_2, \ldots, L_5 \} \qquad \text{and} \qquad \mathcal{S}_2 = \{ L'_1, L'_2, \ldots, L'_5 \} .$$

Every member, or *line*, of $\mathcal{S}_2$ intersects three members of $\mathcal{S}_1$. Without loss of generality, we may assume that the line L'_5 does not intersect the lines

L_1 and L_2. These two lines together contain 6 points. Each of these 6 points is contained in exactly one of the lines of $\mathcal{S}_2$. As a line contains 3 points we get that there must be two lines, say L'_1 and L'_2, of $\mathcal{S}_2$ that meet both L_1 and L_2.

Let $Q = L_1 \cap L'_1$, $Q' = L_2 \cap L'_2$, $R_1 = L_1 \cap L'_2$ and $R_2 = L_2 \cap L'_1$, i.e., with the original notation

$$L_{Q,1} \cap L_{Q',2} = R_1 \qquad \text{and} \qquad L_{Q,2} \cap L_{Q',1} = R_2 \,. \tag{3}$$

Then the line L, that meets the points R_1 and R_2, satisfies the following relation

$$L = \langle R_1, R_2 \rangle = (T_Q \cap W) \cap (T_{Q'} \cap W) \,.$$

If the relation (2) holds for all points Q of W, then L will meet both the spaces $L_{Q,3}$ and $L_{Q',3}$. Note that L contains just three points, the above defined two points R_1 and R_2, and a third point R_3. So from Equation (3), we can infer that both the spaces $L_{Q,3}$ and $L_{Q',3}$ must meet L at the point R_3. This contradicts the fact that $\mathcal{S}_3$ is a spread and the proof is complete.

References

[1] K. Metsch, Bose-Burton type theorems for finite projective, Affine and Polar spaces, *Surveys in Combinatorics*, ed. by Lamb and Preece, London Mathematical Society, Lecture Notes Series 267, 1999.

[2] S. Thomas, Designs and partial geometries over finite fields, *G. Dedicata* **63** (1996), 247–253.

O. Heden, Department of Mathematics, KTH, S-100 44 Stockholm, Sweden. (olohed@math.kth.se)

P. A. Sissokho, 4520 Mathematics Department, Illinois State University, Normal, Illinois 61790-4520, U.S.A.
(psissok@ilstu.edu)