The Relationship between Series $\sum_{i=1}^{n} i^m$ and the Eulerian Numbers

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Abstract

In this paper, I study the Eulerian numbers $\langle A(m,k)\rangle_{k=1}^{m}$ and prove the relationship between $\sum_{i=1}^{n} i^m$ and $\langle A(m,k)\rangle_{k=1}^{m}$ to be $\sum_{i=1}^{n} i^m = \sum_{k=1}^{m} A(m,k)\binom{n+k}{m+1}$.

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1 Introduction

Let $\pi = a_1 a_2 \cdots a_m \in S_m$ be a permutation of length m. The descent number $d(\pi)$ of π is defined as follows

$$d(\pi) = |\{i\colon 1 \le i \le m-1,\ a_i > a_{i+1}\}|.$$

Let m, k be positive integers. For $1 \le k \le m$, the Eulerian numbers $A(m,k)$ are defined as $A(m,k) = |\{\pi \in S_m\colon d(\pi) = k-1\}|$, see Stanley [5, Chaper 1]. Clearly, $A(m,1) = A(m,m) = 1$ and $\sum_{k=1}^{m} A(m,k) = m!$.

By definition, we can easily acquire the following table.

m	$\langle A(m,k)\rangle_{k=1}^{m}$
1	1
2	1 1
3	1 4 1
4	1 11 11 1

Table 1 The Eulerian numbers for $m = 1,2,3,4$

On the other hand, we know

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) = \binom{n+1}{2},$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) = \binom{n+1}{3} + \binom{n+2}{3},$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2 = \binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4},$$

$$\sum_{i=1}^{n} i^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$= \binom{n+1}{5} + 11\binom{n+2}{5} + 11\binom{n+3}{5} + \binom{n+4}{5}.$$

If we let $\left\langle \binom{n+k}{m+1} \right\rangle_{k=1}^{m}$ be a basis, its coefficients coincide with the Eulerian numbers as in Table 1. Then, a natural question is whether $\sum_{i=1}^{n} i^m$ and $\sum_{k=1}^{m} A(m,k)\binom{n+k}{m+1}$ are equal in the general case. The aim of this paper is to confirm this result.

2 The relationship between $\sum_{i=1}^{n} i^m$ and $\langle A(m,k)\rangle_{k=1}^{m}$

For convenience, we let $A(m,k) = 0$ for $k = 0$ or $k > m$. For $\pi = a_1a_2\cdots a_m$, let $\pi' = a'_1a'_2\cdots a'_m$ where $a'_i = m+1-a_i$, and it is easy to see $d(\pi) = m-1-d(\pi')$. Then the following identity is immediate.

$$A(m,k) = A(m, m+1-k). \tag{2.1}$$

To prove our result, we need the following lemma.

Lemma 2.1 *For positive integers m, k ($k \le m+1$), we have*

$$A(m+1,k) = k\,A(m,k) + (m+2-k)A(m,k-1).$$

Proof. There are two ways we can get an $(m+1)$-permutation π with $(k-1)$ descents from an m-permutation π' by inserting the entry $m+1$ into π'. Either π' has $k-1$ descents, and the insertion of $m+1$ does not form a new descent, or π' has $k-2$ descents, and the insertion of $m+1$ does form a new descent.

In the first case, we have to put the entry $m+1$ at the end of π', or we have to insert $m+1$ between two entries that form one of the $k-1$ descents of π'. This means we have k choices for the position of $m+1$. As we have $A(m,k)$ choices for π', the first term of the right-hand side is explained.

In the second case,we have to put the entry $m+1$ at the front of π', or we have to insert $m+1$ between two entries that form one of the $(m-1)-(k-2)$ ascents of π'. This means that we have $m+2-k$ choices for the position of $m+1$. As we have $A(m,k-1)$ choices for π', the second part of the right-hand side is explained, and the lemma is proved. ∎

Theorem 2.2 *For any positive integer m,*

$$\sum_{k=1}^{m} A(m,k)\binom{x+k-1}{m} = x^m.$$

Proof. Let

$$f_m(x) = \sum_{k=1}^{m} A(m,k)\binom{x+k-1}{m}.$$

Then it is easy to see $f_1(x) = x$. For $m \geq 2$, by Lemma 2.1, we have

$$\begin{aligned}
f_m(x) &= \sum_{k=1}^{m}(k\,A(m-1,k) + (m+1-k)A(m-1,k-1))\binom{x+k-1}{m} \\
&= \sum_{k=1}^{m-1} k\,A(m-1,k)\binom{x+k-1}{m} + \sum_{k=2}^{m}(m+1-k)A(m-1,k-1)\cdot \\
&\quad \binom{x+k-1}{m} \\
&= \sum_{k=1}^{m-1} k\,A(m-1,k)\binom{x+k-1}{m} + \sum_{k=1}^{m-1}(m-k)A(m-1,k)\binom{x+k}{m} \\
&= \sum_{k=1}^{m-1} A(m-1,k)\left(k\binom{x+k-1}{m} + (m-k)\binom{x+k}{m}\right) \\
&= \sum_{k=1}^{m-1} A(m-1,k)\left(k\frac{x+k-m}{m} + (m-k)\frac{x+k}{m}\right)\binom{x+k-1}{m-1} \\
&= x\sum_{k=1}^{m-1} A(m-1,k)\binom{x+k-1}{m-1}
\end{aligned}$$

$$= xf_{m-1}(x).$$

By induction, we get the required result. This completes the proof. ∎

Now, we can prove the main result of this paper.

Theorem 2.3 *For any positive integers m, n,*

$$1^m + 2^m + \cdots + n^m = \sum_{k=1}^{m} A(m,k)\binom{n+k}{m+1}$$

Proof. From Theorem 2.2,

$$\begin{aligned}\sum_{i=1}^{n} i^m &= \sum_{i=1}^{n}\sum_{k=1}^{m} A(m,k)\binom{i+k-1}{m}\\ &= \sum_{k=1}^{m}\sum_{i=1}^{n} A(m,k)\binom{i+k-1}{m}\\ &= \sum_{k=1}^{m} A(m,k)\sum_{i=1}^{n}\binom{i+k-1}{m}.\end{aligned}$$

On the other hand, for all $k = 1, \ldots, m$, we have

$$\begin{aligned}\sum_{i=1}^{n}\binom{i+k-1}{m} &= \binom{k}{m} + \binom{k+1}{m} + \cdots + \binom{k+(n-1)}{m}\\ &= \binom{k+1}{m+1} + \binom{k+1}{m} + \cdots + \binom{k+(n-1)}{m}\\ &= \binom{k+n}{m+1},\end{aligned}$$

which implies the theorem. ∎

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