Edge-Maximal Graphs Without θ_5-Graphs

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Abstract

Let $\mathcal{G}(n;\theta_{2k+1})$ denote the class of non-bipartite graphs on n vertices containing no θ_{2k+1}-graph and $f(n;\theta_{2k+1}) = \max\{\varepsilon(G) : G \in \mathcal{G}(n;\theta_{2k+1})\}$. In this paper we determine $f(n;\theta_5)$, by proving that for $n \geq 11$, $f(n;\theta_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$. Further, the bound is best possible. Our result confirm the validity of the conjecture made in [1], "Some extermal problems in graph theory", Ph.D thesis, Curtin University of Technology, Australia (2007).

1 Introduction

For our purposes a graph G is finite, undirected and simple. We denote the vertex set of G by $V(G)$ and the edge set of G by

$E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. Let C be a cycle in a graph G, an edge in G that joins two non-adjacent vertices of C is called a chord of C. Further, a graph G has a θ_k- graph if G has a cycle C of length k and C has a chord in a graph G. Let G be a graph and $u \in V(G)$. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. The neighbor set of a vertex u of G in a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u; we write $d_H(u) = |N_H(u)|$. For vertex disjoint subgraphs H_1 and H_2 of G we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$.

For a proper subgraph H of G we write $G[V(H)]$ and $G - V(H)$ simply as $G[H]$ and $G - H$ respectively.

In this paper, we consider the Turán-type extermal problem with the θ-graph being the forbidden subgraph. Since a bipartite graph contains no odd θ-graph, we consider non-bipartite graphs. First, we recall some notation and terminology. For a positive integer n and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n; \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on n vertices, and

$$f(n; \mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \mathcal{F})\}.$$

Moreover, let $\mathcal{H}(n; \mathcal{F})$ denote the subclass of $\mathcal{G}(n; \mathcal{F})$ consisting of Hamiltonian graphs in $\mathcal{G}(n; \mathcal{F})$. We write

$$h(n; \mathcal{F}) = \max\{\mathcal{E}(H) : H \in \mathcal{H}(n; \mathcal{F})\}.$$

An important problem in extermal graph theory is that of determining the values of the functions $f(n; \mathcal{F})$ and $h(n; \mathcal{F})$. Further, characterize the extermal graphs of $\mathcal{G}(n; \mathcal{F})$ and $\mathcal{H}(n; \mathcal{F})$ where $f(n; \mathcal{F})$ and $h(n; \mathcal{F})$ are attained.

For a given C_r , the edge maximal graphs of $\mathcal{G}(n; C_r)$ have been studied by a number of authors [2, 4, 5, 6, 9]. Bondy [3] proved that a Hamiltonian graph G on n vertices without a cycle of length r has at most $\frac{1}{2}n^2$ edges with equality holding if and only if n is even and r is odd. Höggkvist et al. [8] proved that $f(n;C_r) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$ for all r. This result is sharp only for $r = 3$. Jia [10] proved that $f(n;C_5) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 9$, and he characterized the extermal graphs as well. Jia [10] conjectured that $f(n;C_{2k+1}) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$ for $n \geq 4k+2$. Recently, Bataineh [1] confirm positively the above conjecture for $n > 36k$. Moreover, Bataineh [1] conjectured that $f(n;\theta_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$.

In this paper we establish the above conjecture by proving that for $n \geq 9$,

$$f(n;\theta_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Furthermore, the bound is best possible.

2 Main Results

The following results will be used frequently in the sequel:

Theorem 2.1 ([10]) Let $G \in \mathcal{G}(n;C_5), n \geq 9$. Then

$$f(n;C_5) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3.$$

Furthermore, equality holds if and only if $G \in \mathcal{G}_5^*(n)$ for $n \geq 10$ where $\mathcal{G}_5^*(n)$ denote the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor \frac{1}{2}(n-2) \rfloor, \lceil \frac{1}{2}(n-2) \rceil}$.

Lemma 2.2 ([7]) For $5 \leq n \leq 8$, let G be a graph on n vertices containing no θ_5-graph as a subgraph. Then $\mathcal{E}(G) \leq 7$ for $n = 5$, and $\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ for $6 \leq n \leq 8$.

In the following theorem we determine the maximum number of edges of a graph with n vertices containing no θ_5-graph as a subgraph.

Theorem 2.3 For a positive integer $n \geq 9$, let G be a graph on n vertices containing no θ_5-graph as a subgraph. Then

$$\mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof: We prove the theorem by using strong mathematical induction. For $n = 9$. Let G be a graph with 9 vertices containing no θ_5-graph as a subgraph. If G is a bipartite graph, then $\mathcal{E}(G) \leq 20$ as required. Now, consider that G is a non-bipartite graph. If G has no cycle of length 5, then by Theorem 2.1 we get $\mathcal{E}(G) \leq 15$. So, we need to consider that G has a cycle of length 5. Let $x_1x_2x_3x_4x_5x_1$ be the cycle of length 5 in G, and let y_1, y_2, y_3 and y_4 be the remaining vertices in G. Define $A = G[x_1, x_2, x_3, x_4, x_5]$ and $B = G[y_1, y_2, y_3, y_4]$. Note that A contains no chord as otherwise θ_5 is produced. Thus, $\mathcal{E}(A) = 5$. Also, $\mathcal{E}(y_i, A) \leq 3$ for $i = 1, 2, 3, 4$ with equality hold only if the vertex y_i is adjacent to three consecutive vertices of A, otherwise θ_5 is produced. Define $H = \{y_i \in B : \mathcal{E}(y_i, A) = 3, i = 1, 2, 3, 4\}$. Note that $E(G[H]) = \varnothing$ and $|H| \leq 2$, otherwise G would have θ_5 as a subgraph. We consider three cases according to the value of $|H|$.

Case 1: $|H| = 0$. Note that $\mathcal{E}(B, A) \leq 8$. Thus,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(A) + \mathcal{E}(B) + \mathcal{E}(B, A) \\
&\leq 5 + 6 + 8 \\
&< \left\lfloor \frac{9^2}{4} \right\rfloor.
\end{aligned}$$

Case 2: $|H| = 1$. Suppose that $\mathcal{E}(y_1, A) = 3$, say y_1 is adjacent to x_1, x_2, x_3. Observe that, if y_1y_j is an edge in B, for some $j = 2, 3, 4$, then $\mathcal{E}(y_i, A) \leq 1$ and equality holds when y_j is adjacent to x_2, otherwise G would have θ_5 as a subgraph. Hence

$\mathcal{E}(B)+\mathcal{E}(B,A)\leq 12,$

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A)+\mathcal{E}(B)+\mathcal{E}(B,A)\\ &\leq 5+12\\ &< \left\lfloor \frac{9^2}{4}\right\rfloor.\end{aligned}$$

Case 3: $|H| = 2$. Using the same argument as in case 2, we have

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A)+\mathcal{E}(B)+\mathcal{E}(B,A)\\ &\leq 5+5+8\\ &< \left\lfloor \frac{9^2}{4}\right\rfloor.\end{aligned}$$

Now, we suppose the result holds for $9 < k < n$, so we need to prove it for n. Let G be a graph on n vertices containing no θ_5-graph as a subgraph. We now consider two cases according to parity of G.

Case 1: G is a bipartite graph. Then

$$\mathcal{E}(G)\leq \left\lfloor \frac{n^2}{4}\right\rfloor$$

Case 2: G is a non-bipartite graph. So we need to consider two subcases according to the existence of a cycle of length 5 in G.

Subcase 2.1: G contains no cycle of length 5. Then by Theorem 2.1 we get

$$\begin{aligned}\mathcal{E}(G) &\leq \left\lfloor \frac{1}{4}(n-2)^2\right\rfloor+3\\ &= \left\lfloor \frac{n^2}{4}\right\rfloor-n+4\\ &\leq \left\lfloor \frac{n^2}{4}\right\rfloor \quad \text{(Sinse } n\geq 9).\end{aligned}$$

Subcase 2.2: G has a cycle of length 5. Let $x_1x_2x_3x_4x_5x_1$ be a cycle of length 5 in G. Define $A = G[x_1, x_2, x_3, x_4, x_5]$ and $B = G - A$. Note that A has no chord, otherwise a θ_5-graph is produced. As above, define $H = \{x \in B : \mathcal{E}(x, A) = 3\}$. Note that every vertex of H is adjacent to three consecutive vertices. Further, $E(G(H)) = \varnothing$ and $|H| \leq 2$, otherwise G would have θ_5. Now, we consider 3 cases according to the value of $|H|$. Now, we consider the case $|H| = 0$ (i.e., every vertex of B adjacent to at most two vertices of A). Then by induction step and Lemma 2.2, we have $\mathcal{E}(B) \leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor$ if $n \neq 10$ and $\mathcal{E}(B) \leq 7$ if $n = 10$. Thus, for $n \neq 10$,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(B) + \mathcal{E}(B, A) + \mathcal{E}(A) \\ &\leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 2(n-5) + 5 \\ &\leq \frac{n^2 - 2n + 5}{4} \\ &< \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

And for $n = 10$,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(B) + \mathcal{E}(B, A) + \mathcal{E}(A) \\ &\leq 7 + 10 + 5 \\ &< \left\lfloor \frac{10^2}{4} \right\rfloor. \end{aligned}$$

We now consider the case $|H| = 1$ (i.e., only one vertex of B adjacent to three vertices of A, say x). Suppose that x is adjacent to x_1, x_2, x_3. Set $A_1 = G[A, x]$ and $B_1 = G - A_1$ (see Figure 1).

Let z be a vertex in B_1. If zx is an edge in B, then $\mathcal{E}(z, A) \leq 1$ and equality holds when z is adjacent to x_2, otherwise G would

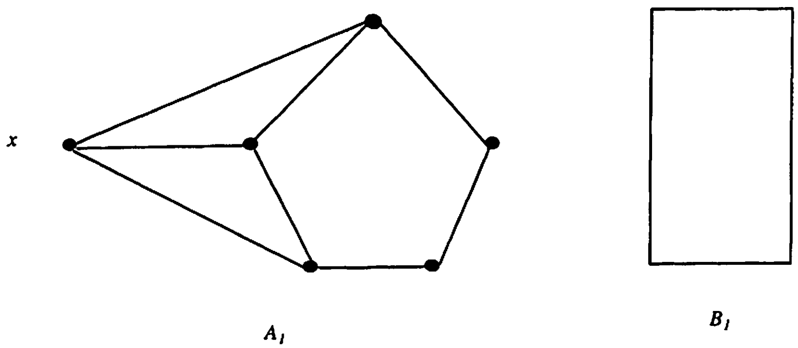

Figure 1: This Figure depicts the situation in case $|H| = 1$.

have θ_5 as a subgraph. Thus, $\mathcal{E}(A_1) + \mathcal{E}(B_1, A_1) \leq 8 + 2(n-6)$. By induction step, $\mathcal{E}(G) \leq \left\lfloor \frac{(n-6)^2}{4} \right\rfloor$ if $n \neq 10, 11$, and so

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(B_1) + \mathcal{E}(B_1, A_1) + \mathcal{E}(A_1) \\ &\leq \left\lfloor \frac{(n-6)^2}{4} \right\rfloor + 2(n-6) + 8 \\ &\leq \frac{n^2 - 4n + 20}{4} \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor . \end{aligned}$$

If $n = 11$, then by Lemma 2.2 $\mathcal{E}(B_1) \leq 7$, and so

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(B_1) + \mathcal{E}(B_1, A_1) + \mathcal{E}(A_1) \\ &\leq 7 + 10 + 8 \\ &< \left\lfloor \frac{11^2}{4} \right\rfloor . \end{aligned}$$

Similarly, if $n = 10$, then it is clear that $\mathcal{E}(B_1) \leq 6$, and so

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(B_1) + \mathcal{E}(B_1, A_1) + \mathcal{E}(A_1) \\ &\leq 6 + 8 + 8 \\ &< \left\lfloor \frac{10^2}{4} \right\rfloor . \end{aligned}$$

Finally we consider the case $|H| = 2$ (i.e., Exactly two vertices of B adjacent to three vertices of A). Suppose that, $\mathcal{E}(x, A) = \mathcal{E}(y, A) = 3$, say x is adjacent to x_1, x_2, x_3. Then y is adjacent to x_1, x_4, x_5 or adjacent to x_3, x_4, x_5. Without loss of generality, we assume that y is adjacent to x_1, x_4, x_5. Set $A_2 = G[A, x, y]$ and $B_2 = G - A_2$ (see Figure 2). Let z be a vertex in B_2.

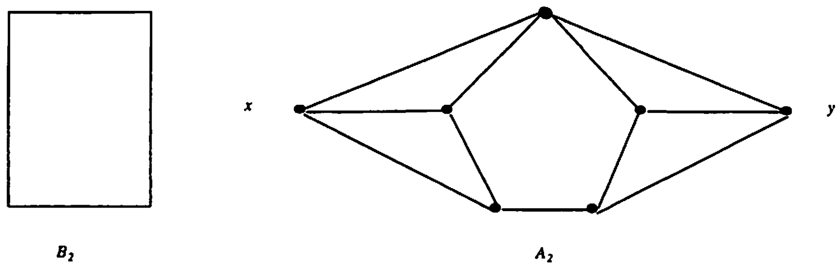

Figure 2: This figure depicts the situation in case $|H| = 2$.

As in the above, if zx or zy is an edge in B, then $\mathcal{E}(z, A) \leq 1$. Moreover, no vertex of B is adjacent to both x and y, (as otherwise if $z \in B$ is adjacent to both x and y, then zyx_1x_2xz is a θ_5, which is a contradiction). Thus, $\mathcal{E}(A_2) + \mathcal{E}(B_2, A_2) \leq 11 + 2(n-7)$. By induction step, $\mathcal{E}(G) \leq \left\lfloor \frac{(n-6)^2}{4} \right\rfloor$ if $n \neq 10, 11, 12$, and so

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(B_2) + \mathcal{E}(B_2, A_2) + \mathcal{E}(A_2) \\
&\leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor + 2(n-7) + 11 \\
&\leq \frac{n^2 - 6n + 37}{4} \\
&\leq \left\lfloor \frac{n^2}{4} \right\rfloor .
\end{aligned}$$

For $n = 10, 11, 12$, we can use the same arguments as above, by taking into account that for $n = 10, \mathcal{E}(B_2) \leq 3$, for $n =$

$11, \mathcal{E}(B_2) \leq 6$ and for $n = 12, \mathcal{E}(B_2) \leq 7$. This completes the proof.

We now determine $f(n;\theta_5)$ and $h(n;\theta_5)$. We begin with the following construction: For odd n, let G_1 be the graph obtained from $K_{\frac{1}{2}(n-1),\frac{1}{2}(n-1)}$ by subdividing an edge. For even $n \geq 8$, let u, v be two vertices in the same bipartition set of $K_{\frac{n}{2},\frac{n}{2}}$. Let G_2 be the graph obtained from $K_{\frac{n}{2},\frac{n}{2}} + uv$ by deleting $\frac{1}{2}n$ edges incident to u or v such that $N_{G_2}(u) \cap N_{G_2}(v) = \varnothing, d_{G_2}(u) + d_{G_2}(v) = \frac{1}{2}n + 2$, $d_{G_2}(u) \geq 2$ and $d_{G_2}(v) \geq 2$. Note that G_1 and G_2 are Hamiltonian graphs containing no θ_5. Examples of the graphs G_1 and G_2 for $n = 7$ and $n = 8$, respectively, are shown below in Figure 3.

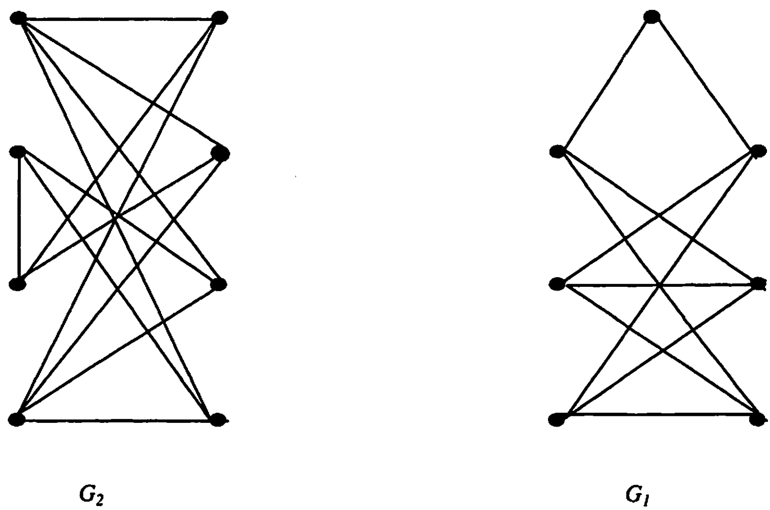

Figure 3: G_1 and G_2 represent examples of the above construction in cases $n = 7, 8$.

Now, in the following theorem we determine $f(n;\theta_5)$.

Theorem 2.4. Let $G \in \mathcal{G}(n;\theta_5)$. Then,

$$f(n;\theta_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$$

for $n \geq 11$. Furthermore, the bound is best possible.

Proof: Let $G \in \mathcal{G}(n;\theta_5)$. If G has no cycle of length 5, then by Theorem 2.1 we have

$$f(n;5) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$$

for $n \geq 9$. Thus,

$$\begin{aligned} \mathcal{E}(G) &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3 \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1. \end{aligned}$$

So we need to consider the case when G has a cycle of length 5. Let $x_1x_2x_3x_4x_5x_1$ be a cycle of length 5 in G and $A = G[x_1,x_2,x_3,x_4,x_5]$. Define $R = G - A$. Observe that A has no chord, as otherwise G would have θ_5, so $\mathcal{E}(A) = 5$. We want to find $\mathcal{E}(R, A)$. Now, as in the argument of proof of Theorem 2.2, any vertex $x \in R$, x is adjacent to A by at most 3 edges. Moreover, if x is adjacent to 3 vertices of A, then they must be consecutive. Now, define $H = \{x \in R : \mathcal{E}(x, A) = 3\}$. Observe that $E(G[H]) = \varnothing$ and $|H| \leq 2$, otherwise G would have θ_5. Now, we consider 3 cases according to the value of $|H|$.

Case 1: $|H| = 0$. Then any vertex in R has at most two neighbors on A. And so, $\mathcal{E}(R, A) \leq 2(n-5)$. By Theorem 2.3, we have

$$\mathcal{E}(R) \leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor.$$

But,

$$\mathcal{E}(G) = \mathcal{E}(R) + \mathcal{E}(R, A) + \mathcal{E}(A)$$

Hence,

$$\begin{aligned}
\mathcal{E}(G) &\leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 2n - 10 + 5 \\
&\leq \left\lfloor \frac{n^2 - 10n + 25 + 8n - 20}{4} \right\rfloor \\
&= \left\lfloor \frac{n^2 - 2n + 5}{4} \right\rfloor \\
&= \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.
\end{aligned}$$

Thus, we have

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Therefore,

$$\begin{aligned}
f(n;\theta_5) &= \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;\theta_5)\} \\
&\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.
\end{aligned}$$

Case 2: $|H| = 1$. Let $H = \{u\}$. Note that, as in Case 2 of Theorem 2.1, every vertex in $N_R(u)$ has at most one neighbor on A. Define $R_1 = R - H$. Then by Lemma 2.2 and Theorem 2.3 $\mathcal{E}(R_1) \leq \left\lfloor \frac{(n-6)^2}{4} \right\rfloor$ if $n \neq 11$ and $\mathcal{E}(R_1) \leq 7$ if $n = 11$. Observe that any vertex in $R_1 - N_{R_1}(u)$ has at most two neighbors on A. Thus,

$$\mathcal{E}(R_1, A) \leq 2(n-6) - |N_{R_1}(u)|.$$

and

$$\mathcal{E}(R_1, H) \leq |N_{R_1}(u)|.$$

Hence,

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}(A)+\mathcal{E}(\{u\})+\mathcal{E}(R_1)+\mathcal{E}(R_1,A)+\mathcal{E}(R_1,\{u\})\\ &\quad +\mathcal{E}(A,\{u\})\\ &\leq 5+0+\left\lfloor\frac{(n-6)^2}{4}\right\rfloor+2n-12-|N_{R_1}(u)|\\ &\quad +|N_{R_1}(u)|+3\\ &= \left\lfloor\frac{(n-6)^2}{4}\right\rfloor+2n-4\\ &\leq \left\lfloor\frac{n^2-12n+36+8n-16}{4}\right\rfloor\\ &= \left\lfloor\frac{n^2-4n+20}{4}\right\rfloor\\ &\leq \left\lfloor\frac{(n-1)^2}{4}\right\rfloor+1.\end{aligned}$$

if $n \neq 11$. For $n = 11$,

$$\begin{aligned}\mathcal{E}(G) &\leq 7+10+8\\ &= 25\\ &< \left\lfloor\frac{10^2}{4}\right\rfloor+1\end{aligned}$$

Case 3: $|H| = 2$. Let $H = \{u, w\}$ and $R_1 = R - H$. If $|N_A(u) \cap N_A(w)| \geq 2$, then θ_5 is produced . So we have

$$|N_A(u) \cap N_A(w)| = 1.$$

Thus, without loss of generality, we assume that $N_A(u) = \{x_1, x_2, x_3\}$ and $N_A(w) = \{x_1, x_4, x_5\}$. By Lemma 2.2 and Theorem 2.3 $\mathcal{E}(R_1) \leq \left\lfloor\frac{(n-7)^2}{4}\right\rfloor$ if $n \neq 11, 12$ and $\mathcal{E}(R_1) \leq 7$ if $n = 12$. Moreover, it is easy to see that $\mathcal{E}(R_1) \leq 6$ if $n = 11$. Note that any vertex in R_1 has at most two neighbors on A. Thus,

$$\mathcal{E}(R_1, A) \leq 2(n-7)$$

Now, we want to find $\mathcal{E}(R_1, H)$. Observe that $|N_{R_1}(u) \cap N_{R_1}(w)| = 0$, otherwise G would have θ_5. Also, $|N_{R_1}(u)| + |N_{R_1}(w)| \leq n-7$. Note that every vertex in $N_{R_1}(u)$ has at most one neighbor on A. Similarly for the vertices in $N_{R_1}(w)$. Thus,

$$\mathcal{E}(R_1, A) \leq 2(n-7) - |N_{R_1}(u)| - |N_{R_1}(w)|$$

and

$$\mathcal{E}(R_1, H) = |N_{R_1}(u)| + |N_{R_1}(w)|.$$

Hence,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(R_1) + \mathcal{E}(A) + \mathcal{E}(H) + \mathcal{E}(R_1, A) + \mathcal{E}(R_1, H) \\
&\quad + \mathcal{E}(A, H) \\
&\leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor + 5 + 0 + 2(n-7) - |N_{R_1}(u)| \\
&\quad - |N_{R_1}(w)| + |N_{R_1}(u)| + |N_{R_1}(w)| + 6 \\
&\leq \left\lfloor \frac{(n-7)^2}{4} \right\rfloor + 2n - 14 + 11 \\
&\leq \left\lfloor \frac{n^2 - 14n + 49 + 8n - 12}{4} \right\rfloor \\
&= \left\lfloor \frac{n^2 - 6n + 37}{4} \right\rfloor \\
&\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.
\end{aligned}$$

if $n \neq 11, 12$. For $n = 11, 12$, we use the same arguments as above, by taking into account that for $n = 11, \mathcal{E}(R_1) \leq 6$, and for $n = 12, \mathcal{E}(R_1) \leq 7$. Note that the bound is achievable by G_1 and G_2 in the above construction. This completes the proof of the theorem.

In the following theorem we determine $h(n; \theta_5)$.

Theorem 2.5. Let $G \in \mathcal{H}(n;\theta_5)$. Then

$$h(n;\theta_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1$$

for $n \geq 11$. Further, the bound is best possible.
Proof: We know that $\mathcal{H}(n;\theta_5) \subseteq \mathcal{G}(n;\theta_5)$. So,

$$\begin{aligned} h(n;\theta_5) &\leq f(n;\theta_5) \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1. \end{aligned}$$

Observe that the graphs G_1 and G_2 are Hamiltonian. Thus,

$$h(n;\theta_5) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Therefore,

$$h(n;\theta_5) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

References

[1] M. Bataineh, "*Some extermal problems in graph theory*", Ph.D thesis, Curtin University of Technology, Australia (2007).

[2] B. Bollobás, "*Extermal Graph Theory*", Academic Press, London (1978).

[3] J. A. Bondy, *Pancyclic Graphs*, J. Combinatorial Theory Ser B 11, 80- 84 (1971).

[4] J.A. Bondy, *Large cycle in graphs*, Discrete Mathematics 1, 121- 132 (1971).

[5] L. Caccetta and R. Jia, *Edge maximal non-bipartite graphs without odd cycles of prescribed length*, Graphs and Combinatorics, 18, 75-92 (2002).

[6] L. Caccetta and K. Vijayan, *Maximal cycles in graphs*, Discrete Mathematics 98, 1-7 (1991).

[7] I.Y. Al-Shboul, "*On edge maximal non-bipartite graphs without* θ_{2k+1}", M. Sc. thesis, Yarmouk university (2010).

[8] R. Häggkvist, R.J. Faudree and R.H Schelp, *Pancyclic graphs– connected Ramsey number*, Ars Combin. 11, 37-49 (1981).

[9] G.R.T. Hendry and S. Brandt, *An extermal problem for cycles in Hamiltonian graphs*, Graphs and Combinatorics. 11, 255-262 (1995).

[10] R. Jia, "*Some extermal problems in graph theory*", Ph.D thesis, Curtin University of Technology, Australia (1998).