On the sum-connectivity spectral radius and sum-connectivity Estrada index of graphs *

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Abstract

Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the sum-connectivity matrix of a graph G. The sum-connectivity spectral radius of G is the largest eigenvalue of its sum-connectivity matrix, and the sum-connectivity Estrada index of G, is defined as $SEE(G) = \sum_{i=1}^{n} e^{\mu_i}$. In this paper, we obtain some results about the sum-connectivity spectral radius of graphs. In addition, we give some upper and lower bounds on sum-connectivity Estrada index of graph G, as well as some relations between SEE and sum-connectivity energy. Moreover, we characterize that the star has maximum sum-connectivity Estrada index among trees on n vertices.

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1. Introduction

In this paper we are concerned with simple finite graphs, without loops, multiple or directed edges. If G is such a graph with n-vertices and m-edges, then G will be called (n, m)-graph. Let G be such a graph, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices v_i and v_j of G are adjacent, then

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we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of the vertices of the vertices adjacent to v_i.

In 1975 Milan Randić [1] invented a molecular structure descriptor defined as

$$R(G) = \sum_{i \sim j} \frac{1}{\sqrt{d_i d_j}}$$

where the sum run over all pairs of adjacent vertices of the underlying (molecular) graph. Nowadays, $R(G)$ is referred to as the Randić index; for detail see[2,3]. The Randić-index-concept suggest that it is purposeful to associate to the graph G a symmetric square matrix $\mathbf{R}(G)$. The Randić matrix $\mathbf{R}(G) = (r_{ij})_{n \times n}$ is defined as [4,5]

$$r_{ij} = \begin{cases} \frac{1}{\sqrt{d_i d_j}}, & if\ v_i \sim v_j \\ 0, & if\ the\ vertices\ v_i\ and\ v_j\ are\ not\ adjacent \\ 0, & if\ i = j. \end{cases}$$

In 2009, a closely related variant of the Randić connectivity index called the sum-connectivity index was introduced by B. Zhou and N. Trinajstić [6]. It was defined as

$$S(G) = \sum_{i \sim j} \frac{1}{\sqrt{d_i + d_j}}.$$

For more mathematical properties of the sum-connectivity index, one may refer to [7,8,9].

The sum-connectivity matrix $\mathbf{S} = \mathbf{S}(G)$ of the graph G is defined as [10]

$$s_{ij} = \begin{cases} \frac{1}{\sqrt{d_i + d_j}}, & if\ v_i \sim v_j \\ 0, & otherwise. \end{cases}$$

Since $\mathbf{S}$ is real symmetric matrices, their eigenvalues are real numbers. Denote the eigenvalues of the sum-connectivity matrix $\mathbf{S} = \mathbf{S}(G)$ by $\mu_1, \mu_2, \ldots,$ μ_n and label them in non-increasing order. The greatest eigenvalue μ_1 is called the sum-connectivity spectral radius of the graph G. The multiset $Sp_S = Sp_S(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ will be called the S-spectrum of the graph G. In addition, $\phi_S(G, \lambda) = det(\lambda I_n - \mathbf{S})$ will be referred to as the S-characteristic polynomial of G.

The sum-connectivity energy of the graph G, denoted by $SE(G)$, defined as [10]

$$SE(G) = \sum_{i=1}^{n} |\mu_i|.$$

B. Zhou *et al.* [10] obtained upper bounds for the maximum eigenvalue of sum-connectivity matrix of a graph G. In the second section of this paper, we get lower bounds for sum-connectivity spectral radius of a graph G. In the section 3, a new index will be defined, namely sum-connectivity Estrada index and then will be obtained lower and upper bounds for this new index. We determine that the star has maximum sum-connectivity Estrada index among trees on n vertices.

2. Lower bounds for sum-connectivity spectral radius of a graph

In order to obtain several lower bounds for the sum-connectivity spectral radius of a connected graph, we need some auxiliary definitions and lemmas.

Definition 2.1 Let G be a graph with n vertices. Then the sum of the i-th row of sum-connectivity matrix $\mathbf{S}(G)$ is defined as

$$S_i = \sum_{j=1}^{n} s_{ij}.$$

Lemma 2.2 (Rayleigh-Ritz)[11] If A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, then for any $X \in \mathbb{R}^n (X \neq 0)$,

$$X^{\mathrm{T}} A X \leq \lambda_1 X^{\mathrm{T}} X.$$

Equality holds if and only if X is an eigenvector of A corresponding to the largest eigenvalue λ_1.

Lemma 2.3 [12] Let A be a nonnegative symmetric matrix and X be a unit vector of $\mathbb{R}^n$. If $\lambda_1(A) = X^{\mathrm{T}} A X$, then $AX = \lambda_1(A) X$.

Lemma 2.4 Let G be a bipartite graph with sum-connectivity matrix $\mathbf{S}$. If μ is an eigenvalue of $\mathbf{S}$ with multiplicity k, then $-\mu$ is also an eigenvalue of $\mathbf{S}$ with multiplicity k.

Proof. Let $V(G) = X \bigcup Y$ be a bipartition of G. It is not hard to see that we may write $\mathbf{S} = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}$. Let $(x, y)^{\mathrm{T}}$ be an eigenvector of $\mathbf{S}$ corresponding to μ. So we get the equation

$$\begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mu \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then it is easy to verify that $(x, -y)^{\mathrm{T}}$ is an eigenvector of $\mathbf{S}$ with eigenvalue $-\mu$. It is also clear that if we have k linearly independent eigenvector for μ, then the above construction will produce k linearly independent eigenvectors for $-\mu$. Thus μ and $-\mu$ are eigenvalues with the same multiplicity. This completes the proof. □

Lemma 2.5 Let G be a simple connected graph with n vertices. Then

$$\mu_1(G) \geq \frac{2S(G)}{n}$$

with equality holds if and only if $S_1 = S_2 = \cdots = S_n$.

Proof. Let $X = \frac{1}{\sqrt{n}}(1,1,\ldots,1)^{\mathrm{T}}$. By Lemma 2.2, we obtain

$$\begin{aligned}\mu_1(G) \geq & \frac{X^{\mathrm{T}}\mathbf{S}X}{X^{\mathrm{T}}X} \\ = & \frac{1}{n}(S_1, S_2, \ldots, S_n)(1,1,\ldots,1)^{\mathrm{T}} \\ = & \frac{1}{n}\sum_{i=1}^{n} S_i \\ = & \frac{2}{n}S(G).\end{aligned}$$

Now we suppose that the equality holds. By Lemma 2.3, we get $\mathbf{S}X = \mu_1(G)X$. This implies that $S_i = \mu_1(G)$ for all i. Conversely, if $S_1 = S_2 = \cdots = S_n = a$ (a is a constant), then $\mathbf{S}X = aX$. It is known that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. Hence $\mu_1(G) = a$. Since $\sum\limits_{i=1}^{n} S_i = na = 2S(G)$, we have $\mu_1(G) = \frac{2}{n}S(G)$. This completes the proof.□

Now we define a sequence

$$\sigma_i^{(1)}, \sigma_i^{(2)}, \ldots, \sigma_i^{(l)}, \ldots$$

where $\sigma_i^{(1)} = a_i \in \mathbb{R}^+$, and $\sigma_i^{(l)} = \sum\limits_{i\sim j} s_{ij}\sigma_j^{(l-1)}$ for each $l \geq 2$, $l \in \mathbb{Z}$.

Theorem 2.6 Let G be a connected graph with n vertices. Then

$$\mu_1(G) \geq \max_{l}\max_{a_i}\sqrt{\frac{\sum\limits_{i=1}^{n}\left(\sigma_i^{(l+1)}\right)^2}{\sum\limits_{i=1}^{n}\left(\sigma_i^{(l)}\right)^2}}.$$

The equality holds if and only if

$$\frac{\sigma_1^{(l+1)}}{\sigma_1^{(l)}} = \frac{\sigma_2^{(l+1)}}{\sigma_2^{(l)}} = \cdots = \frac{\sigma_n^{(l+1)}}{\sigma_n^{(l)}}$$

or G is a bipartite graph with the partition $\{v_1, v_2, \ldots, v_{n_1}\}\bigcup\{v_{n_1+1}, v_{n_1+2}, \ldots, v_n\}$, such that

$$\frac{\sigma_1^{(l+1)}}{\sigma_1^{(l)}} = \frac{\sigma_2^{(l+1)}}{\sigma_2^{(l)}} = \cdots = \frac{\sigma_{n_1}^{(l+1)}}{\sigma_{n_1}^{(l)}}, \frac{\sigma_{n_1+1}^{(l+1)}}{\sigma_{n_1+1}^{(l)}} = \frac{\sigma_{n_1+2}^{(l+1)}}{\sigma_{n_1+2}^{(l)}} = \cdots = \frac{\sigma_n^{(l+1)}}{\sigma_n^{(l)}}.$$

Proof. By Rayleigh quotient, we have

$$\mu_1^2(G) = \mu_1(\mathbf{S}^2(G)) = \max_{X\neq 0}\frac{X^{\mathrm{T}}\mathbf{S}^2(G)X}{X^{\mathrm{T}}X}.$$

Now we take the positive vector

$$\alpha = \left(\sigma_1^{(l)}, \sigma_2^{(l)}, \ldots, \sigma_n^{(l)}\right)^{\mathrm{T}}.$$

Since

$$\begin{aligned}\mathbf{S}\alpha =&\Big(\sum_{j=1}^{n} s_{1j}\sigma_j^{(l)}, \sum_{j=1}^{n} s_{2j}\sigma_j^{(l)}, \ldots, \sum_{j=1}^{n} s_{nj}\sigma_j^{(l)}\Big)^{\mathrm{T}} \\ =&\left(\sigma_1^{(l+1)}, \sigma_2^{(l+1)}, \ldots, \sigma_n^{(l+1)}\right)^{\mathrm{T}}\end{aligned}$$

and

$$\alpha^{\mathrm{T}}\alpha = \sum_{i=1}^{n} \left(\sigma_i^{(l)}\right)^2,$$

we get

$$\mu_1(G) = \sqrt{\max_{X \neq 0} \frac{X^{\mathrm{T}}\mathbf{S}^2 X}{X^{\mathrm{T}}X}} \geq \sqrt{\frac{\sum_{i=1}^{n} \left(\sigma_i^{(l+1)}\right)^2}{\sum_{i=1}^{n} \left(\sigma_i^{(l)}\right)^2}}.$$

Now we assume that the equality holds. By Lemma 2.2, we have α is a positive eigenvector of $\mathbf{S}^2$ corresponding to $\mu_1(\mathbf{S}^2)$. Moreover, if the multiplicity of $\mu_1(\mathbf{S}^2)$ is one, then by Perron-Frobenius theorem, α is an eigenvector of $\mathbf{S}$ corresponding to $\mu_1(G)$. Therefore $\mathbf{S}\alpha = \mu_1(G)\alpha$, which implies $\sigma_i^{(l+1)} = \mu_1(G)\sigma_i^{(l)}$ for all i. If the multiplicity of $\mu_1(\mathbf{S}^2)$ is two, then $-\mu_1(\mathbf{S})$ is also an eigenvalue of $\mathbf{S}$. Then G is a connected bipartite graph. Without loss of generality, we assume that $\mathbf{S} = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix}$, where B is an $n_1 \times (n - n_1)$ matrix.

Let $Y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ be a positive eigenvector of $\mathbf{S}$ corresponding to $\mu_1(G)$. Let $\alpha = (\alpha_1, \alpha_2)^{\mathrm{T}}$ and $Y = (Y_1, Y_2)^{\mathrm{T}}$ where $\alpha_1 = (\sigma_1^{(l)}, \sigma_2^{(l)}, \ldots, \sigma_{n_1}^{(l)})^{\mathrm{T}}$, $\alpha_2 = (\sigma_{n_1+1}^{(l)}, \sigma_{n_1+2}^{(l)}, \ldots, \sigma_n^{(l)})^{\mathrm{T}}$, $Y_1 = (y_1, y_2, \ldots, y_{n_1})^{\mathrm{T}}$ and $Y_2 = (y_{n_1+1}, y_{n_1+2}, \ldots, y_n)^{\mathrm{T}}$. Since

$$\mathbf{S}^2 = \begin{pmatrix} BB^{\mathrm{T}} & 0 \\ 0 & B^{\mathrm{T}}B \end{pmatrix}$$

we have

$$BB^{\mathrm{T}}\alpha_1 = \mu_1(\mathbf{S}^2)\alpha_1, B^{\mathrm{T}}B\alpha_2 = \mu_1(\mathbf{S}^2)\alpha_2$$

and

$$BB^{\mathrm{T}}Y_1 = \mu_1(\mathbf{S}^2)Y_1, B^{\mathrm{T}}BY_2 = \mu_1(\mathbf{S}^2)Y_2.$$

Note that BB^{T} and $B^{\mathrm{T}}B$ are similar matrices. Then $\mu_1(\mathbf{S}^2)$ is the eigenvalue both of the matrices BB^{T} and $B^{\mathrm{T}}B$ with multiplicity one.

Hence $Y_1 = a\alpha_1 (a \neq 0)$ and $Y_2 = b\alpha_2 (b \neq 0)$. Now it follows from $\mathbf{S}Y = \mu_1(G)Y$, thus we have

$$\frac{\sigma_1^{(l+1)}}{\sigma_1^{(l)}} = \cdots = \frac{\sigma_{n_1}^{(l+1)}}{\sigma_{n_1}^{(l)}} = \frac{a}{b}\mu_1(G)$$

and

$$\frac{\sigma_{n_1+1}^{(l+1)}}{\sigma_{n_1+1}^{(l)}} = \cdots = \frac{\sigma_n^{(l+1)}}{\sigma_n^{(l)}} = \frac{b}{a}\mu_1(G).$$

Conversely, considering the similar method in the proof of Theorem 2 in [13], it can be easily seen that the result holds. This completes the proof.□

By setting $a_i = 1$ and $l = 1$ in Theorem 2.6, we have the following result.

Corollary 2.7 Let G be a connected graph with n vertices. Then

$$\mu_1(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} S_i^2}$$

with equality holds if and only if $S_1 = S_2 = \cdots = S_n$.

3. Sum-connectivity Estrada index of graphs

In this section we will mainly introduce and investigate sum-connectivity Estrada index of graph G and also present upper and lower bounds for it. Moreover, some bounds for the sum-connectivity Estrada index involving sum-connectivity energy are also put forward.

We first recall that the Estrada index of a graph G is defined in [14] by

$$EE = EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of the adjacency matrix $A(G)$ of G. Denoting by $M_k = M_k(G)$ the k-th moment of the graph G [15]

$$M_k = M_k(G) = \sum_{i=1}^{n} \lambda_i^k.$$

Recalling the power series expansion of e^x, we have

$$EE = \sum_{k=0}^{\infty} \frac{M_k}{k!}.$$

Estrada index of graphs has an important role in chemistry and physics. For detailed information we refer to the reader [15,16]. In addition, recently much work on the Estrada index of the graph appeared also in the mathematical literature [17,18,24-28].

Now we introduce the sum-connectivity Estrada index of graph G.

Definition 3.1 If G is an (n, m)-graph, then the sum-connectivity Estrada index of graph G, denoted by $SEE(G)$, is equal to

$$SEE = SEE(G) = \sum_{i=1}^{n} e^{\mu_i}$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the S-eigenvalues of G.

Let

$$N_k = \sum_{i=1}^{n} \mu_i^k.$$

Then

$$SEE = \sum_{k\geq 0}^{\infty} \frac{N_k}{k!}.$$

Recall that the harmonic index of G is defined as [19] $H(G) = \sum_{i\sim j} \frac{2}{d_i+d_j}$. The properties of the harmonic index may be found in [20-22].

Theorem 3.2 Let G be a graph with n vertices. Then for any integer $t \geq 3$,

$$\begin{aligned}\sqrt{n^2 + 2H(G) + \sum_{k=3}^{t} \frac{2^k N_k}{k!}} \leq & SEE(G) \\ \leq & n - 1 - \sqrt{H(G)} + e^{\sqrt{H(G)}} \\ & + \sum_{k=3}^{t} \frac{N_k - \left(\sqrt{H(G)}\right)^k}{k!}.\end{aligned}$$

Equality holds in both sides if and only if $G \cong \overline{K_n}$.

Proof. Lower bound: By the definition of the sum-connectivity Estrada index, we have

$$SEE^2(G) = \sum_{i=1}^{n} e^{2\mu_i} + 2 \sum_{1\leq i<j\leq n} e^{\mu_i} e^{\mu_j}.$$

By the arithmetic-geometric mean inequality and using the fact that $\sum_{i=1}^{n} \mu_i =$

0, we get

$$
\begin{aligned}
2\sum_{1\le i<j\le n} e^{\mu_i}e^{\mu_j} \ge& n(n-1)\Big[\big(\prod_{i=1}^{n} e^{\mu_i}\big)^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}\\
=& n(n-1)\Big(e^{\sum_{i=1}^{n}\mu_i}\Big)^{\frac{2}{n}}\\
=& n(n-1).
\end{aligned}
$$

By means of a power series expansion of e^x, and $N_0 = n$, $N_1 = 0$, $N_2 = H(G)$, we have

$$
\begin{aligned}
\sum_{i=1}^{n} e^{2\mu_i} =& \sum_{i=1}^{n}\sum_{k\ge 0}\frac{(2\mu_i)^k}{k!}\\
\ge& \sum_{k=0}^{t}\frac{2^k N_k}{k!}\\
=& n + 2H(G) + \sum_{k=3}^{t}\frac{2^k N_k}{k!}.
\end{aligned}
$$

Therefore

$$
SEE(G) \ge \sqrt{n^2 + 2H(G) + \sum_{k=3}^{t}\frac{2^k N_k}{k!}}.
$$

Upper bound: We will use the following inequality:
For nonnegative $a_1, a_2, \ldots, a_n$ and integer $k \ge 2$,

$$
\sum_{i=1}^{n} a_i^k \le \Big(\sum_{i=1}^{n} a_i^2\Big)^{\frac{k}{2}}.
$$

Note that $\sum_{i=1}^{n} \mu_i^2 = H(G)$. We have

$$
\begin{aligned}
SEE(G) =& \sum_{k=0}^{t} \frac{N_k}{k!} + \sum_{k\geq t+1} \frac{1}{k!} \sum_{i=1}^{n} \mu_i^k \\
\leq& \sum_{k=0}^{t} \frac{N_k}{k!} + \sum_{k\geq t+1} \frac{1}{k!} \sum_{i=1}^{n} |\mu_i|^k \\
\leq& \sum_{k=0}^{t} \frac{N_k}{k!} + \sum_{k\geq t+1} \frac{1}{k!} \Big(\sum_{i=1}^{n} \mu_i^2\Big)^{\frac{k}{2}} \\
=& \sum_{k=0}^{t} \frac{N_k}{k!} + \sum_{k\geq t+1} \frac{\big(\sqrt{H(G)}\big)^k}{k!} \\
=& \sum_{k=0}^{t} \frac{N_k}{k!} + e^{\sqrt{H(G)}} - \sum_{k=0}^{t} \frac{\big(\sqrt{H(G)}\big)^k}{k!} \\
=& n - 1 - \sqrt{H(G)} + e^{\sqrt{H(G)}} + \sum_{k=3}^{t} \frac{N_k - \big(\sqrt{H(G)}\big)^k}{k!}.
\end{aligned}
$$

Now we assume that the equality holds. Then $\mu_1 = \mu_2 = \cdots = \mu_n = 0$. This happens only in the case of the graph $\overline{K_n}$. This completes the proof.□

Remark 3.3 Since $\sum_{k\geq 3} \frac{2^k N_k}{k!} \geq 0$, thus $SEE(G) \geq \sqrt{n^2 + 2H(G)}$. Note that $H(G) \geq \frac{2m}{n}$ (see [23]). Hence

$$SEE(G) \geq \sqrt{n^2 + \frac{4m}{n}}.$$

In the following, we give lower and upper bound for SEE of bipartite graphs.

Theorem 3.4 Let G be a connected bipartite graph with $n \geq 2$ vertices. If n_0 is the multiplicity of its eigenvalue zero of G, then

$$n_0 + (n - n_0)cosh\Big(\sqrt{\frac{H(G)}{n - n_0}}\Big) \leq SEE(G) \leq n - 2 + 2cosh\Big(\sqrt{\frac{H(G)}{2}}\Big),$$

where $cosh$ stands for the hyperbolic cosine $\big[cosh(x) = (e^x + e^{-x})/2\big]$. Moreover, the equality of left-hand side holds if and only if all positive eigenvalues are equal, while the equality of right-hand side holds if and only if G is a complete bipartite graph.

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the S-eigenvalues of G. Since G is a bipartite graph, using Lemma 2.4, we have $\mu_1 = -\mu_n$.

Lower bound: We will use the following inequality:
For positive $a_1, a_2, \ldots, a_n$ and integer $k \geq 0$,

$$\sum_{i=1}^{n} a_i^k \geq n\Big(\frac{1}{n}\sum_{i=1}^{n} a_i\Big)^k$$

with equality for $k \geq 2$ if and only if all a_i are equal.

Note that $\sum\limits_{i:\mu_i \neq 0} \mu_i^2 = H(G)$. We have

$$\begin{aligned}
SEE(G) =& n_0 + \frac{1}{2} \sum_{i:\mu_i \neq 0} (e^{\mu_i} + e^{-\mu_i}) \\
=& n_0 + \sum_{k \geq 0} \frac{\sum\limits_{i:\mu_i \neq 0} (\mu_i^2)^k}{(2k)!} \\
\geq& n_0 + \sum_{k \geq 0} \frac{1}{(2k)!}(n - n_0)\Big(\frac{1}{n - n_0} \sum_{i:\mu_i \neq 0} \mu_i^2\Big)^k \\
=& n_0 + (n - n_0) \sum_{k \geq 0} \frac{\Big(\sqrt{\frac{H(G)}{n - n_0}}\Big)^{2k}}{(2k)!} \\
=& n_0 + (n - n_0) cosh\Big(\sqrt{\frac{H(G)}{n - n_0}}\Big).
\end{aligned}$$

Now we suppose that the equality holds. It is evident that equality will be attained if and only if all the positive eigenvalue are equal.

Upper bound: By n_+ we denote the number of positive eigenvalue of G. Note that $\sum\limits_{i=1}^{n_+} \mu_i^2 = \frac{H(G)}{2}$. We have

$$\begin{aligned}
SEE(G) =& n_0 + \sum_{i=1}^{n_+} (e^{\mu_i} + e^{-\mu_i}) \\
=& n_0 + 2 \sum_{k \geq 0} \frac{\sum\limits_{i=1}^{n_+} (\mu_i^2)^k}{(2k)!} \\
=& n_0 + 2n_+ + 2 \sum_{k=1}^{\infty} \frac{\sum\limits_{i=1}^{n_+} \mu_i^{2k}}{(2k)!}
\end{aligned}$$

$$\leq n+2\sum_{k=1}^{\infty}\frac{\left(\sum_{i=1}^{n_+}\mu_i^2\right)^k}{(2k)!},\ as\ n=n_0+2n_+$$

$$=n-2+2\sum_{k=0}^{\infty}\frac{\left(\sum_{i=1}^{n_+}\mu_i^2\right)^k}{(2k)!}$$

$$=n-2+2\sum_{k=0}^{\infty}\frac{\left(\sqrt{\frac{H(G)}{2}}\right)^{2k}}{(2k)!}$$

$$=n-2+2cosh\left(\sqrt{\frac{H(G)}{2}}\right).$$

Now suppose that equality holds. Then we get $\sum_{i=1}^{n_+}\mu_i^{2k}=\left(\sum_{i=1}^{n_+}\mu_i^2\right)^k$ for $k\geq 1$. Since G is a connected bipartite graph with $n\geq 2$ vertices, μ_1 is nonzero. So we have $n_+\geq 1$. For $k\geq 2$, $\sum_{i=1}^{n_+}\mu_i^{2k}=\left(\sum_{i=1}^{n_+}\mu_i^2\right)^k$ implies that $n_+\leq 1$. Thus $n_+=1$. Since G is bipartite graph, we have $\mu_1=-\mu_n$ and $\mu_2=\mu_3=\cdots=\mu_{n-1}=0$. Then by [10] Proposition 2, we conclude that G is a complete bipartite graph. This completes the proof. □

Using Theorem 3.4, we have the following corollary.

Corollary 3.5 Let T be a tree of order n. Then

$$SEE(T)\leq SEE(S_n).$$

Equality holds if and only if T is a star S_n.

By computing the sum-connectivity characteristic polynomial of the complete bipartite graph K_{n_1,n_2}, we know that the $S-$spectrum of K_{n_1,n_2} is $\left\{0^{(n_1+n_2-2)},\sqrt{\frac{n_1n_2}{n_1+n_2}}^{(1)},-\sqrt{\frac{n_1n_2}{n_1+n_2}}^{(1)}\right\}$. By the definition, we have

$$SEE(K_{n_1,n_2})=n_1+n_2-2+2cosh\left(\sqrt{\frac{n_1n_2}{n_1+n_2}}\right).$$

By the monotonicity of $f(x)=cosh(x)$, we obtain the Corollary 3.6.

Corollary 3.6 $SEE(K_{1,n-1})<SEE(K_{2,n-2})<\cdots<SEE(K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil})$.

In the following, we present some bounds for the sum-connectivity Estrada index involving sum-connectivity energy.

Theorem 3.7 Let G be a graph with n vertices. Then

$$n-n_++\frac{1}{2}(e-1)SE(G)\leq SEE(G)\leq n-1+e^{\frac{SE(G)}{2}}$$

where n_+ denotes the number of positive sum-connectivity eigenvalues of G. Moreover, the equality holds in both sides if and only if $G \cong \overline{K_n}$.

Proof. Lower bound: Let n_0 be the number of eigenvalue zero of G. Considering the inequalities $e^x \geq ex$ and $e^x \geq 1+x$, we have

$$\begin{aligned}
SEE(G) =& \sum_{i=1}^{n} e^{\mu_i} \\
=& n_0 + \sum_{\mu_i>0} e^{\mu_i} + \sum_{\mu_i<0} e^{\mu_i} \\
\geq& n_0 + \sum_{\mu_i>0} e\mu_i + \sum_{\mu_i<0} (1+\mu_i) \\
=& n_0 + (e-1)(\mu_1 + \mu_2 + \cdots + \mu_{n_+}) + (n - n_+ - n_0), \ as \sum_{i=1}^{n} \mu_i = 0 \\
=& n - n_+ + \frac{e-1}{2} SE(G).
\end{aligned}$$

Upper bound: Since $f(x) = e^x$ monotonically increases in $(-\infty, +\infty)$, we obtain

$$\begin{aligned}
SEE(G) =& n_0 + \sum_{\mu_i>0} e^{\mu_i} + \sum_{\mu_i<0} e^{\mu_i} \\
\leq& n_0 + (n - n_+ - n_0) + \sum_{\mu_i>0} e^{\mu_i} \\
=& n - n_+ + \sum_{i=1}^{n_+} \sum_{k\geq 0} \frac{\mu_i^k}{k!} \\
=& n + \sum_{k\geq 1} \frac{1}{k!} \sum_{i=1}^{n_+} \mu_i^k \\
\leq& n + \sum_{k\geq 1} \frac{1}{k!} \left(\sum_{i=1}^{n_+} \mu_i \right)^k \\
=& n - 1 + \sum_{k\geq 0} \frac{\left(\frac{SE(G)}{2}\right)^k}{k!} \\
=& n - 1 + e^{\frac{SE(G)}{2}}.
\end{aligned}$$

Moreover, the equality holds if and only if $G \cong \overline{K_n}$. This completes the proof. □

In the following, we present another lower bound for the $SEE(G)$.

Theorem 3.8 Let G be a nonempty simple connected graph of order n. Then

$$SEE(G) \geq e^{\sqrt{\frac{\sum_{i=1}^{n}\left(\sigma_i^{(l+1)}\right)^2}{\sum_{i=1}^{n}\left(\sigma_i^{(l)}\right)^2}}} + \frac{n-1}{e^{\frac{1}{n-1}\sqrt{\frac{\sum_{i=1}^{n}\left(\sigma_i^{(l+1)}\right)^2}{\sum_{i=1}^{n}\left(\sigma_i^{(l)}\right)^2}}}}.$$

Moreover, the equality holds if and only if $G \cong K_n$.
Proof. Using arithmetic-geometric mean inequality, we get

$$\begin{aligned} SEE(G) &= \sum_{i=1}^{n} e^{\mu_i} \\ &\geq e^{\mu_1} + (n-1)\left(\prod_{i=2}^{n} e^{\mu_i}\right)^{\frac{1}{n-1}} \\ &= e^{\mu_1} + (n-1)\left(e^{-\mu_1}\right)^{\frac{1}{n-1}}. \end{aligned}$$

Since $f(x) = e^x + (n-1)e^{-\frac{x}{n-1}}$ is an increasing function for $x > 0$. By Theorem 2.6, we obtain the required lower bound.

Now we assume that the equality holds. Then $\mu_1 = \sqrt{\frac{\sum_{i=1}^{n}\left(\sigma_i^{(l+1)}\right)^2}{\sum_{i=1}^{n}\left(\sigma_i^{(l)}\right)^2}}$ and $\mu_2 = \mu_3 = \cdots = \mu_n$. Note that G has only eigenvalue if and only if G is an empty graph. Then G has exactly two distinct sum-connectivity eigenvalues. By [10] Proposition 1, we know that G is the complete graph K_n. Conversely, it can be easily seen that the equality holds for the complete graph K_n. This completes the proof of theorem. □

The following corollary states a lower bound for the sum-connectivity Estrada index involving the sum-connectivity index $S(G)$.
Corollary 3.9 Let G be a connected graph with n vertices. Then

$$SEE(G) \geq e^{\frac{2S(G)}{n}} + \frac{n-1}{e^{\frac{2S(G)}{n(n-1)}}}.$$

Equality holds if and only if $S_1 = S_2 = \cdots = S_n$.
Proof. The result is obvious from Theorem 3.8 and Theorem 2.5. □

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