A COMBINATORIAL IDENTITY AND ITS RELATED CONJECTURE

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ABSTRACT. In recent researches on a discriminant for polynomials, I faced with a recursive (combinatorial) sequence $\lambda_{n,m}$ that its first four terms and identities are $\lambda_{0,m} := \binom{m}{0} = \binom{m-1}{m-1}$, $\lambda_{1,m} := \binom{m}{1} = \binom{m}{m-1}$, $\lambda_{2,m} := \binom{m}{1}^2 - \binom{m}{2} = \binom{m+1}{m-1}$, $\lambda_{3,m} := \binom{m}{1}^3 - 2\binom{m}{1}\binom{m}{2} + \binom{m}{3} = \binom{m+2}{m-1}$. In this paper I introduce it, prove an identity thereabout, and leave a problem and a conjecture concerning it.

1. INTRODUCTION

In [2], I faced to a new recursive doubled sequence as follows. For every integer numbers $n, m \geq 0$, define the sequence $\lambda_{n,m}$ by the following recursive definition

$$\lambda_{0,m} := 1 \ , \ \lambda_{n,m} := \sum_{j=0}^{n-1} (-1)^{n+j+1} \binom{m}{n-j} \lambda_{j,m}.$$

Therefore

$$\lambda_{0,m} := \binom{m}{0}, \lambda_{1,m} := \binom{m}{1}, \lambda_{2,m} := \binom{m}{1}^2 - \binom{m}{2}$$

$$, \lambda_{3,m} := \binom{m}{1}^3 - 2\binom{m}{1}\binom{m}{2} + \binom{m}{3}$$

$$, \lambda_{4,m} := \binom{m}{1}^4 - 3\binom{m}{1}^2\binom{m}{2} + 2\binom{m}{1}\binom{m}{3} + \binom{m}{2}^2 - \binom{m}{4}, \cdots .$$

As regards finding the coefficients of the above summations a problem and related conjecture have been raised. A simplified formula and an explicit definition for $\lambda_{n,m}$ are aimed for this interesting sequence.

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2. Main theorem and problem

Now, we prove the first identity for $\lambda_{n,m}$.

Theorem 2.1. *For every integer numbers $n, m \geq 0$, we have* $\lambda_{n,m} = \binom{m-1+n}{m-1}$.

Proof. For $n = 0$ the formula is true. Next, assume that the formula holds for every $1 \leq k \leq n$. Consider the the following well-known combinatorial identity (see [1])

$$\sum_{j=0}^{k}(-1)^j\binom{m}{k-j}\binom{m-1+j}{m-1} = 0$$

Upon replacing k by $k+1$ in the above identity we get

$$\sum_{j=0}^{k}(-1)^j\binom{m}{k+1-j}\binom{m-1+j}{m-1} = (-1)^k\binom{m+k}{m-1}.$$

Employing the above identity we obtain

$$\lambda_{n+1,m} = (-1)^n\binom{m}{n+1} + \sum_{j=0}^{n-1}(-1)^{n+j+1}\binom{m}{n-j}\binom{m+j}{m-1}$$

$$= (-1)^n\binom{m}{n+1} + (-1)^n((-1)^n\binom{m+n}{m-1} - \binom{m}{n+1})) = \binom{m-1+(n+1)}{m-1}.$$

Therefore, the proof is complete. □

Now, let m be a fixed positive integer. By using the above theorem, we observe that

$\binom{m}{0} = \binom{m-1}{m-1}$

$\binom{m}{1} = \binom{m}{m-1}$

$\binom{m}{1}^2 - \binom{m}{2} = \binom{m+1}{m-1}$

$\binom{m}{1}^3 - 2\binom{m}{1}\binom{m}{2} + \binom{m}{3} = \binom{m+2}{m-1}$

$\binom{m}{1}^4 - 3\binom{m}{1}^2\binom{m}{2} + 2\binom{m}{1}\binom{m}{3} + \binom{m}{2}^2 - \binom{m}{4} = \binom{m+3}{m-1}$

$\binom{m}{1}^5 - 4\binom{m}{1}^3\binom{m}{2} + 3\binom{m}{1}^2\binom{m}{3} - 2\binom{m}{4}\binom{m}{1} + 3\binom{m}{1}\binom{m}{2}^2 - 2\binom{m}{3}\binom{m}{2} + \binom{m}{5} = \binom{m+4}{m-1}$

...

In general, we have

$$\binom{m-1+n}{m-1} = \sum_{\substack{n_1+\cdots+n_k=n \\ 0\leq n_1\leq\cdots\leq n_k}} C_{n_1,\ldots,n_k}\binom{m}{n_1}\cdots\binom{m}{n_k}$$

$$= \sum_{\substack{i_1 + 2i_2 + \cdots + n i_n = n \\ i_j \geq 0}} d_{i_1,\ldots,i_n} \binom{m}{1}^{i_1} \cdots \binom{m}{n}^{i_n},$$

where $C_{n_1,\ldots,n_k}$ and $d_{i_1,\ldots,i_n}$ are integers subject to the next conjecture. Therefore we have the following problem .

Problem. Find the values of $C_{n_1,\ldots,n_k}$, $d_{i_1,\ldots,i_n}$.

Following the above problem, we guess the conjecture bellow.

Conjecture. The values of $C_{n_1,\ldots,n_k}$, $d_{i_1,\ldots,i_n}$ are dependent only on n (independent from m), $d_{n,0,\ldots,0} = 1$, $d_{0,\ldots,0,1} = (-1)^{n+1}$, and the summation of all coefficients $d_{i_1,\ldots,i_n}$ is zero for all $n > 1$.

References

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