Packing and covering λ-fold complete symmetric digraphs with 6-circuits

Jenq-Jong Lin
Ling Tung University, Taichung 408, Taiwan

Abstract

In this paper we give the solutions of finding maximum packings and minimum coverings of λ-fold complete symmetric digraphs with 6-circuits.

1 Introduction and preliminary

If G is a graph and λ is a positive integer, let λG^* be the λ-*fold symmetric digraph* obtained by replacing each edge uv of G with λ arcs $\overrightarrow{uv}$ and λ arcs $\overrightarrow{vu}$. A 1-fold symmetric digraph $1G^*$ will be simply denoted G^* and called symmetric digraph. In particular, we use λK_n^* to denote the λ-*fold complete symmetric digraph* with n vertices where any two distinct vertices u and v are joined by λ arcs $\overrightarrow{uv}$ and λ arcs $\overrightarrow{vu}$. For an integer $k \geq 2$, by a k-*circuit* $\overrightarrow{C_k}$ we mean an elementary circuit (directed cycle) of length k with vertex set $\{v_1, v_2, \ldots, v_k\}$ and the arc set $\{\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{k-1}v_k}, \overrightarrow{v_kv_1}\}$; and it is denoted by $(v_1, v_2, \ldots, v_k)$. In particular, (v_1, v_2) denotes the 2-circuit with arcs $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_2v_1}$. A *decomposition* of a digraph G is a family $\{H_i : i \in L\}$ of subdigraphs of G such that each arc of G contained in exactly one member of $\{H_i : i \in L\}$. Suppose that $G, H_1, H_2, \ldots, H_r$ are digraphs, we will write $G = H_1 \oplus H_2 \oplus \cdots \oplus H_r$ if G has a decomposition into subdigraphs $H_1, H_2, \ldots, H_r$. Let a digraph H be given. An H-*decomposition* of a digraph G is a decomposition of G into subdigraphs isomorphic to H. J. C. Bermond et al. [4] solved the k-circuit decompositions problem of λK_n^* when $k \in \{4, 6, 8, 10.12, 14, 16\}$. The existence problem for k-circuit decompositions of complete symmetric digraphs K_n^* has been settled by B. Alspach et al. [2].

A k-circuit decomposition of λK_n^* may not exist, however, it is of interest to see just how "close" we can come to a k-circuit decomposition. Let $G = (V(G), A(G))$ be a digraph with vertex set $V(G)$ and arc set $A(G)$.

For a set $B \subseteq A(G)$, the *arc-deletion* (resp. *arc-addition*) of B from G is the digraph $G - B$ (resp. $G + B$) by removing (resp. adding) all arcs of B. A *maximum k-circuit packing* of G is a set $\{G_1, G_2, \ldots, G_\ell\}$ where $G_i \cong \overrightarrow{C_k}$, $V(G_i) \subseteq V(G)$ for all i, and $\cup_{i=1}^{\ell} A(G_i) = A(G - L)$ such that $|L|$ is minimum. The arc set L is called the *minimum leave*. A *minimum k-circuit covering* is a set $\{F_1, F_2, \ldots, F_m\}$ where $F_i \cong \overrightarrow{C_k}$, $V(F_i) \subseteq V(G)$ for all i, and $\cup_{i=1}^{m} A(F_i) = A(G + P)$ such that $|P|$ is minimum. The arc set P is called the *minimum padding*. Note that any arc set mentioned in this paper may be a multiset. When there is no chance of confusion, we also refer to the subdigraph induced by the arcs in L (resp. P) as the leave (resp. padding).

For undirected graphs, maximum k-cycle packings and minimum k-cycle coverings of K_n have been found for all values of n when $k \in \{3, 4, 5, 6, 8\}$ (see [5, 6, 7, 8, 10, 11, 13]). For digraphs, the problem of packings and coverings of the λ-fold complete symmetric digraph with 3- and 4-circuits was investigated by F. E. Bennett et al. [3]. H. C. Lee [9] studied the maximum packings and minimum coverings of $\lambda K_{m,n}^*$ with 6-circuits. In this paper we give the solutions of finding maximum packings and minimum coverings of λ-fold complete symmetric digraphs with 6-circuits, respectively.

In order to state our results, we introduce some notations and terminologies. Consider two digraphs $G = (V(G), A(G))$ and $G' = (V(G'), A(G'))$, the *union* of G and G' is the digraph $G \cup G'$ with vertex set $V(G) \cup V(G')$ and arc set $A(G) \cup A(G')$. The union of G and G' is *disjoint* if $V(G) \cap V(G') = \emptyset$, denoted by $G \uplus G'$. Let $K_n^*[v_1, v_2, \ldots, v_n]$ be the complete symmetric digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $K_{m,n}^*[a_1, a_2, \ldots, a_m; b_1, b_2, \ldots, b_n]$ the complete symmetric bipartite digraph with bipartition $(\{a_1, a_2, \ldots, a_m\}, \{b_1, b_2, \ldots, b_n\})$, respectively. In addition, denote by C_n a *cycle* with n vertices and P_n a *path* with n vertices.

For a vertex v of a multidigraph G, the *outdegree* $d_G^+(v)$ (resp. *indegree* $d_G^-(v)$) of v is the number of arcs incident from (resp. incident to) v.

Lemma 1.1. *For each vertex v of a leave L (respectively, padding P) of a 6-circuit packing (resp. 6-circuit covering) of λK_n^*, then $d_L^+(v) = d_L^-(v)$ (resp. $d_P^+(v) = d_P^-(v)$).*

Proof. It follows from that every vertex v in λK_n^*, $d_{\lambda K_n^*}^+(v) = d_{\lambda K_n^*}^-(v)$, and so does the vertex of $\overrightarrow{C_6}$ in λK_n^*. □

2 The case of $n = 6$

Let us begin with a theorem concerning the k-circuit decomposition of K_n^*.

Theorem 2.1. ([2]) *For positive integers n and k, with $2 \leq k \leq n$, there exists a k-circuit decomposition of K_n^* if and only if k divides the number of arcs in K_n^* and $(n,k) \neq (4,4),(6,3),(6,6)$.*

Since $|A(K_6^*)|$ is a multiple of 6 and K_6^* can not be decomposed into 6-circuits (Theorem 2.1), either the leave of a packing or the padding of a covering should have sizes a multiple of 6. Note that $\overrightarrow{C_6}$ is a spanning subdigraph of K_6^*. By Lemma 1.1, the minimum possible leaves of K_6^* are $\overrightarrow{C_2} \uplus \overrightarrow{C_2} \uplus \overrightarrow{C_2}$, $\overrightarrow{C_4} \uplus \overrightarrow{C_2}$ and $\overrightarrow{C_3} \uplus \overrightarrow{C_3}$, so are the minimium possible paddings.

Lemma 2.2. *There exists a 6-circuit packing of K_6^* with leave $\overrightarrow{C_2} \uplus \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ and there exists a 6-circuit covering of K_6^* with padding $\overrightarrow{C_2} \uplus \overrightarrow{C_2} \uplus \overrightarrow{C_2}$.*

Proof. Let $V(K_6^*) = \{1,2,3,4,5,6\}$. A suitable packing $\mathscr{P}_1$ is given by $(1,3,5,2,6,4),(4,6,2,5,3,1),(1,5,4,2,3,6),(6,3,2,4,5,1)$ with leave $(1,2)$, $(3,4),(5,6)$. On the other hand, since $(1,2) \cup (3,4) \cup (5,6) \cup (2,3) \cup (4,5) \cup (6,1) = (1,2,3,4,5,6) \cup (6,5,4,3,2,1)$, this implies that $\mathscr{P}_1, (1,2,3,4,5,6),(6,5,4,3,2,1)$ is a required covering with padding $(2,3),(4,5),(6,1)$. □

Lemma 2.3. *There exists a 6-circuit packing of K_6^* with leave $\overrightarrow{C_4} \uplus \overrightarrow{C_2}$ and there exists a 6-circuit covering of K_6^* with padding $\overrightarrow{C_4} \uplus \overrightarrow{C_2}$.*

Proof. Let $V(K_6^*) = \{1,2,3,4,5,6\}$. A suitable packing $\mathscr{P}_2$ is given by $(1,4,6,5,2,3),(1,6,4,5,3,2),(1,3,5,6,2,4),(1,2,5,4,3,6)$ with leave $(2,6,3,4),(1,5)$. On the other hand, since $(2,6,3,4) \cup (1,5) \cup (2,5,4,6) \cup (1,3) = (1,5,4,2,6,3) \cup (1,3,4,6,2,5)$, this implies that $\mathscr{P}_2, (1,5,4,2,6,3),(1,3,4,6,2,5)$ is a required covering with padding $(2,5,4,6),(1,3)$. □

Lemma 2.4. *There does not exist a 6-circuit packing of K_6^* with leave $\overrightarrow{C_3} \uplus \overrightarrow{C_3}$.*

Proof. Suppose on the contrary that there exists a 6-circuit packing of $K_6^*[1,2,\ldots,6]$ with leave $\overrightarrow{C_3} \uplus \overrightarrow{C_3}$, namely $(1,2,3),(4,5,6)$. Since $K_6^*[1,2,\ldots,6] = K_3^*[1,2,3] \oplus K_{3,3}^*[1,2,3;4,5,6] \oplus K_3^*[4,5,6]$, this implies that there exists a 6-circuit decomposition of the digraph $(3,2,1) \cup K_{3,3}^*[1,2,3;4,5,6] \cup (6,5,4)$. We consider the following two cases.

Case 1. The arcs in $(3,2,1)$ are contained in three distinct 6-circuits, namely A_1, A_2 and A_3. Note that the arcs in $(6,5,4)$ must be contained in the same three 6-circuits. Without loss of generality, we may assume that $\overrightarrow{21}$ and $\overrightarrow{54}$ are in A_1. We shall deal with the case where $\overrightarrow{32}$ and $\overrightarrow{65}$ are in A_2, $\overrightarrow{13}$ and $\overrightarrow{46}$ are in A_3; the other cases are treated similarly.

The possible 6-circuit A_1 that contains both $\overrightarrow{21}$ and $\overrightarrow{54}$ is $(2,1,5,4,3,6)$ or $(2,1,6,3,5,4)$; A_2 that contain both $\overrightarrow{32}$ and $\overrightarrow{65}$ is $(3,2,6,5,1,4)$ or

$(3,2,4,1,6,5)$; A_3 that contain both $\overrightarrow{13}$ and $\overrightarrow{46}$ is $(1,3,4,6,2,5)$ or $(1,3,5,2,4,6)$. It is not difficult to check that none of the eight possibilities can happen.

Case 2. The arcs in $(3,2,1)$ are contained in two distinct 6-circuits, namely B_1 and B_2. Note that the arcs in $(6,5,4)$ must be contained in the same two 6-circuits and one arc in B_1, the other two arcs in B_2. Without loss of generality, we assume that $\overrightarrow{21}$ and $\overrightarrow{54}$ in B_1, $\overrightarrow{13} \cup \overrightarrow{32}$ and $\overrightarrow{46} \cup \overrightarrow{65}$ in B_2, and the other cases are similar.

The possible 6-circuit B_1 that contain both $\overrightarrow{21}$ and $\overrightarrow{54}$ is $(2,1,5,4,3,6)$ or $(2,1,6,3,5,4)$ and one possibility for the 6-circuit B_2 that contains $\overrightarrow{13} \cup \overrightarrow{32}$ and $\overrightarrow{46} \cup \overrightarrow{65}$ is $(1,3,2,4,6,5)$. Hence the subdigraph induced by the arcs of $B_1 \cup B_2$ in $K^*_{3,3}[1,2,3;4,5,6]$ is $(1,5) \cup (2,4,3,6)$ or $(2,4) \cup (1,6,3,5)$. It follows that there exists a 6-circuit decomposition of the digraph $K^*_{3,3}[1,2,3;4,5,6] - A((1,5) \cup (2,4,3,6))$ or $K^*_{3,3}[1,2,3;4,5,6] - A((2,4) \cup (1,6,3,5))$. We consider the case of $(1,5) \cup (2,4,3,6)$ and the case of $(2,4) \cup (1,6,3,5)$ is similar. Since there exist no arcs between vertices 1 and 5, this implies that one 6-circuit must be of the type $(1,a,b,5,c,d)$ where $\{a,d\} = \{4,6\}$ and $\{b,c\} = \{2,3\}$. We consider the case of $a = 4$, $b = 2$, $c = 3$, $d = 6$ and the other cases are similar. In this case, the 6-circuit is $(1,4,2,5,3,6)$, but the arc $\overrightarrow{36}$ has been removed. This contradiction finishes the proof. □

Lemma 2.5. *There exists a 6-circuit covering of K^*_6 with padding $\overrightarrow{C_3} \uplus \overrightarrow{C_3}$.*

Proof. Let $V(K^*_6) = \{1,2,3,4,5,6\}$. By Lemma 2.3, there exists a 6-circuit packing $\mathscr{P}_2$ with leave $(2,6,3,4),(1,5)$. On the other hand, since $(2,6,3,4) \cup (1,5) \cup (1,4,6) \cup (2,5,3) = (1,5,3,4,2,6) \cup (1,4,6,3,2,5)$, this implies that $\mathscr{P}_2, (1,5,3,4,2,6), (1,4,6,3,2,5)$ is the required covering with padding $(1,4,6),(2,5,3)$. □

Proposition 2.6. ([4]) *For integers $n \geq 6$ and $\lambda \geq 1$, there exists a 6-circuit decomposition of λK^*_n if and only if $\lambda n(n-1) = 0 \pmod 6$ except for $n = 6$, $\lambda = 1$.*

Now we have the following results for $n = 6$.

Theorem 2.7. *Let λ be a positive integer.*
(a) *There exists a maximum 6-circuit packing of λK^*_6 with minimum leave L, where*

$$L \cong \begin{cases} \overrightarrow{C_2} \uplus \overrightarrow{C_2} \uplus \overrightarrow{C_2} \text{ or } \overrightarrow{C_4} \uplus \overrightarrow{C_2}, & \text{if } \lambda = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

(b) *There exists a minimum 6-circuit covering of* λK_6^* *with minimum padding* P, *where*

$$P \cong \begin{cases} \overrightarrow{C_2} \uplus \overrightarrow{C_2} \uplus \overrightarrow{C_2},\ \overrightarrow{C_4} \uplus \overrightarrow{C_2} \text{ or } \overrightarrow{C_3} \uplus \overrightarrow{C_3}, & \text{if } \lambda = 1; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proof. This result follows from Theorem 2.1, Lemmas 2.2– 2.5 and Proposition 2.6. □

Corollary 2.8. *For an integer* λ, *there exists a 6-circuit decomposition of* λK_6^* *if and only if* $\lambda \geq 2$.

3 Small cases of maximum packings and minimum coverings

In view of the above result, we shall assume that $n \geq 7$ in the sequel. We will discuss necessary conditions for maximum packings and minimum coverings. In dealing with minimum possible leaves and minimum possible paddings of undirected graphs, one has to consider both the parity of degrees and the number of edges. However, in the case of digraphs, only the number of arcs is relevant because the arcs which are oriented outward from v and inward to v appear in pairs for any vertex $v \in V(\lambda K_n^*)$.

If $n \equiv 0, 1 \pmod 3$, then clearly 6 divides $|A(\lambda K_n^*)|$, so minimum possible leaves and minimum possible paddings are empty.

For the case when $n \equiv 2 \pmod 3$, if $\lambda \equiv 0 \pmod 3$, then $|A(\lambda K_n^*)|$ is a multiple of 6. This implies that the minimum possible leaves and minimum possible paddings are empty. If $\lambda \equiv 1 \pmod 3$, then $|A(\lambda K_n^*)| = 6\alpha + 2$ for some positive integer α, the minimum possible leave is $\overrightarrow{C_2}$ and the minimum possible paddings are $\overrightarrow{C_4}, P_3^*, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ and $2P_2^*$ (the case of $2P_2^*$ doesn't exist when $\lambda = 1$), in view of the divisibility requirement for the number of arcs in λK_n^*. If $\lambda \equiv 2 \pmod 3$, arguing in the same way, the minimum possible leaves are $\overrightarrow{C_4}, P_3^*, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$ and $2P_2^*$, and the minimum possible padding is $\overrightarrow{C_2}$.

We may summarize our results in Tables 1 and 2.

The well-known results about the decomposition of K_n into isomorphic cycles are due to M. Šajna [1] and B. Alspach et al. [12], respecively.

Theorem 3.1. ([1]) *For positive even integers* m *and* n *with* $4 \leq m \leq n$, *the graph* $K_n - I$ *can be decomposed into cycles of length* m *if and only if the number of edges in* $K_n - I$ *is a multiple of* m, *where* I *is a 1-factor of* K_n.

Table 1: The minimum possible leaves in 6-circuit packings of λK_n^*

$n \pmod 3$	1	2	0
$\lambda = 1 \pmod 3$	$\emptyset$	$\overrightarrow{C_2}$	$\emptyset$
$\lambda = 2 \pmod 3$	$\emptyset$	$\overrightarrow{C_4}, P_3^*, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$, or $2P_2^*$,	$\emptyset$
$\lambda = 0 \pmod 3$	$\emptyset$	$\emptyset$	$\emptyset$

Table 2: The minimum possible paddings in 6-circuit coverings of λK_n^*

$n \pmod 3$	1	2	0
$\lambda = 1 \pmod 3$	$\emptyset$	$\overrightarrow{C_4}, P_3^*, \overrightarrow{C_2} \uplus \overrightarrow{C_2}$, or $2P_2^*$,	$\emptyset$
$\lambda = 2 \pmod 3$	$\emptyset$	$\overrightarrow{C_2}$	$\emptyset$
$\lambda = 0 \pmod 3$	$\emptyset$	$\emptyset$	$\emptyset$

(the case of $2P_2^*$ doesn't exist when $\lambda = 1$)

Theorem 3.2. ([12]) *Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n.*

The following criterion for the $2k$-circuit decomposition of complete bipartite symmetric digraphs is needed for our discussions.

Theorem 3.3. ([14]) *There exists a $2k$-circuit decomposition of $K_{m,n}^*$ if and only if $m \geq k$, $n \geq k$, and k divides mn.*

The following lemma provides a useful tool for the problems of k-circuit packing and covering.

Lemma 3.4. *If there exists a k-cycle packing (resp. covering) of K_n with leave L (resp. padding P), then there exists a k-circuit packing (resp. covering) of K_n^* with leave L^* (resp. padding P^*).*

Proof. For each C_k in a k-cycle packing (resp. covering) of K_n, obtain two $\overrightarrow{C_k}$ by giving C_k the two possible orientations in each edge. Hence there exists a k-circuit packing (resp. covering) of K_n^* with leave L^* (resp. padding P^*). □

Next, we will give a collection of the necessary lemmas of maximum packings and minimum coverings for the general construction to follow.

Theorem 3.5. ([2]) *There exists a 6-circuit decomposition of K_n^* when* $n = 7, 9, 10, 12$.

Lemma 3.6. *Let λ be a positive integer with $\lambda \leq 3$.*
(a) *There exists a 6-circuit packing of λK_8^* with leave L, where*

$$L \cong \begin{cases} \overrightarrow{C_2}, & \text{if } \lambda = 1\,; \\ \overrightarrow{C_4}, P_3^*, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, \text{ or } 2P_2^*, & \text{if } \lambda = 2\,; \\ \emptyset, & \text{if } \lambda = 3. \end{cases}$$

(b) *There exists a 6-circuit covering of λK_8^* with padding P, where*

$$P \cong \begin{cases} \overrightarrow{C_4}, P_3^*, \text{ or } \overrightarrow{C_2} \uplus \overrightarrow{C_2}, & \text{if } \lambda = 1\,; \\ \overrightarrow{C_2}, & \text{if } \lambda = 2\,; \\ \emptyset, & \text{if } \lambda = 3. \end{cases}$$

Proof. Let $V(K_8^*) = \{1, 2, 3, \ldots, 8\}$. The case of $\lambda = 3$ follows from Proposition 2.6. We distinguish two cases by the values of λ.

Case 1. $\lambda = 1$. By Theorem 3.1, we have that $K_8 - I$ can be decomposed into cycles of length 6, where I is a 1-factor of K_8. Thus, by Lemma 3.4, there exists a 6-circuit packing $(1,2,5,6,3,8), (2,3,5,4,7,8), (1,3,4,6,7,5), (1,7,2,4,8,6), (8,3,6,5,2,1), (8,7,4,5,3,2), (5,7,6,4,3,1), (6,8,4,2,7,1)$ of K_8^* with leave $(1,4), (3,7), (5,8), (2,6)$. In addition, we can see that $(1,2,5,6,3,8) \cup (2,3,5,4,7,8) \cup (1,4) \cup (3,7) \cup (5,8) = (1,2,3,8,5,4) \cup (1,4,7,3,5,8) \cup (2,5,6,3,7,8)$. It follows that there exists a 6-circuit packing of K_8^* with leave $\overrightarrow{C_2} : (2,6)$. On the other hand, we have

$(1,3,4,6,7,5) \cup (2,6) \cup (2,8,5,3) = (1,3,2,6,7,5) \cup (2,8,5,3,4,6)$;
$(1,3,4,6,7,5) \cup (2,6) \cup (1,8) \cup (2,8) = (1,3,4,6,2,8) \cup (1,8,2,6,7,5)$;
$(1,7,2,4,8,6) \cup (2,6) \cup (1,4) \cup (7,8) = (1,7,8,6,2,4) \cup (1,4,8,7,2,6)$.

Hence there exist 6-circuit coverings of K_8^* with paddings $\overrightarrow{C_4} : (2,8,5,3)$, $P_3^* : (1,8), (2,8)$ and $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (1,4), (7,8)$, respectively.

Case 2. $\lambda = 2$. Note that $2K_8^* = K_8^* \oplus K_8^*$. Since the leave of $2K_8^*$ can be composed of the leave of K_8^* together with the leave of the other K_8^*, by the result of case 1, it suffices to show the existence of the leave $\overrightarrow{C_4}$. First, we choose two 6-circuits $(1,3,4,6,7,5)$, $(1,4,7,3,5,8)$ and the leave $(2,6)$ from the 6-circuit packing of K_8^* in case 1. Next, by the result of case 1 again, we assume that there exists a 6-circuit packing of the other K_8^* with leave $(2,5)$. Then we have $(1,3,4,6,7,5) \cup (2,6) \cup (2,5) = (1,3,4,6,2,5) \cup (2,6,7,5)$. It follows that there exists a 6-circuit packing of $2K_8^*$ with leave $\overrightarrow{C_4} : (2,6,7,5)$. On the other hand, we have $(1,4,7,3,5,8) \cup (2,6,7,5) \cup (1,2) = (1,2,6,7,5,8) \cup (1,4,7,3,5,2)$. Hence there exists a 6-circuit covering of $2K_8^*$ with padding $\overrightarrow{C_2} : (1,2)$. □

Lemma 3.7. *Let λ be a positive integer with $\lambda \leq 3$.*
(a) *There exists a 6-circuit packing of λK_{11}^* with leave L, where*

$$L \cong \begin{cases} \overrightarrow{C_2}, & \text{if } \lambda = 1\,; \\ \overrightarrow{C_4}, P_3^*, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, \text{ or } 2P_2^*, & \text{if } \lambda = 2\,; \\ \emptyset, & \text{if } \lambda = 3. \end{cases}$$

(b) *There exists a 6-circuit covering of λK_{11}^* with padding P, where*

$$P \cong \begin{cases} \overrightarrow{C_4}, P_3^*, \text{ or } \overrightarrow{C_2} \uplus \overrightarrow{C_2}, & \text{if } \lambda = 1\,; \\ \overrightarrow{C_2}, & \text{if } \lambda = 2\,; \\ \emptyset, & \text{if } \lambda = 3. \end{cases}$$

Proof. Let $V(K_{11}^*) = \{1,2,3,\ldots,11\}$. The case of $\lambda = 3$ follows from Proposition 2.6. We distinguish two cases by the values of λ.

Case 1. $\lambda = 1$. A suitable packing is given by $(1,3,7,11,6,9)$, $(2,6,11,4,8,10)$, $(2,11,5,8,7,3)$, $(1,7,9,4,11,2)$, $(6,1,5,9,7,4)$, $(9,11,8,5,7,2)$, $(8,11,9,10,3,1)$, $(5,11,7,1,6,2)$, $(3,11,1,10,7,6)$, $(1,11,3,10,8,4)$, $(2,10,6,7,8,1)$, $(6,10,9,3,4,5)$, $(2,3,6,8,9,5)$, $(4,10,5,3,8,2)$, $(5,10,4,9,2,7)$, $(5,4,3,9,8,6)$, $(2,8,3,5,1,4)$, $(7,10,1,9,6,4)$ with leave $(10,11)$. On the other hand, since
$(1,3,7,11,6,9) \cup (10,11) \cup (2,9,3,10) = (1,3,10,11,6,9) \cup (2,9,3,7,11,10)$;
$(1,3,7,11,6,9) \cup (10,11) \cup (1,2) \cup (2,10) = (1,3,7,11,10,2) \cup (1,2,10,11,6,9)$;
$(2,6,11,4,8,10) \cup (10,11) \cup (2,4) \cup (6,8) = (2,4,8,6,11,10) \cup (2,6,8,10,11,4)$.
Hence there exist 6-circuit coverings of K_{11}^* with paddings $\overrightarrow{C_4} : (2,9,3,10)$, $P_3^* : (1,2),(2,10)$ and $\overrightarrow{C_2} \uplus \overrightarrow{C_2} : (2,4),(6,8)$, respectively.

Case 2. $\lambda = 2$. Note that $2K_{11}^* = K_{11}^* \oplus K_{11}^*$. Since the leave of $2K_{11}^*$ can be composed of the leave of K_{11}^* together with the leave of the other K_{11}^*, by the result of case 1, it suffices to show the existence of the leave $\overrightarrow{C_4}$. First, we choose two 6-circuits $(2,11,5,8,7,3)$, $(1,7,9,4,11,2)$ and the leave $(10,11)$ from the 6-circuit packing of K_{11}^* in case 1. Next, by the result of case 1 again, we assume that there exists a 6-circuit packing of K_{11}^* with leave $(8,10)$. Then $(2,11,5,8,7,3) \cup (10,11) \cup (8,10) = (2,11,10,8,7,3) \cup (5,8,10,11)$. It follows that there exists a 6-circuit packing of $2K_{11}^*$ with leave $\overrightarrow{C_4} : (5,8,10,11)$. On the other hand, we have $(1,7,9,4,11,2) \cup (5,8,10,11) \cup (7,8) = (1,7,8,10,11,2) \cup (4,11,5,8,7,9)$. Hence there exists a 6-circuit covering of $2K_{11}^*$ with padding $\overrightarrow{C_2} : (7,8)$. □

4 Main results

Lemma 4.1. *For an integer $n \geq 6$, if there exists a 6-circuit packing (resp. covering) of λK_n^* with leave L (resp. padding P), then there exists* *a 6-circuit packing (resp. covering) of* λK^*_{n+6} *with leave* L *(resp. padding* P*).*

Proof. Let $V(\lambda K^*_{n+6}) = \{\infty, 1, 2, 3, \ldots, n+4, n+5\}$, we have a decomposition of λK^*_{n+6} as follows.

$$\begin{aligned}
&\lambda K^*_{n+6}[\infty, 1, 2, \ldots, n+5] \\
=&\lambda K^*_n[\infty, 1, 2, \ldots, n-1] \\
&\quad \oplus \lambda K^*_{n-1,6}[1, 2, \ldots, n-1; n, n+1, \ldots, n+5] \\
&\qquad \oplus \lambda K^*_7[\infty, n, n+1, \ldots, n+5].
\end{aligned}$$

Hence the result follows from the fact that $K^*_{n-1,6}$ and K^*_7 have 6-circuit decompositions by Theorems 3.3 and 3.5, respectively. □

Now we prove the main result of this paper.

Theorem 4.2. *Let* n *and* λ *be positive integers with* $n \geq 7$.
(a) *There exists a maximum 6-circuit packing of* λK^*_n *with minimum leave* L*, where*

$$L \cong \begin{cases} \overrightarrow{C_2}, & \textit{if } n = 2 \pmod 3 \textit{ and } \lambda = 1 \pmod 3; \\ \overrightarrow{C_4}, P^*_3, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, \textit{or } 2P^*_2, & \textit{if } n = 2 \pmod 3 \textit{ and } \lambda = 2 \pmod 3; \\ \emptyset, & \textit{otherwise.} \end{cases}$$

(b) *There exists a minimum 6-circuit covering of* λK^*_n *with minimum padding* P*, where*

$$P \cong \begin{cases} \overrightarrow{C_4}, P^*_3, \textit{or } \overrightarrow{C_2} \uplus \overrightarrow{C_2}, & \textit{if } n = 2 \pmod 3 \text{ and } \lambda = 1; \\ \overrightarrow{C_4}, P^*_3, \overrightarrow{C_2} \uplus \overrightarrow{C_2}, \textit{or } 2P^*_2, & \textit{if } n = 2 \pmod 3 \\ & \qquad \textit{and } \lambda = 1 \textit{ (mod 3)}, \lambda \geq 4; \\ \overrightarrow{C_2}, & \textit{if } n = 2 \pmod 3 \textit{ and } \lambda = 2 \pmod 3; \\ \emptyset, & \textit{otherwise.} \end{cases}$$

Proof. By Proposition 2.6, Theorem 3.5, Lemmas 3.6, 3.7 and 4.1, it suffices to show the existence of the minimum possible paddings $2P^*_2$ of $4K^*_8$ and $4K^*_{11}$. Let $V(K^*_8) = \{1, 2, 3, \ldots, 8\}$. By Lemma 3.6, there exists a 6-circuit covering of $2K^*_8$ with padding $\overrightarrow{C_2} : (1, 2)$. Since $4K^*_8 = 2K^*_8 \oplus 2K^*_8$, there exists a 6-circuit covering of $4K^*_8$ with padding $2P^*_2 : (1, 2), (1, 2)$. Similarly, $4K^*_{11}$ has the same result and the proof is complete. □

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