Maximally connected and super arc-connected Bi-Cayley digraphs *

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Abstract Let $X = (V, E)$ be a digraph. X is *maximally connected*, if $\kappa(X) = \delta(X)$. X is *maximally arc-connected*, if $\lambda(X) = \delta(X)$. And X is *super arc-connected*, if every minimum arc-cut of X is either the set of inarcs of some vertex or the set of outarcs of some vertex. In this paper, we prove that the strongly connected *Bi-Cayley digraphs* are maximally connected and maximally arc-connected, and the most of strongly connected Bi-Cayley digraphs are super arc-connected.

Keywords: Bi-Cayley digraph, atom, $\lambda-$atom, $\lambda-$superatom

1 Introduction

All graphs considered in this paper are finite and simple, unless otherwise stated. We follow the notation and terminology, not defined here, of Bondy and Murty [1].

A *digraph* is a pair $X = (V, E)$, where V is a finite set and E is an irreflexive relation on V. Thus E is a set of ordered pairs $(u, v) \in V \times V$ such that $u \neq v$. The elements of V are called the *vertices* or *nodes* of X and the elements of E are called the *arcs* of X. Arc (u, v) is said to be an *inarc* of v and an *outarc* of u; we also say that (u, v) originates at u and terminates at v. If u is a vertex of X, then the *outdegree* of u in X is the

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number $d_X^+(u)$ of arcs of X originating at u and the *indegree* of u in X is the number $d_X^-(u)$ of arcs of X terminating at u. The minimum outdegree of X is $\delta^+(X)=\min\{d_X^+(u) \mid u \in V\}$ and the minimum indegree of X is $\delta^-(X)=\min\{d_X^-(u) \mid u \in V\}$. We denote by $\delta(X)$ the minimum of $\delta^+(X)$ and $\delta^-(X)$.

The *reverse* digraph of digraph $X = (V, E)$ is the digraph $X^{(r)} = (V, \{(v, u) \mid (u, v) \in E\})$. Digraph $X = (V, E)$ is *symmetric* if $E=E^{(r)}$ and is *antisymmetric* if $E \bigcap E^{(r)}=\varnothing$. An undirected graph is a pair $X = (V, E)$, where V is a finite set and E is a collection of two-element subsets of V. We can identify an undirected graph $X = (V, E)$ with the symmetric digraph $X_s = (V, E_s)$ where $E_s=\{(u, v) \mid \{u, v\} \in E\} \cup \{(v, u) \mid \{u, v\} \in E\}$. A digraph with exactly one vertex is called a *trivial* digraph. We denote by K_n^* the digraph with vertices the integers from 1 to n and arcs all pairs (i, j) of such integers with $i \neq j$. A digraph isomorphic to K_n^* is said to be a *complete symmetric digraph.*

For a digraph $X = (V, E)$ and a subset A of V, we can get a subdigraph $X[A]$ of X whose vertex set is A and whose arc-set consists of all arcs of X which have both ends in A. And we call the subdigraph $X[A]$ is an induced subdigraph of X.

Definition 1.1. *Let G be a group and T_0, $T_1 \subseteq G$. Then we define the Bi-Cayley digraph $X = BD(G, T_0, T_1)$ to be the bipartite digraph with vertex set $G \times \{0, 1\}$ and arc set $\{((g,\ 0),\ (t_0 \cdot g,\ 1)),\ ((t_1 \cdot g,\ 1),\ (g,\ 0)) \mid g \in G,$ $t_0 \in T_0,\ t_1 \in T_1\}$.*

By definition we observe that $d_X^+((g,0))=|T_0|$, $d_X^-((g, 0)) = |T_1|$, $d_X^+((g, 1)) = |T_1|$, $d_X^-((g, 1)) = |T_0|$, for any $g \in G$.

In this paper, we always denote $X_0 = G \times \{0\}$ and $X_1 = G \times \{1\}$. Some new results on the Bi-Cayley graph are referred to [2, 5, 6, 7], and the related knowledge about groups can be found in the book of Xu [10]. Let $R(G) = \{R(a) \mid R(a) : (g, i) \to (ga, i)$, for $a, g \in G$ and i=0, 1$\}$. Then we have the following proposition.

Proposition 1.2. *Let $X = BD(G, T_0, T_1)$. Then*

(1) *$R(G) \leqslant Aut(X)$, furthermore $Aut(X)$ acts transitively both on X_0 and X_1.*

(2) *X is strongly connected if and only if $|T_0| \geq 1$, $|T_1| \geq 1$ and $G = \langle T_1^{-1}T_0 \rangle$.*

Proof. (1) For any $R(a) \in R(G)$ and $((g_1, 0), (g_2, 1)) \in E(X)$, there exists some $t_0 \in T_0$ such that $g_2 = t_0 g_1$, then $g_2 a = t_0 g_1 a$. Thus $((g_1, 0), (g_2, 1))^{R(a)}$

$= ((g_1 a, 0), (g_2 a, 1)) \in E(X)$. Similarly, if $((g_2, 1), (g_1, 0)) \in E(X)$, then $((g_2, 1), (g_1, 0))^{R(a)} \in E(X)$. So R(a) is an automorphism of the Bi-Cayley digraph X, thus $R(G) \leq Aut(X)$. Since $(g_1, i)^{R(g_1^{-1} g_2)} = (g_2, i)$ for any $g_1, g_2 \in G$, $Aut(X)$ acts transitively both on X_0 and X_1.
(2) If X is strongly connected, then $|T_0| \geq 1$, $|T_1| \geq 1$ and there exists a directed path from $(1_G, 0)$ to $(g, 0)$ for any $g \in G$. Thus there exists an integer n, $t_0^{(i)} \in T_0$ and $t_1^{(i)} \in T_1 (1 \leq i \leq n)$ such that $1_G \to t_0^{(1)} \to (t_1^{(1)})^{-1} t_0^{(1)} \to \cdots \to (t_1^{(n)})^{-1} t_0^{(n)} \cdots (t_1^{(2)})^{-1} t_0^{(2)} (t_1^{(1)})^{-1} t_0^{(1)} = g$, that is $G = < T_1^{-1} T_0 >$. On the other hand, for any h, $g \in G$, $h^{-1} g$ is in $G = < T_1^{-1} T_0 >$ if and only if it can be written as a product of elements of $T_1^{-1} T_0 \cup (T_1^{-1} T_0)^{-1}$. Thus we can easily know there exists a path from (g, i) to (h, i). And since $|T_0| \geq 1$ and $|T_1| \geq 1$, $(g, 0)$ has both outarcs and inarcs for any $g \in G$. So X is strongly connected. □

2 Connectivity

Let $X = (V, E)$ be a strongly connected digraph. An *arc disconnecting set* of X is a subset W of E such that $X \backslash W = (V, E \backslash W)$ is not strongly connected. An arc disconnecting set is *minimal* if no proper subset of W is an arc disconnecting set of X and is a *minimum arc disconnecting set* if no other arc disconnecting set has smaller cardinality than W. The *arc connectivity* $\lambda(X)$ of a nontrivial digraph X is the cardinality of a minimum arc disconnecting set of X.

The *positive arc neighborhood* of a subset A of V is the set $\omega_X^+(A)$ of all arcs which initiate at a vertex of A and terminate at a vertex of $V \backslash A$. The *negative neighborhood* of subset A of V is the set $\omega_X^-(A)$ of all arcs which initiate in $V \backslash A$ and terminate in A. Thus $\omega_X^-(A) = \omega_X^+(V \backslash A)$. Arc neighborhoods of proper, nonempty subsets of V, often called *arc-cuts*, are clearly arc disconnecting sets. Thus for any proper, nonempty subset A of V, $|\omega^+(A)| \geq \lambda(X)$. If we consider the cases where A consists of a single vertex or the complement of a single vertex, we easily see that $\lambda(X) \leq \delta(X)$.

A nonempty subset A of V is called a *positive (respectively, negative) arc fragment* of X if $|\omega^+(A)| = \lambda(X)$ (respectively, $|\omega^-(A)| = \lambda(X)$). An arc fragment A with $2 \leq |A| \leq |V(X)| - 2$ is called a *strict arc fragment* of X. An arc fragment of minimum cardinality is called λ-*atom* of X and a strict arc fragment of least possible cardinality is called a λ-*superatom* of X. Note that a λ-atom (respectively, λ-superatom) may be either a positive arc fragment or a negative arc fragment or both. A $\lambda-$atom which is a positive (respectively, negative) arc fragment is called a *positive (respectively, nega-*

tive) λ*-atom* and a λ—superatom which is a positive (respectively, negative) arc fragment is called a *positive (respectively, negative)* λ*-surperatom.*

A *vertex disconnecting set* of X is a subset F of $V(X)$ such that $X\backslash F$ is either trivial or is not strongly connected. We often call F a *vertex-cut.* The *connectivity* $\kappa(X)$ of a nontrivial digraph X is the cardinality of a minimum vertex disconnecting set of X.

The *positive neighborhood* of a subset F of V is the set $N^+(F)$ of all vertices of $V \setminus F$ which are targets of arcs initiating at a vertex of F. The *positive closure* $C^+(F)$ of F is the union of F and $N^+(F)$. The *negative neighborhood* of subset F of V is the set $N^-(F)$ of all vertices of $V \setminus F$ which are the initial vertices of arcs which terminate at a vertex of F. The *negative closure* $C^-(F)$ of F is the union of F and $N^-(F)$.

If F is a nonempty subset of V with $C^+(F) \neq V$, then the positive neighborhood of F is clearly a vertex disconnecting set for X. Thus for each such set F, $|N^+(F)| \geq \kappa(X)$. If we consider the cases where F consists of a single vertex or the complement of a single vertex, we easily see that $\kappa(X) \leq \delta(X)$. There are many elegant and powerful results on connectivity in graph theory, see [8, 9, 11] for example.

A nonempty subset F of V is called a *positive* (respectively, *negative*) *fragment* of X if $|N^+(F)| = \kappa(X)$ and $C^+(F) \neq V$ (respectively, $|N^-(F)| = \kappa(X)$ and $C^-(F) \neq V$). A fragment of minimum cardinality is called *atom.* Note that an atom may be either a positive fragment or a negative fragment or both. A atom which is a positive (respectively, negative) fragment is called a *positive (respectively, negative) atom* .

A digraph X is *maximally arc connected* (respectively, *maximally connected*), or more simply, *max*-λ (respectively, *max*-κ), if $\kappa(X) = \delta(X)$ (respectively, $\lambda(X) = \delta(X)$). And X is *super arc connected,* or more simply, *super*-λ if every minimum arc-cut of X is either the set of inarcs of some vertex or the set of outarcs of some vertex. The relationship of $\lambda(X)$ and $\kappa(X)$ is well known: $\kappa(X) \leq \lambda(X) \leq \delta(X)$. So if $\kappa(X) = \delta(X)$, then $\lambda(X) = \delta(X)$. In the following of this section, we try to prove that $\kappa(X) = \delta(X)$ for Bi-Cayley digraphs.

A desirable property one wishes any type of atom to have is that, if nontrivial, they form *imprimitive blocks* for the automorphism group of the digraph. To be precise. an *imprimitive block* for a group Φ of permutations of a set T is a proper, nontrivial subset A of T such that if $\varphi \in \Phi$ then either $\varphi(A) = A$ or $\varphi(A) \cap A = \emptyset$. In the following proposition, Hamidoune has proved that the positive (respectively, negative) atoms of X are imprimitive blocks of X. The following proposition also indicates why imprimitivity is so useful.

Proposition 2.1. [4] *Let $X = (V, E)$ be a graph or digraph and let Y be the subgraph or subdigraph induced by an imprimitive block A of X. Then*
1. If X is vertex-transitive then so is Y;
2. If X is a strongly connected arc-transitive digraph or a connected edge-transitive graph and A is a proper subset of V, then A is an independent subset of X.

Proposition 2.2. [3] *Let $X = (V, E)$ be a strongly connected digraph which is not a complete symmetric digraph and let A be a positive (respectively, negative) atom of X. If B is a positive (respectively, negative) fragment of X with $A \cap B \neq \emptyset$, then $A \subset B$.*

Proposition 2.3. *Let X be a strongly digraph with $\kappa(X) < \delta(X)$, and A be an atom of X. Then $X[A]$ is strongly connected.*

Clearly if $X = BD(G, T_0, T_1)$ is a strongly connected Bi-Cayley digraph with $\kappa(X) < \delta(X)$ and A is an atom of X, then $A_i = A \cap X_i \neq \varnothing$ for $i = 0, 1$.

Lemma 2.4. *Let $X = BD(G, T_0, T_1)$ be a strongly connected Bi-Cayley digraph with $\kappa(X) < \delta(X)$. If A is an atom of X, then*
(1) *$V(X)$ is a disjoint union of distinct positive(or, negative) atoms of X;*
(2) *Let $Y = X[A]$. Then $Aut(Y)$ acts transitively both on A_0 and A_1;*
(3) *If $(1, i) \in A_i = H_i \times \{i\}$, then H_i is the subgroup of G for i=0,1;*
(4) $|A_0| = |A_1|$.

Proof. (1) and (2) follow from the results that the distinct positive (negative) atoms are disjoint and $Aut(X)$ acts transitively both on X_0 and X_1.
(3) For any $g \in H_0$, Ag is also a positive atom since R(g)$\in$ Aut(X). And $g \in A \cap Ag$, then we get that $A = Ag$, thus $A_0 g = A_0$ and $A_1 g = A_1$. The former equality means that H_0 is a subgroup of G.
(4) From Proposition 1.2(1) and Proposition 2.2, we can get $V(X) = \cup_{i=1}^{k} \varphi_i(A)$ where $\varphi_i \in Aut(X)$ such that $\varphi_i(A) \cap \varphi_j(A) = \emptyset$ if $i \neq j$, then $X_i = \cup_{i=1}^{k} \varphi_i(A_i)$. Since $|X_0| = |X_1|$, we have $|A_0| = |A_1|$. □

From the proof of Lemma 2.4, $Y = X[A]$ has the property that $d_Y^+((g_i, 0)) = d_Y^+((g_j, 0))$ and $d_Y^-((g_i, 0)) = d_Y^-((g_j, 0))$ for any vertices $(g_i, 0)$, $(g_j, 0) \in A_0$. And if $(1, 0) \in A_0$,then $A_1 g = A_1$ is right for any $g \in H_0$, so $H_1 H_0 = H_1$. It means H_1 is a left coset of H_0 since $|H_0| = |H_1|$. We have the following lemma.

Lemma 2.5. *Let $X = BD(G, T_0, T_1)$ be a strongly connected Bi-Cayley digraph with $\kappa(X) < \delta(X)$, and A be a positive atom. Let $A_0 = \{g_1, g_2, ..., g_m\} \times \{0\} = H_0 \times \{0\}$ and $A_1 = \{g_1', g_2', ..., g_m'\} \times \{1\} = H_1 \times \{1\}$. Then*
(1) *If $t_i g_j \in H_1$ for some $t_i \in T_i$ (i=0, 1) and some some $j (1 \leq j \leq m)$,*

then $t_i g_k \in H_1$ *for any* $k(1 \le k \le m)$;
(2) *If* $t_1^{-1} g_j' \in H_0$ *for some* $t_1 \in T_1$ *and some* $j(1 \le j \le m)$, *then* $t_1^{-1} g_k' \in H_0$ *for any* $k(1 \le k \le m)$.

Proof. (1) Assume $(1,0) \in A_0$, then $H_1 H_0 = H_1$. If $t_i g_j \in H_1$, then $t_i g_j H_0 = t_i H_0 = H_1$. It means $t_i g_k \in H_1$ for any $k(1 \le k \le m)$.
(2) Similarly, assume $(1,0) \in A_0$, then $H_1 H_0 = H_1$. If $t_1^{-1} g_j' \in H_0$, then $g_j' \in t_1 H_0$. So $H_1 = t_1 H_0$. it means that $t_1^{-1} g_k' \in H_0$ for any $k(1 \le k \le m)$. □

Theorem 2.6. *Let* $X = BD(G, T_0, T_1)$ *be a strongly connected Bi-Cayley digraph. Then* $\kappa(X) = \delta(X)$

Proof. Suppose X is not max-κ. Without loss of generality, assume that $A = A_0 \cup A_1$ is a positive atom. Denote $A_0 = H_0 \times \{0\}$ and $A_1 = H_1 \times \{1\}$. If $|N^+(A_0) \setminus A_1| \ne 0$, then by Lemma 2.5 we have $|N^+(A_0) \setminus A_1| \ge |H_0|$. Thus $|N^+(A)| = |N^+(A_0) \setminus A_1| + |N^+(A_1) \setminus A_0| = |N^+(A_0) \setminus A_1| + |\{T_1^{-1} H_1 \setminus H_0\} \times \{0\}| \ge |H_0| + |T_1^{-1} \setminus H_0| \ge | T_1^{-1}| \ge \delta(X)$, a contradiction. If $|N^+(A_1) \setminus A_0| \ne 0$, then by Lemma 2.5 we have $|N^+(A_1) \setminus A_0| \ge |H_1|$. Thus $|N^+(A)| = |N^+(A_0) \setminus A_1| + |N^+(A_1) \setminus A_0| = |\{T_0 H_0 \setminus H_1\} \times \{1\}| + |N^+(A_0) \setminus A_1| \ge |T_0 \setminus H_1| + |H_1| \ge |T_0| \ge \delta(X)$, a contradiction. Therefore $N^+(A) = \emptyset$, it is a contradiction . □

Corollary 2.7. *Let* $X = BD(G, T_0, T_1)$ *be a strongly connected Bi-Cayley digraph. Then* $\kappa(X) = \lambda(X) = \delta(X)$.

3 Super arc-connectivity

A *weak path* of a digraph X is a sequence $u_0, ..., u_r$ of distinct vertices such that for $i = 1, ..., r$, either (u_{i-1}, u_i) or (u_i, u_{i-1}) is an arc of X. A directed graph is *weakly connected* if any two vertices can be joined by a weak path.

Proposition 3.1. *Let* $X = BD(G, T_0, T_1)$ *be a strongly connected Bi-Cayley digraph and* A *be a* $\lambda-$*superatom. Then*
(1) $Y = X[A]$ *is weakly connected;*
(2) $|A| \ge \delta(X)$.

Proof. Suppose A is a positive $\lambda-$superatom.
(1) If $|A| = 2$, then we obtain that A is not an independent set since $|N^+(A)| = \delta(X)$ and $N^+(u) \ne 0$ for any $u \in V(X)$. Now assume $|A| \ge 3$.

If $Y = X[A]$ is not weakly connected, we can get a $\lambda-$superatom with cardinality less than A, a contradiction.
(2) $\lambda(X) = |\omega_X^+(A)| \geq |A|(\delta(X) - (|A| - 1)) = |A|(\delta(X) - |A| + 1)$,
we can verify that $\lambda(X) > \delta(X)$ when $2 \leq |A| < \delta(X)$, a contradiction. □

Any digraph with $d^+(x) = d^-(x)$ for every vertex x of X is said to be a *balanced* digraph.

Proposition 3.2. [4] *Let $X = (V, E)$ be a strongly connected, balanced digraph and let A and B be arc fragments of X such that $A \not\subseteq B$ and $B \not\subseteq A$. If $A \cap B \neq \emptyset$ and $A \cup B \neq V$. Then each of the sets $A \cap B$, $A \cup B$, $A \setminus B$ and $B \setminus A$ is an arc fragments of X.*

Theorem 3.3. [4] *Let $X = (V, E)$ be a strongly connected balanced digraph which is not a symmetric cycle, is not super arc-connected and has $\delta(X) \geq 2$. If $\delta(X) > 2$ or X is vertex-transitive, then distinct $\lambda-$superatoms of X are vertex disjoint.*

Similarly, we can also achieve the analogous results.

Proposition 3.4. *Let $X = (V, E)$ be a strongly connected digraph and let A and B be positive (respectively, negative) arc fragments of X such that $A \not\subseteq B$ and $B \not\subseteq A$. If $A \cap B \neq \emptyset$ and $A \cup B \neq V$, then each of the sets $A \cap B$, $A \cup B$, $A \setminus B$ and $B \setminus A$ is a positive (respectively, negative) arc fragments of X.*

Theorem 3.5. *Let $X = (V, E)$ be a strongly connected digraph which is not a symmetric cycle, is not super arc-connected and has $\delta(X) \geq 2$. If $\delta(X) > 2$ or X is vertex-transitive, then distinct positive (respectively, negative) $\lambda-$superatoms of X are vertex disjoint.*

Lemma 3.6. *Let $X = BD(G, T_0, T_1)$ be strongly connected but not super$-\lambda$. If X is neither a directed cycle nor a symmetric cycle , then distinct positive (respectively, negative) $\lambda-$superatoms of X are vertex disjoint.*

Proof. Suppose to the contrary that there are distinct positive $\lambda-$superatoms A, B of X with $A \cap B \neq \emptyset$. By Proposition 3.4, each of $A \cap B, A \cup B, A \setminus B, B \setminus A$ is a positive arc fragment which is a proper subset of a $\lambda-$superatom. Therefore, each of these sets must have cardinality 1 so that we may assume $A = \{u, v\}$, $B = \{v, w\}$ with $u \neq w$. Thus we have $d^+_{X[A]}(u) = d^-_{X[A]}(v) \leq 1$, $d^-_{X[A]}(u) = d^+_{X[A]}(v) \leq 1$, $d^+_{X[B]}(v) = d^-_{X[B]}(w) \leq 1$ and $d^-_{X[B]}(v) = d^+_{X[B]}(w) \leq 1$.
Case 1 $\delta(X) = 1$.

$d_X^+(u) = d_X^+(v) = d_X^+(w) = 1$, so $|T_0| = 1$, $|T_1| = 1$. And because X is a strongly connected digraph, we can get X is a directed cycle, a contradiction.

Case 2 $\delta(X) = 2$.

$d_X^+(u) = d_X^+(v) = d_X^+(w) = 2.$

Because $X[A]$ and $X[B]$ are weakly connected and A, B and $A \cup B$ are arc fragments, we can deduce

$|T_0| = |T_1| = 2$ and $T_0 = T_1$.

Because X is strongly connected, X is a cycle, a contradiction.

Case 3 $\delta(X) \geq 3$.

It is true by Theorem 3.5. □

For the rest of the paper we set $A_i = A \cap X_1 = H_i \times i$, i = 0,1. Similarly to Lemma 2.4, we can derive the following theorem.

Lemma 3.7. *Let $X = BD(G, T_0, T_1)$, which is neither a directed cycle nor a symmetric cycle, be strongly connected but not super-λ . Let A be a λ-superatom of X. Then*
(1) $V(X)$ is a disjoint union of distinct positive (negative) λ-superatoms;
(2) Let $Y = X[A]$. Then $\mathrm{Aut}(Y)$ *acts transitively both on A_0 and A_1;*
(3) If A_i contains $(1, i)(i = 0, 1)$, then H_i is a subgroup of G;
(4) $|A_0| = |A_1|$.

Similarly as Lemma 2.4, we also have $H_1H_0 = H_1$ if $(1, 0) \in A_0$ and $H_0H_1 = H_0$ if $(1, 1) \in A_1$. The following proposition is easy to get.

By a similar argument as Lemma 2.5, the following lemma is obtained.

Lemma 3.8. *Let $X = BD(G, T_0, T_1)$, which is neither a directed cycle nor a symmetric cycle, be strongly connected but not super-λ . Let A be a λ-superatom of X and set $A_0 = \{g_1, g_2, g_3, ..., g_m\} \times \{0\} = H_0 \times \{0\}$ and $A_1 = \{g_1', g_2', g_3', ..., g_m'\} \times \{1\} = H_1 \times \{1\}$. Then*
(1) If $t_ig_j \in H_1$ for some $t_i \in T_i$ (i=0, 1) and some some $j(1 \leq j \leq m)$, then $t_ig_k \in H_1$ for any $k(1 \leq k \leq m)$;
(2) If $t_1^{-1}g_j' \in H_0$ for some $t_1 \in T_1$ and some $j(1 \leq j \leq m)$, then $t_1^{-1}g_k' \in H_0$ for any $k(1 \leq k \leq m)$.

Theorem 3.9. *Let $X = BD(G, T_0, T_1)$ be strongly connected. If X is neither a directed cycle nor a symmetric cycle, then X is not super-λ if and only if X satisfies one of the following conditions:*
(1) There exists a subgroup $H \leq G$ and there distinct elements $t_0, t_0', t_0'' \in T_0$ such that

$$|H| = \delta(X),\ T_1^{-1}t_0 \subseteq H\ ,t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H \text{ and } t_0^{-1}t_0' \notin H.$$

or

$|H| = \delta(X)/2$, $T_1^{-1}t_0 \subseteq H$, $t_0^{-1}(T_0 \setminus \{t_0', t_0''\}) \subset H$ *and* $t_0^{-1}t_0', t_0^{-1}t_0'' \notin H$. *Where* $t_0' \neq t_0$ *and* $t_0'' \neq t_0$

(2) *There exists a subgroup* $H \leq G$ *and two distinct elements* $t_1, t_1', \in T_1$ *and some element* $t_0 \in T_0$ *such that*

$|H| = \delta(X)$, $t_0^{-1}T_0 \subset H$, $(T_1 \backslash \{t_1'\})^{-1}t_0 \subset H$ *and* ${t_1'}^{-1}t_0 \notin H$.

or

$|H| = \delta(X)/2$, $t_0^{-1}T_0 \subset H$, $(T_1 \backslash \{t_1, t_1'\})^{-1}t_0 \subset H$ *and* $t_1^{-1}t_0, {t_1'}^{-1}t_0 \notin H$.

(3) *There exists a subgroup* $H \leq G$ *and two distinct elements* $t_0, t_0' \in T_0$ *and some element* $t_1 \in T_1$ *such that* $|H| = \delta(X)/2$, $t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H$, $t_0^{-1}t_0' \notin H$, $(T_1 \backslash \{t_1\})^{-1}t_0 \subset H$ *and* $t_1^{-1}t_0 \notin H$, *where* $t_0' \neq t_0$.

Proof. Necessity. Without loss of generality assume A is a positive $\lambda-$superatom of X and $(1,0) \in A$. From Lemma 3.7, H_0 is a subgroup of G and $Y = X[A]$ is a bipartite digraph with $d_Y^+((g_k, i))=d_Y^+((g_l, i))$ and $d_Y^-((g_k, i))=d_Y^-((g_l, i))$ for any vertices $(g_k, i), (g_l, i) \in A$(i=0, 1). Furthermore, $d_Y^+((g_j, 0)) = d_Y^-((g_t, 1))$ and $d_Y^-((g_j, 0)) = d_Y^+((g_t, 1))$. Denote $d_Y^+((g_j, 0)) = d_Y^-((g_t, 1)) = p$, $d_Y^-((g_j, 0)) = d_Y^+((g_t, 1)) = q$. Let $H = H_0$.

Claim: There exist at least an element $t_0 \in T_0$ such that $H_1 = t_0H_0$ if $(1,0) \in A$.

The proof of the Claim: If $p = 0$, then $\delta(X) = \lambda(X) = |\omega_X^+(A)| = |A_0|(|T_0| - p) + |A_1|(|T_1| - q) = |A_0||T_0| + |A_1|(|T_1| - q) \geq |T_0| + |A_1|(|T_1| - q) \geq |T_0| \geq \delta(X)$. So $|A_0| = |A_1| = 1$ and $|T_1| - q = 0$. So $\omega^+(A) = \omega^+(A_0)$, thus X is super$-\lambda$. By a similar argument we can prove X is super$-\lambda$ when $q = 0$. A contradiction. So $pq \neq 0$.

If $(1,0) \in A_0$, then $H_1 = H_1H_0$. And because $p \neq 0$, there exist at least an element $t_0 \in T_0$ such that $t_0 \in H_1$. Thus $H_1 = t_0H_0$. So the Claim is true.

Since $\delta(X) = \lambda(X) = |\omega_X^+(A)| = |A_0|(|T_0| - p) + |A_1|(|T_1| - q)$, $|A| \geq \delta(X)$ and $|A_0| = |A_1|$, we have $|A_0| = |A_1| \geq \delta(X)/2$ and $|T_0| - p + |T_1| - q \leq 2$. Now we consider fine cases.

Case 1 $|T_0| - p = 1$ and $|T_1| - q = 0$.

(i) $\lambda(X) = |\omega_X^+(A)| = |A_0| = |H_0| = |H| = \delta(X)$, since $|T_0| - p = 1$ and $|T_1| - q = 0$.

(ii) Since $|T_0| - p = 1$, there exists an element $t_0' \in T_0$ such that $(T_0 \setminus \{t_0'\})H_0 \subset H_1$ and $t_0'H_0 \cap H_1 = \emptyset$. It means $(T_0 \setminus \{t_0'\})H_0 \subset t_0H_0$ and $t_0'H_0 \cap t_0H_0 = \emptyset$, so $t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H_0$ and $t_0^{-1}t_0' \notin H_0$.

(iii) since $|T_1| - q = 0$, we have that $T_1^{-1}H_1 \subseteq H_0$. It means $T_1^{-1}t_0H_0 \subseteq H_0$, so $T_1^{-1}t_0 \subseteq H_0$.

Case 2 $|T_0| - p = 0$ and $|T_1| - q = 1$.

(i) $\lambda(X) = |\omega_X^+(A)| = |A_1| = |H_1| = |H_0| = |H| = \delta(X)$, since $|T_0| - p = 0$ and $|T_1| - q = 1$.

(ii) Since $|T_0| - p = 0$, we have that $T_0H_0 \subseteq H_1$. It means $T_0H_0 \subseteq t_0H_0$, so $t_0^{-1}T_0 \subseteq H_0$.

(iii) since $|T_1| - q = 1$, there exists an element $t_1 \in T_1$ such that $(T_1 \setminus \{t_1\})^{-1}H_1 \subset H_0$ and $t_1^{-1}H_1 \cap H_0 = \emptyset$. It means $(T_1 \setminus \{t_1\})^{-1}t_0H_0 \subset H_0$ and $t_1^{-1}t_0H_0 \cap H_0 = \emptyset$, so $(T_1 \setminus \{t_1\})^{-1}t_0 \subset H_0$ and $t_1^{-1}t_0 \notin H_0$.

Case 3 $|T_0| - p = 2$ and $|T_1| - q = 0$.

It is similar to Case 1, we have

(i) $|H| = |H_0| = \delta(X)/2$.

(ii) $t_0^{-1}(T_0 \setminus \{t_0', t_0''\}) \subset H_0$ and $t_0^{-1}t_0', t_0^{-1}t_0'' \notin H_0$ for some $t_0', t_0'' \in T_0$.

(iii) $T_1^{-1}t_0 \subseteq H_0$.

Case 4 $|T_0| - p = 0$ and $|T_1| - q = 2$.

It is similar to Case 2, we have

(i) $|H| = |H_1| = \delta(X)/2$.

(ii) $t_0^{-1}T_0 \subset H_0$.

(iii) $(T_1 \backslash \{t_1', t_1''\})^{-1}t_0 \subset H_0$ and $t_1'^{-1}t_0, t_1''^{-1}t_0 \notin H_0$ for some $t_1', t_1'' \in T_1$.

Case 5 $||T_0| - p = 1$ and $|T_1| - q = 1$.

(i) $\lambda(X) = |\omega_X^+(A)| = |A_0| = |H_0| = |H| = \delta(X)/2$, since $|T_0| - p = 1$, and $|T_1| - q = 1$.

(ii) since $|T_0| - p = 1$, then $t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H_0$ and $t_0^{-1}t_0' \notin H_0$ for some element $t_0' \in T_0$ and $t_0' \neq t_0$.

(iii) since $|T_1| - q = 1$, $(T_1^{-1} \backslash \{t_1{}^{-1}\})t_0 \subset H_0$ and $t_1^{-1}t_0 \notin H_0$ for some $t_1 \in T_1$.

Sufficiency. Set $A = H \times \{0\} \cup (t_0H) \times \{1\}$. Thus $(1,0) \in A$, $H_0 = H$ and $H_1 = t_0H_0$.

(1) If $t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H$ and $t_0^{-1}t_0' \notin H$, then $t_0^{-1}(T_0 \setminus \{t_0'\})H = H$ and $t_0' \notin t_0H$, it is $(T_0 \setminus \{t_0'\})H = t_0H = H_1$ and $t_0'H_0 \cap H_1 = \emptyset$. So $|T_0| - p = 1$. And if $T_1^{-1}t_0 \subseteq H$, then $T_1^{-1}t_0H \subseteq H$. It is $T_1^{-1}H_1 \subseteq H$. So $|T_1| - q = 0$. Associate with the condition $|H| = \delta(X)$, we have $\lambda(X) = |\omega_X^+(A)| = |A_0| = |H| = \delta(X)$. So A is a $\lambda-$superatom of X.

Similarly, If $|H| = \delta(X)/2$, $T_1^{-1}t_0 \subseteq H$, $t_0^{-1}(T_0 \setminus \{t_0', t_0''\}) \subset H$ and $t_0^{-1}t_0', t_0^{-1}t_0'' \notin H$, we can prove A is a $\lambda-$superatom of X.

(2) If $(T_1 \backslash \{t_1'\})^{-1}t_0 \subset H$ and $t_1'^{-1}t_0 \notin H$, then $(T_1 \backslash \{t_1'\})^{-1}t_0H \subset H$ and $t_0 \notin t_1H$, it is $(T_1 \backslash \{t_1'\})^{-1}H_1 \subset H$ and $t_0H \cap t_1H = \emptyset$. So $|T_1| - q = 1$. And if $t_0^{-1}T_0 \subset H$, then $t_0^{-1}T_0H \subset H$, it is $T_0H \subset t_0H$. So $|T_0| - p = 0$. Associate with the condition $|H| = \delta(X)$, we get $\lambda(X) = |\omega_X^+(A)| = |A_1| = |H_1| = |H_0| = \delta(X)$. So A is a $\lambda-$superatom of X.

Similarly, If $|H| = \delta(X)/2$, $t_0^{-1}T_0 \subset H$, $(T_1 \backslash \{t_1, t_1'\})^{-1}t_0 \subset H$ and $t_1^{-1}t_0, t_1'^{-1}t_0$

H, we can prove A is a $\lambda-$superatom of X.
(3) If $t_0^{-1}(T_0 \setminus \{t_0'\}) \subset H$, $(T_1\setminus\{t_1\})^{-1}t_0 \subset H$, and $t_0^{-1}t_0', t_1^{-1}t_0 \notin H$ then $t_0^{-1}(T_0 \setminus \{t_0'\})H \subset H$, $(T_1\setminus\{t_1\})^{-1}t_0H \subset H$, $t_0' \notin t_0H$ and $t_0 \notin t_1H$, it is $(T_0 \setminus \{t_0'\})H \subset t_0H = H_1$, $(T_1\setminus\{t_1\})^{-1}H_1 \subset H_0$, $t_0'H \cap t_0H = \emptyset$ and $t_0H \cap t_1H = \emptyset$. So $|T_0| - p = 1$ and $|T_1| - q = 1$. Thus associate with the condition $|H| = \delta(X)/2$, we have $\lambda(X) = |\omega_X^+(A)| = |A_0| + |A_1| = |H_0| + |H_1| = \delta(X)$. So A is a $\lambda-$superatom of X. □

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