On the page number of triple-loop networks with even cardinality *

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Abstract: A book-embedding of a graph G consists of placing the vertices of G on a spine and assigning edges of the graph to pages so that edges assigned to the same page without crossing. In this paper, we propose schemes to embed the connected triple-loop networks with even cardinality in books, then we give upper bounds of page number of some multi-loop networks.

Keywords: Book embedding; Page number; Triple-loop networks; Multi-loop networks.

1 Introduction

In this paper, we investigate embedding of graph in structures called books. Let G be a graph, denote the vertex set of G by $V(G)$ and edge set by $E(G)$. A *book* consists of a *spine* which is just a line and some number of *pages* each of which is a half-plane with the spine as boundary. A book-embedding of a graph G consists of placing the vertices of G on the line in order and assigning edges of the graph to pages so that edges are assigned to same pages without crossing. *Page number*, denoted by $pn(G)$, is a measure of the quality of a book embedding which is the minimum number of pages in which G can be embedded. For an easier understanding of page number, it is helpful to have a look at the example in Fig. 1.

Ollmann [7] first introduce the page number problem and the problem is NP-complete, even if the order of nodes on the spine is fixed ([1, 2]).

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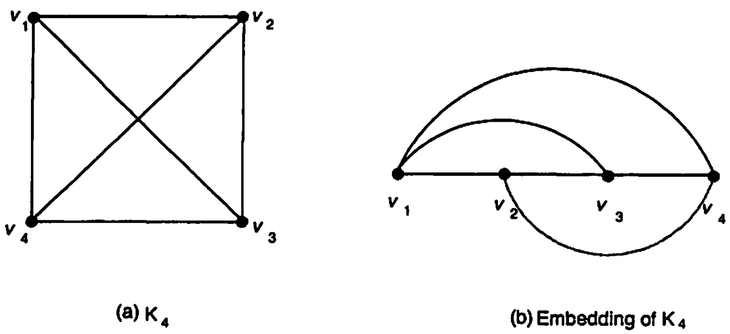

Fig.1 Embedding of K_4, $pn(K_4) = 2$. Ordering of $V(G) = \{v_1, v_2, v_3, v_4\}$. In (b), dashed line represents one page, black lines represent another.

The book embedding problem has been motivated by several areas of computer science such as sorting with parallel stacks, single-row routing, fault-tolerant processor arrays and turning machine graphs, see [1]. Book embeddings have applications in several contexts, such as VLSI design, fault-tolerant processing, sorting networks and parallel matrix multiplication ([1, 4, 5, 6]).

A multi-loop network, denoted by $ML(N; a_1, a_2, \ldots, a_l)$, can be represented by a directed graph with N nodes, $0, 1, \ldots, N-1$ and lN links of l types, where the type -a_i links (we call the type -s_i links s_i-arcs if there is no confusion) are

$$v \to v + a_i (\text{mod } N), v = 0, 1, \ldots, N-1 \text{ and } i = 0, 1, \ldots, l.$$

A triple-loop networks are denoted by $TL(N; a_1, a_2, a_3)$. In [8], Yang embed double-loop networks with even cardinality in books.

Theorem 1.1. *[8] Let $gcd(N; s) = d_1, gcd(N, t) = d_2$. Then $DL(N; s, t)$ can be embedded in a 4-page-book if d_1 (or d_2) is even. In particular, $DL(N; s, t)$ can be embedded in a 3-page-book if $N|d_1 t$ (or $N|d_2 t$).*

Theorem 1.2. *[8] Let $gcd(N; s) = d_1, gcd(N, t) = d_2$. Then $DL(N; s, t)$ can be embedded in a 7-page-book if d_1 and d_2 are odd. Furthermore, $DL(N; s, t)$ can be embedded in a 6-page book if $d_1 = 1$ (or $d_2 = 1$).*

In this paper, we propose schemes to embed the connected triple-loop networks with even cardinality in books, then we give upper bounds of page number of some multi-loop networks.

2 Preliminaries

Theorem 2.1. *[9] $ML(N; s_1, s_2, \dots, s_l)$ is strongly connected if and only if $gcd(N, s_1, s_2, \dots, s_l) = 1$.*

If $gcd(N, a_1, a_2, a_3) = d$ then we can decompose $TL(N; a_1, a_2, a_3)$ to d copies of $TL(\frac{N}{d}; \frac{a_1}{d}, \frac{a_2}{d}, \frac{a_3}{d})$. Since if G_1 and G_2 are two components of G, then $pn(G_1 \cap G_2) = max\{pn(G_1), pn(G_2)\}$ ([10]). So, we always assume that $gcd(N, a_1, a_2, a_3) = 1$ in the following.

C. Godsil and G. Royle[11] have shown the next theorem. We use this theorem to prove a lemma which is important to this paper.

Theorem 2.2. *[11] If θ is an automorphism of the group G, then $X(G, C)$ and $X(G, \theta(C))$ are isomorphic.*

In Theorem 2.3, X is a Cayley graph, and C is an inverse-closed subset of $G \backslash e$. We use $a \equiv b$ to denote $a \equiv b \ (mod\ N)$ if there is no confusion. Let $gcd(N, a_i) = d_i$ for $i = 1, 2, 3$. By Theorem 2.3, we can draw the following lemma.

Lemma 2.3. *If $d_i = 1$ for some $i \in \{1, 2, 3\}$, then there are two integers b and c such that $TL(N; a_1, a_2, a_3) \cong TL(N; 1, b, c)$.*

Proof. Since Z_N^* is an automorphism of Z_N, by Theorem 2.3, we have $TL(N; a_1, a_2, a_3) \cong TL(N; ua_1, ua_2, ua_3)$ with $u \in Z_N^*$. Since $d_i = 1$, without loss of generality, we assume that $i = 1$, there are two nonnegative integers u and v such that $ua_1 + vN \equiv 1$. Clearly, $gcd(u, N) = 1$ and $ua_1 \equiv 1$. Let $b \equiv ua_2$ and $c \equiv ua_3$, $TL(N; a_1, a_2, a_3) \cong TL(N; ua_1, ua_2, ua_3) \cong TL(N; 1, b, c)$. □

3 Main results

In this section, we consider the upper bounds of page number of triple-loop networks and some multi-loop networks.

Theorem 3.1. *If d_i is even, $\frac{d_i}{2}$ is odd, and $a_j = (\frac{d_i}{2})a_l$, where i, j and l are distinct, and $i, j, l \in \{1, 2, 3\}$, then $pn(TL(N; a_1, a_2, a_3)) \leq 6$. In particulars, $pn(TL(N; a_1, a_2, a_3))$ is reduced one if $d_i a_l \equiv 0$. Further more, $pn(TL(N; a_1, a_2, a_3))$ can be reduced two if $N = 2a_j$*

Proof. Without loss of generality, we assume that d_1 is even, $\frac{d_1}{2}$ is odd, and $a_3 = (\frac{d_1}{2})a_2$. Next we derive the method of book embedding.

Let $C_i (i \in \{0, 1, \dots, d_1 - 1\})$ be an ordered $\frac{N}{d}$-element array (mod N is omitted) and

$C_0 = \big(0, a_1, 2a_1 \dots, (\frac{n}{d_1} - 1)a_1\big),$

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$C_i = \big(0 + ia_2, a_1 + ia_2, 2a_1 + ia_2, \dots, (\frac{N}{d_1} - 1)a_1 + ia_2\big)$, i is even and $i < \frac{d_1}{2}$,

$C_i = \big((\frac{N}{d_1} - 1)a_1 + ia_2, (\frac{N}{d_1} - 2)a_1 + ia_2, (\frac{N}{d_1} - 3)a_1 + ia_2 \dots, ia_2\big)$, i is odd and $i < \frac{d_1}{2}$,

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$C_i = \big(0 + (\frac{3d_1}{2} - i - 1)a_2, a_1 + (\frac{3d_1}{2} - i - 1)a_2, 2a_1 + (\frac{3d_1}{2} - i - 1)a_2, \dots, (\frac{N}{d_1} - 1)a_1 + (\frac{3d_1}{2} - i - 1)a_2\big)$, i is even and $i \geq \frac{d_1}{2} + 1$,

$C_i = \big((\frac{N}{d_1} - 1)a_1 + (\frac{3d_1}{2} - i - 1)a_2, (\frac{N}{d_1} - 2)a_1 + (\frac{3d_1}{2} - i - 1)a_2, (\frac{N}{d_1} - 3)a_1 + (\frac{3d_1}{2} - i - 1)a_2, \dots, 0 + (\frac{3d_1}{2} - i - 1)a_2\big)$, i is odd and $i \geq \frac{d_1}{2}$,

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$C_{d_1-1} = \big((\frac{N}{d_1} - 1)a_1 + \frac{d_1}{2}, (\frac{N}{d_1} - 2)a_1 + \frac{d_1}{2}, (\frac{N}{d_1} - 3)a_1 + \frac{d_1}{2}, \dots, 0 + \frac{d_1}{2}\big),$

Thus $\cup_{i=0}^{d-1} C_i = V(G)$, because $|C_i| = \frac{N}{d_1}$ and $C_i \cap C_j = \emptyset$.

Put C_i in the line with the ordering of $C_0, C_1, \dots, C_{d-1}$, then all vertices of $V(G)$ are assigned. Use $E(C_i)$ to denote an arc set containing all arcs induced by vertex set C_i and use $E(C_i, C_j)$ to denote an arc set containing all arcs from C_i to C_j.

There are some properties as follows.

1. The ordering of $V(TL(N; a_1, a_2, a_3))$ is $C_0 \to C_1 \to \dots \to C_{d_1-1}$.
2. The arc set $\{E(C_i) |\ i = 0, 1, \dots, d-1\}$ contains no a_2-arcs and a_3-arcs, and $\{E(C_i, C_j) |\ i, j = 0, 1, \dots, d-1, i \neq j\}$ contains no a_1-arcs.
3. Arc set $(\cup_{i=0}^{\frac{d_1}{2}-2} E(C_i, C_{i+1})) \bigcup E(C_{\frac{d_1}{2}-1}, C_{d_1-1}) \bigcup (\cup_{i=\frac{d_1}{2}}^{d_1-2} E(C_{i+1}, C_i))$ contains no a_3-arcs. Arc set $(\cup_{i=0}^{\frac{d_1}{2}-1} E(C_i, C_{d_1-i-1})) \bigcup (\cup_{i=\frac{d_1}{2}}^{d_1-1} E(C_i, C_{d_1-i-1}))$ contains no a_2-arcs.

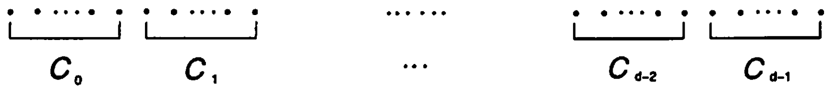

Fig.2 Property 1.

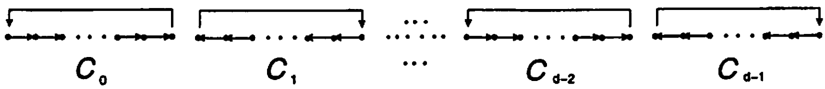

Fig.3 a_1-arcs.

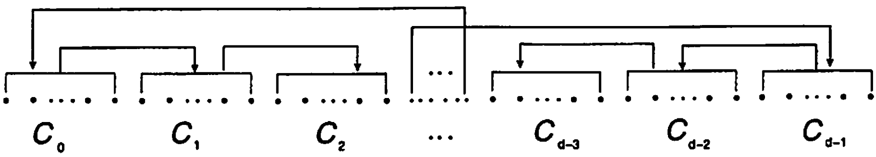

Fig.4 a_2-arcs.

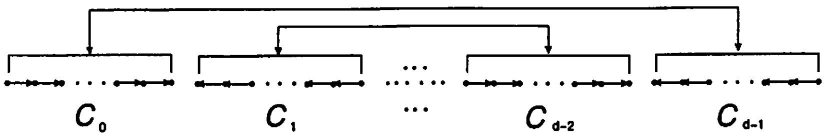

Fig.5 a_3-arcs.

For an easier understanding of the Property, it is helpful to have a look at Fig.2-5.

By Property 1 and each ordering of C_i for $i \in \{0, 1, \ldots, d_1 - 1\}$, we have that a_1-arcs is embedded in one page. Thus we only need to embed a_2-arcs and a_3-arcs in book.

Claim 1. Arc set a_2-arcs can be embedded in three pages with out crossing. In particular, it only need two pages to be embedded if $d_1 a_2 \equiv 0$.

Proof. In the ordering of $V(TL(N; a_1, a_2, a_3))$, if i is even and $i < \frac{d_1}{2} - 1$, then $E(C_i, C_{i+1})$ contains $\frac{N}{d_1}$ a_2-arcs and they are $\{(ja_1 + ia_2, ja_1 + (i+1)a_2) | j \text{ from } 0 \text{ to } \frac{N}{d_1} - 1\}$ which can be embedded in one page denoted by page-I without crossing. If i is odd and $i < \frac{d_1}{2}$, then arcs of $E(C_i, C_{i+1})$ are $\{(ja_1 + ia_2, ja_1 + (i+1)a_2) | j \text{ from } \frac{N}{d_1} - 1 \text{ to } 0\}$ which can be embedded in another page denoted by page-II without crossing.

Similarly, when $i \geq \frac{d_1}{2}$. If i is even, then $E(C_{i+1}, C_i)$ can be embedded in one page without crossing. Since $E(C_{i+1}, C_i)$ does not cross with $E(C_j, C_{j+1})$ for $j < \frac{d_1}{2}$, $E(C_{i+1}, C_i)$ can be embedded in page-I. If i is odd, then $E(C_{i+1}, C_i)$ can be embedded in one page. Since $E(C_{i+1}, C_i)$ does not cross with $E(C_j, C_{j+1})$ for $j < \frac{d_1}{2}$, $E(C_{i+1}, C_i)$ can be embedded in page-II.

Arc set $E(C_{\frac{d_1}{2}-1}, C_{d_1-1})$ contains $\frac{N}{d_1}$ a_2-arcs, and they are $\{(ja_1+(\frac{d_1}{2}-1)a_2, ja_1+\frac{d_1}{2}a_2)|j$ from 0 to $\frac{N}{d_1}-1\}$ which can be embedded in page-I.

Arc set $E(C_{\frac{d_1}{2}}, C_0)$ contains $\frac{N}{d_1}$ a_2-arcs, and they are $\{(ja_1+(d_1-1)a_2, ja_1+d_1a_2)|j$ from $\frac{N}{d_1}-1$ to $0\}$ which can be embedded in two pages. We assign $\{((d_1-1)a_2+ia_2, d_1a_2+ia_2)|i$ from 0 to $\frac{N-d_1a_2}{a_1}\}$ in page-II, and the other arcs of $E(C_{\frac{d_1}{2}}, C_0)$ can be embedded in another page. Clearly, this is an arrangement without crossing. So a_2-arcs can be embedded in three pages. In particular, if $N|d_1a_2$, then $E(C_{\frac{d_1}{2}}, C_0) = \{((d_1-1)a_2, 0), ((d_1-1)a_2+a_1, a_1), \ldots, ((d_1-1)a_2+(\frac{N}{d_1}-1)a_1, (\frac{N}{d_1}-1)a_1)\}$ only need one page. That is a_2-arcs can be embedded in two pages.

Claim 2. Arc set a_3-arcs can be embedded in two page without crossing. In particulars, it can be embedded in one page if $N = 2a_3$.

Proof. Since $a_3 = \frac{d_1}{2}a_2$, and $\frac{d_1}{2}$ are odd, in the vertex ordering of $TL(N; a_1, a_2, a_3)$, $E(C_i, C_{d_1-1-i}) = \{(ja_1+ia_2, ja_1+(i+\frac{d_1}{2})a_2)|0 \le j \le \frac{N}{d_1}-1, 0 \le i \le \frac{d_1}{2}-1\}$ can be embedded in one page. Arc set $E(C_i, C_{d_1-1-i}) = \{(ja_1+ia_2, ja_1+(i+\frac{d_1}{2})a_2)|0 \le j \le \frac{N}{d_1}-1, \frac{d_1}{2} \le i \le d_1-1\}$ can be embedded in two pages, and $\{(ja_1+ia_2, ja_1+(i+\frac{d_1}{2})a_2)|0 \le j \le \frac{N-2a_3}{a_1}, \frac{d_1}{2} \le i \le d_1-1\}$ can be embedded in one page and other arcs need another one. For $i, j \in \{0, 1, \ldots, \frac{d_1}{2}-1\}$ and $i \ne j$, arcs in $E(C_i, C_{d_1-1-i})$ do not cross with arcs in $E(C_j, C_{d_1-1-j})$. Since any a_3-arcs belong to $(\cup_{i=1}^{\frac{d_1}{2}-1} E(C_i, C_{d_1-1-i})) \bigcup (\cup_{i=1}^{\frac{d_1}{2}-1} E(C_{d_1-1-i}, C_i))$, a_3-arcs can be embedded in two pages.

When $N = 2a_3$, in the vertex ordering of $TL(N; a_1, a_2, a_3)$, for $i, j \in \{0, 1, \ldots, \frac{d_1}{2}-1\}$ and $i \ne j$, arcs in $E(C_i, C_{d_1-1-i})$ and arcs in $E(C_{d_1-1-i,C_i})$, where these arcs have same end vertices, have reverse direction, and they can be embedded in one page.

Combining Claim 1 and Claim 2, we have that $pn(TL(N; a_1, a_2, a_3)) \le 6$. In particulars, $pn(TL(N; a_1, a_2, a_3))$ is reduced one if $d_i a_l \equiv 0$. Further more, $pn(TL(N; a_1, a_2, a_3))$ can be reduced two if $N = 2a_j$. □

Let N and a_i are even, and assume $a_i \equiv q_i$ for $i = 1, 2, 3$. When $q_i|N$, let $s_i = \frac{q_i}{2}$. When $q_i \nmid N$, let $s_i = \frac{q_i}{2}+1$. When $q_i \nmid N$, assume $N = kq_i + t$, where k and t are positive integer. Thus let $s_i = \frac{q_i-t}{2}(l+1)$, where l is the minimum positive integer such that $q_i|lN$. For arc set E, we use $G[E]$ to denote induced subgraph by E. Since symmetry of triple-loop networks, $G[a_i\text{-arcs}] \cong G[(N-a_i)\text{-arcs}]$. Thus we can assume $q_i \le \frac{N}{2}$, so $s_i \le \lceil \frac{N}{2} \rceil$.

Lemma 3.2. *For positive integers N, l, a_1 and a_2, if N and a_1 are even, $a_1 \nmid lN$ $(l > 1)$, a_2 is odd, and $gcd(N, a_1) = d \ne 2$, then single-loop*

$(SL(N;a_1))$ *can be embedded in* s_1 *pages in vertex ordering* $(0, 2, 4, \ldots, N-2, a_2-2, \ldots, a_2+4, a_2+2, a_2)$.

Proof. Let the vertex ordering of single-loop $SL(N;a_1)$ be $(0, 2, \ldots, N-2, a_2-2, \ldots, a_2+2, a_2)$, where a_2 is an arbitrary odd integer and $a_2 < N$. Assume $a_1 \equiv q_1$, if $q_1|N$, then $s_1 = \frac{q_1}{2}$, where s is an positive integer. If $q_1 \nmid N$, then $s_1 = \frac{q_1}{2} + 1$. Let $V_1 = \{0, 2, \ldots, N-2\}$ and $V_2 = \{a_2 - 2, a_2 - 4, \ldots, a_2\}$. Thus we have $SL(N;a_1) = G[V_1] \cup G[V_2]$, $G[V_1] \cong G[V_2]$ and $G[V_1] \cap G[V_2] = \varnothing$. When $q_1|N$, let $N = kq_1$, where k is an integer, $E(G[V_1]) = \cup_{j=0}^{\frac{q_1}{2}-1} E_j$, where $E_j = \{(2j + iq_1, 2j + (i+1)q_1)|i = 0, 1, \ldots, k-1\}$ and each E_j can be embedded in one page without crossing. For $j_1 \neq j_2$, $E_{j_1} \cap E_{j_2} = \varnothing$, thus $E(G[V_1])$ needs s_1 pages to be embedded. So $pn(SL(N;a_1)) \leq s_1$.

When $q_1 \nmid N$ and $q_1 \nmid lN$ $(l > 1)$, let $N = kq_1 + t$ with $t > 0$, $E(G[V_1]) = \{(0, q_1), (q_1, 2q_1), \ldots, (N-q, 0)\}$ can be embedded in $\frac{q}{2} + 1$ pages. Staring from $(0, q_1)$ up to $((k-1)q_1, kq_1)$, every k arcs can be embedded in one page without crossing, total required $\frac{q_1}{2}$ pages because $|E(G[V_1])| = \frac{N}{2}$ and remain $\frac{t}{2}$ arcs unassigned. The remain $\frac{t}{2}$ arcs need another page to be embedded. So $pn(G[V_1]) \leq s_1$. Since $G[V_1] \cong G[V_2]$ and $G[V_1] \cap G[V_2] = \varnothing$, $pn(SL(N;a_1)) \leq s_1$.

When $q_1 \nmid N$ and $q_1|lN$ $(l > 1)$, let $N = kq_1 + t$ with $t > 0$, $E(G[V_1]) = \cup_{j=0}^{\frac{q_1-t}{2}} E_j = \{(2j + ia_1, 2j + (i+1)a_1)|0 \leq i \leq \frac{lN}{a_1} - 1\}$. Each E_j can be embedded in $(l+1)$ pages because every arc set $\{(mt + na_1, mt + (n+1)a_1)|0 \leq m \leq l-1, 0 \leq n \leq k-1\}$ can be embedded in one page, and arc $(2j - a_1, 2j)$ with $0 \leq j \leq \frac{q_1-t}{2}$ needs another page. So $pn(G[V_1]) \leq s_1$. Since $G[V_1] \cong G[V_2]$ and $G[V_1] \cap G[V_2] = \varnothing$, $pn(SL(N;a_1)) \leq s_1$. □

In next lemma, we do not discuss these cases which are $a_1 = a_3 = 1, a_2 \neq 2$, and $a_1 \neq a_3$, $a_1 = 1$ or $a_3 = 1$ (mod N is omitted).

Lemma 3.3. *For positive integer* N, a_1, a_2 *and* a_3, *if* $gcd(N, a_2) = 2$, *and* a_1, a_3 *are odd, then single-loop* $(SL(N;a_1))$ *can be embedded in four pages or three pages or one page in vertex ordering* $(0, a_2, 2a_2, \ldots, N-a_2, N-a_2+a_3, \ldots, 2a_2+a_3, a_2+a_3, a_3)$.

Proof. Let the vertex ordering of $SL(N;a_1)$ be $(0, a_2, 2a_2, \ldots, N-a_2, N-a_2+a_3, \ldots, 2a_2+a_3, a_2+a_3, a_3)$, where a_2 is even, and a_3 is odd.

If $a_1 = a_3 = 1$ and $a_2 = 2$, then 1-arcs can be embedded in two pages.

If $a_1 \neq a_3$, $a_1 \neq 1$ and $a_3 \neq 1$, then there is m_i, such that $a_1 + m_i a_2 \equiv a_3$ and denote the minimum m_i by m. Likewise, there is also a n_i, such that $a_3 + n_i a_2 + a_1 \equiv N - a_2$ and denote the minimum n_i by n. It is easily to see that all a_1 -arcs can be embedded in to four pages as follows.

page-1 : $\{(ia_2, ia_2 + a_1)|i = 0, 1, \ldots, m-1\}$.
page-2 : $\{(ia_2, ia_2 + a_1)|i = m, m+1, \ldots, \frac{N}{2} - 1\}$.

page-3 : $\{(a_3 + ia_2, a_3 + ia_2 + a_1)|i = 0, 1, \ldots, n\}$.
page-4 : $\{(a_3 + ia_2, a_3 + ia_2 + a_1)|i = n, n+1, \ldots, \frac{N}{2} - 1\}$.

If $a_1 = a_3 \neq 1$, then there is a k_i, such that $a_3 + k_i a_2 + a_1 \equiv N - a_2$ and denote the minimum k_i by k. Arc set a_1-arcs need only there pages to be embedded as follows.

page-1 : $\{(ia_2, ia_2 + a_1)|i = 0, 1, \ldots, \frac{N}{2} - 1\}$.
page-2 : $\{(a_3 + ia_2, a_3 + ia_2 + a_1)|i = 0, 1, \ldots, k\}$.
page-3 : $\{(a_3 + ia_2, a_3 + ia_2 + a_1)|i = k+1, k+2, \ldots, \frac{N}{2} - 1\}$. □

Theorem 3.4. *If d_i and d_j are even, $d_i \neq 2$, $d_j \neq 2$, and d_l is odd, where i, j and l are distinct, and $i, j, l \in \{1, 2, 3\}$, then $pn(TL(N; a_1, a_2, a_3)) \leq s_i + s_j + 3$. In particular, b and c are positive integer, if $d_l = 1$, then $TL(N; a_1, a_2, a_3) \cong TL(N; 1, b, c)$ and $pn(TL(N; a_1, a_2, a_3)) \leq min\{s_i + s_j + 2, s_b + s_c + 2\}$.*

Proof. Without loss of generality, we assume that d_1 and d_2 are even, $d_1 \neq 2$, $d_2 \neq 2$ and d_3 is odd. Let the vertex ordering of $TL(N; a_1, a_2, a_3)$ is $(0, 2, 4, \ldots, N-2, N-2+a_3, \ldots, a_3+4, a_3+2, a_3)$. By Lemma 3.2, a_1-arcs can be embedded in s_1 pages, and a_2-arcs can be embedded in s_2 pages. By Lemma 3.3, a_3-arcs can be embedded in three pages. Furthermore, if $d_3 = 1$, by Lemma 2.3, $TL(N; a_1, a_2, a_3) \cong TL(N; 1, b, c)$. Clearly, b and c are even. By Lemma 3.3, 1-arcs can be embedded in two pages. So, $pn(TL(N; 1, b, c)) \leq s_b + s_c + 2$.

Above all, if d_i and d_j are even, $d_i \neq 2$, $d_j \neq 2$, and d_l is odd, where i, j and l are distinct, and $i, j, l \in \{1, 2, 3\}$, then $pn(TL(N; a_1, a_2, a_3)) \leq s_i + s_j + 3$. In particular, if $d_l = 1$, $pn(TL(N; a_1, a_2, a_3)) \leq min\{s_i + s_j + 2, s_b + s_c + 2\}$. □

Theorem 3.5. *If d_i and $\frac{d_i}{2}$ are even, d_j and d_l are odd, where i, j and l are distinct, and $i, j, l \in \{1, 2, 3\}$, then $pn(TL(N; a_1, a_2, a_3)) \leq s_i + 7$. In particular, b is positive integer, if $d_j = 1$, then $TL(N; a_1, a_2, a_3) \cong TL(N; 1, b, c)$ and $pn(TL(N; a_1, a_2, a_3)) \leq min\{s_i + 6, s_b + 6\}$.*

Proof. Without loss of generality, we assume that d_1 is even, $d_1 \neq 2$, d_2 and d_3 are odd. Let the vertex ordering of $TL(N; a_1, a_2, a_3)$ be $(0, 2, \ldots, N-2, N-2+a_3, \ldots, a_3+2, a_3)$. By Lemma 3.2, a_1-arcs needs s_1 pages to be embedded. By Lemma 3.3, a_2-arcs need four pages to be embedded and a_3-arcs can be embedded in three pages. So, $pn(TL(N; a_1, a_2, a_3)) \leq s_i + 7$. Specially, if $d_2 = 1$ (or $d_3 = 1$), then $TL(N; a_1, a_2, a_3) \cong TL(N; 1, b, c)$, where b is even, and c is odd. Therefore, $pn(TL(N; a_1, a_2, a_3)) \leq min\{s_i + 6, s_b + 6\}$. □

Theorem 3.6. *If $d_i = 2$, d_j and d_l are odd, where i, j and l are distinct, and $i, j, l \in \{1, 2, 3\}$, then $pn(TL(N; a_1, a_2, a_3)) \leq 8$. In particular, if $d_j = 1$, then $pn(TL(N; a_1, a_2, a_3)) \leq 7$.*

Proof. Without loss of generality, we assume that $d_1 = 2$, d_2 and d_3 are odd. Let the vertex ordering of $TL(N; a_1, a_2, a_3)$ be $(0, a_1, 2a_1, \dots, N - a_1, N - a_1 + a_2, \dots, 2a_1 + a_2, a_1 + a_2, a_2)$. Clearly, a_1-arcs can be embedded in one page. Next, we embed a_2-arcs and a_3-arcs. By Lemma 3.3, a_2-arcs need three pages to be embedded, and a_3-arcs can be embedded in four pages. Therefore, $pn(TL(N; a_1, a_2, a_3)) \leq 8$. In particular, if $d_2 = 1$ (or $d_3 = 1$), then $TL(N; a_1, a_2, a_3) \cong TL(N; 1, b, c)$. By Lemma 3.3, 1-arc need two pages to be embedded. So, $pn(TL(N; a_1, a_2, a_3)) \leq 7$. □

The next Corollary is a simple application of Theorem 3.6.

Corollary 3.7. *In networks $ML(N; a_1, a_2, \dots, a_l)$, if $d_i = 2$ for $i \in \{0, 1, \dots, l\}$, and d_j is odd for any $j \in \{0, 1, \dots, l\}$ with $i \neq j$, then $pn(ML(N; a_1, a_2, \dots, a_l)) \leq 4(l-1)$. In particular, if $d_j = 1$, then $pn(ML(N; a_1, a_2, \dots, a_l)) \leq 4(l-1) - 1$.*

Theorem 3.8. *If d_1, d_2 and d_3 are odd, then $pn(TL(N; a_1, a_2, a_3)) \leq 11$. In particular, if $d_i = 1$ for $i \in \{1, 2, 3\}$, then $pn(TL(N; a_1, a_2, a_3)) \leq 10$.*

Proof. Let the vertex ordering of $TL(N; a_1, a_2, a_3)$ be $(0, 2, 4, \dots, N - 2, N - 2 + a_1, \dots, a_1 + 4, a_1 + 2, a_1)$. By Lemma 3.2, a_1-arcs can be embedded in three pages. a_2-arcs and a_3-arcs can be embedded in four pages respectively. So, $pn(TL(N; a_1, a_2, a_3)) \leq 11$. In particular, if $d_1 = 1$ (or $d_2 = 1$ or $d_3 = 1$), by Lemma 2.4 and 3.3, $pn(TL(N; a_1, a_2, a_3)) \leq 10$. □

The next Corollary is a simple application of Theorem 3.8.

Corollary 3.9. *In networks $ML(N; a_1, a_2, \dots, a_l)$, if d_i is odd for any $i \in \{0, 1, \dots, l\}$, then $pn(ML(N; a_1, a_2, \dots, a_l)) \leq 4l - 1$. In particular, if $d_i = 1$, for $i \in \{1, 2, 3\}$, then $pn(ML(N; a_1, a_2, \dots, a_l)) \leq 4l - 2$.*

4 Concluding remarks

In this work, we give the upper bounds of tipple-loop networks with even cardinality. For $TL(N; a_1, a_2, a_3)$, double-loop network $DL(N; a_1, a_2)$ is its subgraph. So, $pn(TL(N; a_1, a_2, a_3)) \geq pn(DL(N; a_1, a_2))$. For example, $DL(N; 18, 6, 5)$ is a subgraph of $TL(18; 6, 5, 15)$. By Theorem 1.1, $pn(DL(N; 18, 6, 5)) \leq 4$. By Theorem 3.1, $pn(TL(18; 6, 5, 15)) \leq 6$. The difference between $pn(DL(N; 18, 6, 5))$ and $pn(TL(18; 6, 5, 15))$ is two. As triple-loop networks are more complicated than double-loop networks, the upper bounds we give here are not bad. We leave for future study seeing whether these bounds can be improved.

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