Equitable and List Equitable Colorings of Graphs with Bounded Maximum Average Degree *

Aijun Dong[1†] Qingsong Zou[2] Guojun Li[3]

[1] School of Science, Shandong Jiaotong University, Jinan, 250023, P. R. China

[2] Department of Mathematics, Xidian University, Xi'an, 710071, P. R. China

[3] School of Mathematics, Shandong University, Jinan, 250100, P. R. Cina

Abstract

A graph is said to be equitably k-colorable if the vertex set $V(G)$ can be partitioned into k independent subsets V_1, V_2, $\cdots$, V_k such that $||V_i| - |V_j|| \leq 1$ $(1 \leq i,j \leq k)$. A graph G is equitably k-choosable if, for any given k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. In this paper, we prove that if G is a graph such that $mad(G) \leq 3$, then G is equitably k-colorable and equitable k-choosable where $k \geq \max\{\Delta(G), 5\}$.

Keywords: Graph coloring; Equitable choosability; Maximum average degree

MSC(2000): 05C15

1 Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let $G = (V, E)$ be a graph. We use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of G, respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (face) x in G. A vertex x is called a *k-vertex*, *k^+-vertex*,

*This work was supported by the National Natural Science Foundation of China (Grant No. 61103022). It was also supported by China Postdoctoral Science Foundation Funded Project and Nature Science Foundation of Shandong Province of China (Grant No. ZR2014AM028).

†The corresponding author.

if $d(x) = k$, $d(x) \geq k$. Let $n_i(v)$ denote the number of i-vertices incident to v for each $v \in V(G)$. The *girth* of a planar graph is the length of a smallest cycle in the graph, denote the girth of a graph G by $g(G)$. The *average degree* of a graph G is $\frac{\sum_{v \in V(G)} d(v)}{|V(G)|}$, denote it by $ad(G)$. The *maximum average degree* $mad(G)$ of G is the maximum of the average degree of its subgraphs.

A proper k-coloring of a graph G is a mapping π from the vertex set $V(G)$ to the set of colors $\{1, 2, \cdots, k\}$ such that $\pi(x) \neq \pi(y)$ for every edge $xy \in E(G)$. A graph G is equitable k-colorable if G has a proper k-coloring such that the size of the color classes differ by at most 1. The equitable chromatic number of G, denoted by $\chi_e(G)$, is the smallest integer k such that G is equitably k-colorable. The equitable chromatic threshold of G, denoted by $\chi_e^*(G)$, is the smallest integer k such that G is equitably l-colorable ($l \geq k$). It is obvious that $\chi_e(G) \leq \chi_e^*(G)$ for any graph G. They might not be equal. For example, if $K_{2n+1,2n+1}$ (n is a positive integer) is a complete bipartite graph, then $\chi_e(K_{2n+1,2n+1}) = 2$, $\chi_e^*(K_{2n+1,2n+1}) = 2n + 2$.

In many application of graph coloring, it is desirable that the color classes are not too large. For example, when using a coloring model to find an optimal final exam schedule, one would like to have approximately equal number of final exams in each time slot because the whole exam period should be as short as possible and the number of classrooms available is limited. Recently, Pemmaraju [14] and Janson and Ruciński [7] used equitable colorings to derive deviation bounds for sums of dependent random variables that exhibit limited dependence. In all of these applications, the fewer colors we use, the better the deviation bound is. Equitable coloring has a well-known property that restricts the size of each color class by its definition.

In 1970, Hajnál and Szemerédi proved that $\chi_e^*(G) \leq \Delta(G) + 1$ for any graph G [6]. This bound is sharp as shows the example of $K_{2n+1,2n+1}$. In 1973, Meyer introduced the notion of equitable coloring and made the following conjecture [12].

Conjecture 1 *If G is a connected graph which is neither a complete graph nor odd cycle, then $\chi_e(G) \leq \Delta(G)$.*

In 1994, Chen et al. put forth the following conjecture [2].

Conjecture 2 *For any connected graph G, if it is different from a complete graph, a complete bipartite graph and an odd cycle, then $\chi_e^*(G) \leq \Delta(G)$.*

Chen et al. proved the conjecture for graphs with $\Delta(G) \leq 3$ or $\Delta(G) \geq \frac{|V(G)|}{2}$ or a tree [2, 3]. Yap and Zhang proved that the conjecture holds for outer planar graphs and planar graphs with $\Delta(G) \geq 13$ [17, 18]. Lih and Wu verified $\chi_e^*(G) \leq \Delta(G)$ for bipartite graphs other than a complete bipartite graph [10]. Wang et al. proved the conjecture for line graphs [16], and Kostochka et al. proved it for d-generate graphs with $\Delta(G) \geq 14d+1$ [9].

For a graph G and a list assignment L assigned to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors, a L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. A list assignment L for G is k-*uniform* if $|L(v)| = k$ for all $v \in V(G)$. A graph G is equitably k-choosble if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices.

In 2003, Kostochka, Pelsmajer and West investigated the equitable list coloring of graphs. They proposed the following conjecture[8].

Conjecture 3 *Every graph G is equitably k-choosable whenever $k > \Delta(G)$.*

Conjecture 4 *If G is a connected graph with maximum degree at least 3, then G is equitably $\Delta(G)$-choosable, unless G is a complete graph or is $K_{k,k}$ for some odd k.*

It is proved that Conjecture 3 holds for graphs with $\Delta(G) \leq 3$ in [13, 15]. Kostochka, Pelsmajer and West proved that a graph G is equitably k-choosable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in $K_{k,k}$) and $k \geq \max\{\Delta, \frac{|V(G)|}{2}\}$, or G is a connected inteval graph and $k \geq \Delta(G)$ or G is a 2-degenerate graph and $k \geq \max\{\Delta(G), 5\}$ [8]. Pelsmajer proved that every graph is equitably k-choosable for any $k \geq \frac{\Delta(G)(\Delta(G)-1)}{2} + 2$ [13]. There are several results for planar graphs without short cycles [11, 19, 4].

In this paper, we show that if G is a graph such that $mad(G) \leq 3$, then G is equitably k-colorable and equitable k-choosable where $k \geq \max\{\Delta(G), 5\}$.

2 Graphs with $mad(G) \leq 3$

First let us introduce some important lemma.

Lemma 2.1 *([19]) Let $S = \{v_1, v_2, \cdots, v_k\}$ be a set of k different vertices in G such that $G - S$ has an equitable k-coloring. If $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable k-coloring.*

Lemma 2.2 *([8]) Let G be a graph with a k-uniform list assignment L. Let $S = \{v_1, v_2, \cdots, v_k\}$, where $\{v_1, v_2, \cdots, v_k\}$ are distinct vertices in G. If $G - S$ has an equitable L-coloring and $|N_G(v_i) - S| \le k - i$ for $1 \le i \le k$, then G has an equitable L-coloring.*

Lemma 2.3 *([6]) Every graph has an equitable k-coloring whenever $k \ge \Delta(G) + 1$.*

Lemma 2.4 *([13, 15]) Every graph G with maximum degree $\Delta(G) \le 3$ is equitably k-choosable whenever $k \ge \Delta(G) + 1$.*

Lemma 2.5 *Let G be a graph with $mad(G) \le 3$. Then G is 3-degenerate.*

Proof. By contradiction, there is subgraph G' of G such that $\delta(G') \ge 4$. It is clear that $mad(G') \ge 4$, a contradiction. ∎

Lemma 2.6 *Let G be a connected graph with order at least 5 and $mad(G) \le 3$. Then G has at least one of the structures in Figure 1.*

Proof. Let G be a counterexample. Then G does not contain $H_1 \sim H_{17}$ in Figure 1.

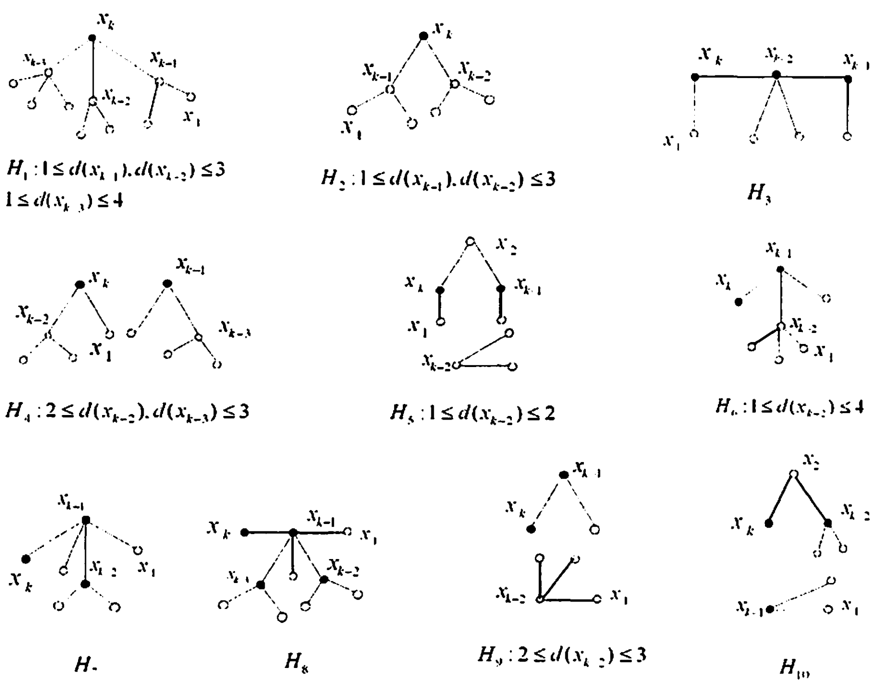

Figure 1

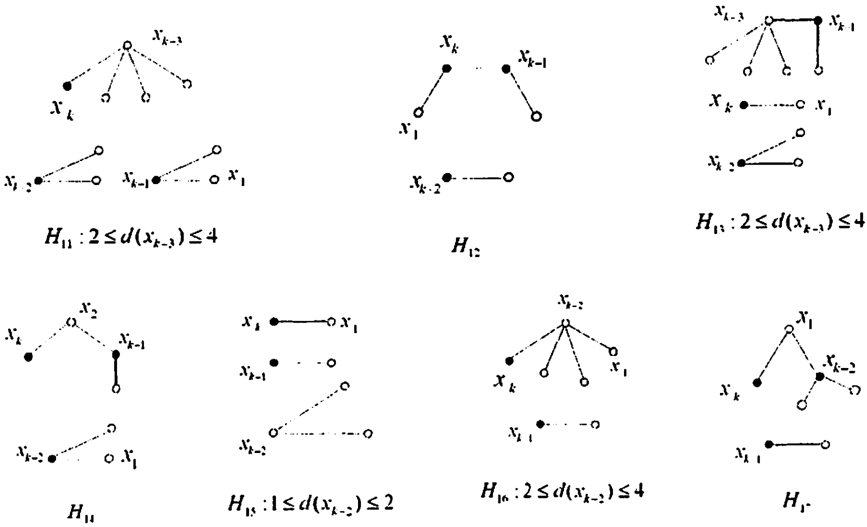

Figure 1

Each configuration in Figure 1 represents subgraphs for which: (1)hollow vertices may be not distinct while solid vertices are distinct. (2)the degree of the solid vertices is fixed, and (3)except for special pointed, the degree of a hollow vertices may be any integer from $[d, \Delta(G)]$, where d is the number of edges incident to the hollow vertex in the configuration.

We use a discharging procedure. For every $v \in V(G)$, we define the original charge of v to be $w(v) = d(v) - 3$. The total charge of the vertices of G is equal to

$$\sum_{v \in V(G)} (d(v) - 3) = |V(G)| \times (ad(G) - 3) \le |V(G)| \times (mad(G) - 3) \le 0.$$

In the following, we can redistribute the charge according to the given discharging rules. Let $w'(v)$ be the new charge of a vertex $v \in V(G)$. If $\sum_{v \in V(G)} w'(v) > 0$ can be deduced, we can show that the assumption is wrong.

We divide the proof into the following three cases by Lemma 2.5.

Case 1 $\delta(G) = 3$.

Since $mad(G) \le 3$, it is clear that G is a 3-regular graph. A contradiction for G contains no structure H_1.

Case 2 $\delta(G) = 2$.

Suppose $\delta(G) = 2$ and there is only one 2-vertex.

For the reason that G contains no structure H_2, we have $\Delta(G) \geq 4$. Since $mad(G) \leq 3$, we have $\Delta(G) = 4$ and there is only one 4-vertex in G. So G must contain the structure H_1, a contradiction.

Suppose $\delta(G) = 2$ and there are two 2-vertices in G.

For the reason that G contains no structure H_2, we have $\Delta(G) \geq 4$.

If $\Delta(G) = 4$, from $mad(G) \leq 3$, we deduce that there are at most two 4-vertices in G. Since G contains no structures H_2, H_3, G must contain the structure H_4, a contradiction.

If $\Delta(G) \geq 5$, from $mad(G) \leq 3$, we have $\Delta(G) = 5$ and there is only one 5-vertex in G, the other vertices of G are 3-vertices. So G must contain the structure H_4, a contradiction.

Suppose $\delta(G) = 2$ and there are at least three 2-vertices in G.

Since G contains no structure H_2, we have $\Delta(G) \geq 4$.

If $\Delta(G) = 4$, from $mad(G) \leq 3$, we deduce that the number of 4-vertices is no less than the number of 2-vertices. For the reason that G contains no structure H_3, G must contain the structure H_4, a contradiction.

If $\Delta(G) \geq 5$, we redistribute the charge according to the following discharging rule.

$R1$ Each 2-vertex receive charge 1 from its adjacent 4^+-vertices.

In the following, we check the new charge of the vertex $v \in V(G)$.

If $d(v) = 2$, then $w(v) = -1$. Since G contains no structure H_2, the vertex v is adjacent to at least one 4^+-vertex. So $w'(v) \geq -1 + 1 = 0$ by $R1$.

If $d(v) = 3$, then $w'(v) = w(v) = 0$.

If $d(v) = 4$, from G contains no structure H_3, the vertex v is adjacent to at most one 2-vertex. We have $w'(v) \geq 1 - 1 = 0$ by $R1$.

If $d(v) \geq 5$, then $w(v) = d(v) - 3$. Since G contains no structure H_5, the vertex v is adjacent to at most one 2-vertex. We have $w'(v) \geq d(v) - 3 - 1 \geq 5 - 3 - 1 = 1 > 0$ by $R1$.

From the above discussion, we have $\sum_{v \in V(G)} w'(v) > 0$, a contradiction.

Case 3 $\delta(G) = 1$.

Suppose $\delta(G) = 1$ and there is one 1-vertex and no 2-vertex in G.

For the reason that G contains no structure H_1, we have $\Delta(G) \geq 4$.

If $\Delta(G) = 4$, then there are at most two 4-vertices in G for the reason that $mad(G) \leq 3$. Since G contains no structure H_6, i.e. the 1-vertex is not adjacent to any 3-vertex, then G must contain the structure H_7, a contradiction.

If $\Delta(G) \geq 5$, then there is at most one 5-vertex and no 4- and 6^+-vertex in G for the reason that $mad(G) \leq 3$. Since G contains no structure H_6, then G must contain the structure H_8, a contradiction.

Suppose $\delta(G) = 1$ and there is one 1-vertex and one 2-vertex in G.

For the reason that G contains no structure H_1, we have $\Delta(G) \geq 4$.

If $\Delta(G) = 4$, then there are at most three 4-vertices in G for the reason that $mad(G) \leq 3$. Since G contains no structure H_9 and H_6, i.e. the 1-vertex is not adjacent to the 2-vertex and any 3-vertex, then G must contain the structure H_7, a contradiction.

If $\Delta(G) \geq 5$, then there is at most one 5^+-vertex and at most one 4-vertex in G for the reason that $mad(G) \leq 3$. Since G contains no structure H_9, H_6, then G must contain the structure H_{10}, a contradiction.

Suppose $\delta(G) = 1$ and there is one 1-vertex and at least two 2-vertices in G.

For the reason that G contains no structure H_1, we have $\Delta(G) \geq 4$.

If $\Delta(G) = 4$, for the reason that G contains no structure H_9, i.e. the 1-vertex is not adjacent to any 2-vertex, then the 1-vertex must be adjacent to a 3- or 4-vertex. Then G must contain the structure H_{11}, a contradiction.

If $\Delta(G) \geq 5$, for the reason that G contains no structure H_9 and H_{11}, i.e. the 1-vertex is not adjacent to any 2-, 3- or 4-vertex, then the 1-vertex must be adjacent to a 5^+-vertex. Since G contains no structure H_{12} and H_{13}, i.e. any 2-vertex is not adjacent to any 2-, 3- or 4-vertex, then 2-vertices must be adjacent 5^+-vertices.

Define discharging rules as the following statements.

*D*1 Every 1-vertex receives charge 2 from its neighbors of degree 5^+-vertex.

*D*2 Every 2-vertex receives charge 1 from its neighbors of degree 5^+-vertex.

In the following, let's check the charge of each element v for $v \in V(G)$.

If $d(v) = 1$, then $w(v) = -2$. From the above discussion, we have $w'(v) \geq -2 + 2 = 0$ by *D*1.

If $d(v) = 2$, then $w(v) = -1$. From the above discussion, we have $w'(v) \geq -1 + 1 \times 2 = 1 > 0$ by *D*2.

If $d(v) = 3$, then $w'(v) = w(v) = 0$.

If $d(v) = 4$, then $w'(v) = w(v) = 1$.

If $d(v) \geq 5$, then $w(v) = d(v) - 3$ and v is adjacent to at most one 1-vertex or 2-vertex for the reason that G contains no structure H_{14}. We have $w'(v) \geq d(v) - 3 - 2 \geq 5 - 3 - 2 = 0$ by *D*1 and *D*2.

From the above discussion, we have $\sum_{v \in V(G)} w'(v) > 0$, a contradiction.

Suppose $\delta(G) = 1$ and there are at least two 1-vertices in G.

Since G contains no structure H_{15}, there are at most two 1-vertices and no 2-vertex in G.

For the reason that G contains no structure H_{16}, we have $\Delta(G) \geq 5$. There are at most two 5^+-vertices in G from $mad(G) \leq 3$. Since G contains no structure H_{16}, we can obtain that G must contain the structure H_{17}, a contradiction. ∎

In the following, let us give the proof of the main theorems.

Theorem 2.7 *If G is a graph such that $mad(G) \leq 3$, then G is equitably k-colorable where $k \geq \max\{\Delta(G), 5\}$.*

Proof. Let G be a counterexample with fewest vertices. If each component of G has at most 4 vertices, then $\Delta(G) \leq 3$. So G is equitably k-colorable by Lemma 2.3. Otherwise, there is at least one component with at least five vertices. By Lemma 2.6, G has one of the structures $H_1 \sim H_{17}$, taking one and the vertices are labeled as they are in Figure 1. If there are vertices labeled repeatedly, then we take the larger (x_i is larger than x_{i-1}). In the following, we show how to find S in Lemma 2.1. If G has H_2, H_3, H_6, H_7, H_9, H_{12}, $H_{15} \sim H_{17}$, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. If G has H_5, H_{10}, H_{14}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$. If G has H_1, H_4, H_8, H_{11}, H_{13}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$. By Lemma 2.5, G is 3-degenerate, then we can find the remaining unspecified positions in S from highest to lowest indices by choosing a vertex with minimum degree in the graph obtained from G by deleting the vertices already being chosen for S at each step. By the minimality of $|V(G)|$ and $k \geq \Delta(G) \geq \Delta(G-S)$, $G-S$ is equitably k-colorable. So G is also equitably k-colorable by Lemma 2.1. ∎

Corollary 2.8 *Let G be a graph such that $mad(G) \leq 3$. If $\Delta(G) \geq 5$, then $\chi_e(G) \leq \Delta(G)$.*

Corollary 2.9 *Let G be a graph such that $mad(G) \leq 3$. If $\Delta(G) \geq 5$, then $\chi_e^*(G) \leq \Delta(G)$.*

Theorem 2.10 *If G is a graph such that $mad(G) \leq 3$ and $k \geq \max\{5, \Delta(G)\}$, then G is equitably k-choosable.*

Proof. Let G be a counterexample with the fewest vertices. If each component of G has at most 4 vertices, then $\Delta(G) \leq 3$. So G is equitably k-choosable by Lemma 2.4. Otherwise, the proof is similar to the proof of Theorem 2.7 by Lemma 2.5 and Lemma 2.2. ∎

Corollary 2.11 *Let G be a graph such that $mad(G) \leq 3$. If $\Delta(G) \geq 5$, then G is equitable $\Delta(G)$-choosable.*

For a planar graph with girth g, by $mad(G) < \frac{2g}{g-2}$, we have the following corollary.

Corollary 2.12 *Let G be a planar graph with girth $g \geq 6$. If $\Delta(G) \geq 5$, then G is equitably $\Delta(G)$-colorable and equitable $\Delta(G)$-choosable.*

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North-Holland, New York, 1976.

[2] B. L. Chen, K. W. Lih, P. L. Wu, Equitable coloring and the maximum degree, European J. Combin. 15 (1994) 443–447.

[3] B. L. Chen, K. W. Lih, Equitable coloring of trees, J. Combin. Theory Ser. B 61 (1994) 83–87.

[4] A. J. Dong, G. J. Li, etc. Equitable coloring and equitable choosability of planar graphs without 6- and 7-cycles, Ars Combinatoria, to appear.

[5] G. Fijavž, M. Juvan, B. Mohar, R. Škrekovki, Planar graphs without cycles of specific lengths, European. J. Combin. 23 (2002) 377–388.

[6] A. Hajnal, E. Szemerédi, Proof of a conjecture of Erdös, in: A. Rényi, V.T. Sós(Eds.), in: Combin Theory and Its Applications, vol. II, North-Holland, Amsterdam, 1970, 601–623.

[7] S. Janson, A. Ruciński The infamous upper tail. Random Structure and Algorithms, 20 (2002) 317–342.

[8] A. V. Kostochka, M. J. Pelsmajer, D. B. West, A list analogue of equitable coloring, J. Graph Theory 47 (2003) 166–177.

[9] A. V. Kostochka, K. Nakprasit, Equitable colorings of k-degenerate graphs, Combin. Probab. Comput. 12 (2003) 53–60.

[10] K. W. Lih, P. L. Wu, On equitable coloring of bipartite graphs, Discrete Math. 151 (1996) 155–160.

[11] Q. Li, Y. H. Bu, Equitable list coloring of planar graphs without 4- and 6-cycles, Discrete Mathematics 309 (2009) 280–287.

[12] W. Meyer, Equitable coloring, Amer. Math. Monthly 80 (1973) 920–922.

[13] M. F. Pelsmajer, Equitable list coloring for graphs of maximum degree 3, J. Graph Theory 47 (2004) 1–8.

[14] S. V. Pemmaraju, Equitable colorings extend Chernoff-Hoeffding bounds,In Proceedings of the 5th International Workshop on Randomization and Approximation Techniques in Computer Science (APPROX-RANDOM 2001), (2001) 285–296.

[15] W. F. Wang, K. W. Lih, Equitable list coloring of graphs, Taiwanese J. Math. 8 (2004) 747–759.

[16] W. F. Wang, K. M. Zhang, Equitable colorings of line graphs and complete r-partite graphs, System Sci. Math. Sci. 13 (2000) 190–194.

[17] H. P. Yap, Y. Zhang, The equitable Δ-coloring conjecture holds for outerplanar graphs, Bull. Inst. Math. Acad. Sin. 25 (1997) 143–149.

[18] H. P. Yap, Y. Zhang, Equitable colorings of planar graphs, J. Combin. Math. Combin. Comput. 27 (1998) 97–105.

[19] J. L. Zhu, Y. H. Bu, Equitable list colorings of planar graphs without short cycles, Theoretical Computer Science 407 (2008) 21–28.