THE N-FIXED POINT PROPERTY IN POSETS

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Abstract: In this paper, a generalized notion of the fixed point property, namely the n-fixed point property, for posets is discussed. The n-fixed point property is proved to be equivalent to the fixed point property in lattices. Further, it is shown that a poset of finite width has the n-fixed point property for some natural number n if and only if every maximal chain in it is a complete lattice.

Keywords: poset, lattice, completeness, fixed point property.

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1. Introduction

A *partially ordered set* (poset) P is a nonempty set together with a binary relation, which is reflexive, antisymmetric and transitive. For basic notions in posets refer to [7]. A map f of a poset P into itself is said to have a *fixed point* if there is an element $a \in P$ such that $f(a) = a$. If every order preserving map $f : P \to P$ has a fixed point, then P is said to have the *fixed point property*. The famous results of Tarski [9] and Davis [4] assert the equivalence of completeness and the fixed point property in lattices. From then onwards there has been much interest among mathematicians to characterize the fixed point property for more general posets. Recently, this problem has been shown to be computationally intractable [5].

In this paper, a generalized notion of the fixed point property, called the n-fixed point property, for posets is introduced and certain classes of posets having this property are characterized. Even though, several sufficient conditions for a poset to have the fixed point property have been discussed by many, the only one notable necessary condition determined so far is that every maximal chain in the poset is a complete lattice. In this context, an interesting observation is made in this paper that in a poset of finite width, the n-fixed point property for some $n \in N$ is equivalent to the completeness of all maximal chains in it.

Let P be a poset. Two elements $x, y \in P$ are said to be *comparable* if either $x \leq y$ or $y \leq x$ and it is denoted by $x \sim y$. A subset A of P is called an *antichain* if no two distinct elements of A are comparable. The supremum of cardinalities of all antichains in P is called the *width* of P. A nonempty subset C of P is called a *chain* if $x \sim y$ for every $x, y \in C$. A poset is called *(maximal) chain complete* if every (maximal) chain in P has a supremum. For chains C and D in P, $C \leq D$ means that $c \leq d$ for all $c \in C$ and $d \in D$. The set $\{p \in P : C \leq \{p\} \leq D\}$ is called a $(C, D)-$*core*. Sets C and D could be empty also. A subset W of P is said to be *well ordered* if each nonempty subset A of W has a least element. *Dually well ordered* sets are defined dually.

A subset R of P is said to be retract of P if there exists an order preserving map $r : P \to R$ such that the restriction of r to R is the identity map. The map r is called a retraction.

2. The n-fixed point property

Let P be a poset and n be any fixed positive integer. Then P is said to have the *n-fixed point property* if for every order preserving map $f : P \to P$, the map f^n has a fixed point, where f^n denotes the composition of the map f, n times. The existence of fixed points of the composition powers has been intensively investigated in [8].

If $f : P \to P$ is order preserving, then so is the map f^k. Therefore, if P has the n-fixed point property, then there exists $x \in P$ such that $(f^k)^n(x) = x$. Since $(f^k)^n = f^{k.n}$, the poset P has the kn-fixed point property for every natural number k. In particular, if P has the fixed point property, then P has the n-fixed point property for every natural number n.

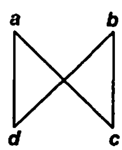

Figure 1

However, the n-fixed point property does not imply the $(n+1)$-fixed point property. For example, consider the poset P represented by Figure 1. The function f given by $f(a) = b, f(b) = a, f(c) = d$ and $f(d) = c$ is order preserving and does not fix any points. Moreover, we have $f^3 = f$. Hence P does not have the fixed point property and the 3-fixed point property. On the other hand, let $g : P \to P$ be any order preserving map. If g does not fix any element, then $g = f$. Thus P has the 2-fixed point property.

Proposition 2.1. *Let P be a poset with n components. If each of the components $P_1, P_2, ..., P_n$ has the fixed point property, then P has the $n!$-fixed point property.*

Proof. Let $f : P \to P$ be an order preserving map and $x \in P$. Then at least two elements of $\{x, f(x), f^2(x), ..., f^n(x)\}$ must lay in P_i, for some $1 \le i \le n$. Thus there exists $y \in P$ such that $y, f^k(y) \in P_i$, where $1 \le k \le n$. Thus $f^k(P_i) \subseteq P_i$. Since P_i has the fixed point property, it follows that $f^k(z) = z$ for some $z \in P_i$. Thus P has the $n!$-FPP. □

Proposition 2.2. *A poset has the n-fixed point property if and only if every retract of it has the n-fixed point property.*

Proof. The proof of sufficiency part is trivial as P is a retract of itself. On the other hand, let P be a poset with the n-fixed point property and R be a retract of P, with a retraction $r : P \to R$. Let $g : R \to R$ be an order preserving map. Then there exists an element $x \in P$ such that $(g \circ r)^n(x) = x$. Clearly, $x \in R$. Further, note that $(g \circ r)(y) = g(y)$ for every $y \in R$. Thus by induction, it follows that $(g \circ r)^n(x) = g^n(x)$. Hence R has the n-fixed point property. □

Proposition 2.3. *If a poset P has the n-fixed point property for some natural number n, then every maximal chain in P is a complete lattice.*

Proof. The proof of this theorem is an adaptation, with some modifications, of the proof of Theorem 2.5 in [3]. Suppose there is a maximal chain S which is not a complete lattice. Then there exists a dually well ordered chain C in S that does not have an infimum in S. Let C^∇ denote the set of all lower bounds of C in S. Let $\wp$ denote the set of all well ordered chains in C^∇. Then $\wp$ is a poset under the relation $\tilde{\le}$ defined by, $X \tilde{\le} Y$ if $X = Y$ or $X = \{y \in Y : y < z\}$ for some $z \in Y$. By Zorn's lemma, $\wp$ has a maximal element, say D.

Since S is a maximal chain in P, by Theorem 1 in [6], S is a retract of P. Hence by Proposition 2.2, S has the n-fixed point property for some n. Now, define a map $f : S \to S$ as follows:

$$f(x) = \begin{cases} min\{d \in D : d > x\} & \text{if } x \in C^\nabla; \\ max\{c \in C : c < x\} & \text{otherwise.} \end{cases}$$

Clearly, f is well defined and order preserving. Hence there exists $x \in P$ such that $f^n(x) = x$. As $f(P) \subseteq C \cup D$, either $x \in C$ or $x \in D$. Suppose $x \in C$. Then $x \notin C^\nabla$ so that $f(x) \in C$. But then $x > f(x)$. Continuing this procedure we get $x > f(x) > ... > f^n(x) = x$, a contradiction. On the other hand, if $x \in D$, then by duality $x < f(x) < ... < f^n(x) = x$, a contradiction.

Thus P does not have the n-fixed point property for any n. □

Since it is well known that a lattice is complete if and only if every maximal chain is complete (Proposition 5.1.7, [7]), the following corollary follows directly from Proposition 2.3.

Corollary 2.4. *If a lattice has the n-fixed point property for some natural number n, then the lattice is complete.*

Thus we have the following theorem.

Theorem 2.5. *The following statements are equivalent for a lattice L:*

(a) *L has the fixed point property.*
(b) *L has the n-fixed point property for every natural number n.*
(c) *L has the n-fixed point property for some natural number n.*
(d) *L is complete.*

Proposition 2.6. *Let P be a poset and $f : P \to P$ be any order preserving map. If P has exactly m minimal elements which form a maximal antichain, then there is a subset $S = \{x_1, x_2, ..., x_k\}$ of the minimal element set M of P satisfying $x_{i+1} \leq f(x_i)$, $i = 1, 2, ..., k-1$ and $x_1 \leq f(x_k)$, where $k < m$.*

Proof. Choose an element $x_1 \in M$. If $x_1 \leq f(x_1)$, then the result follows with $k = 1$. Otherwise there exists an element $x_2 \in M$, such that $x_2 \leq f(x_1)$. Suppose $x_1, x_2, ..., x_{n-1}$ are distinct elements chosen. Then there exists $x_n \in M$ such that $x_n \leq f(x_{n-1})$. If $x_i \leq f(x_n)$, for some $i = 1, 2, ..., n-1$, then relabeling the elements of the set $\{x_i, x_{i+1}, ..., x_n\}$, if necessary, the result follows. As there are only m minimal elements, the above procedure has to stop before m steps, to yield the required result. □

Corollary 2.7 (Corollary 5.3, [2]). *Let P be a poset with finitely many minimal elements M. Assume that*

(a) *for every nonempty subset S of M the subposet $S^{\Delta} = \{x \in P : x \geq s \text{ for all } s \in S\}$ has the fixed point property.*
(b) *for every $x \in P$ there there is an $m \in M$ such that $m \leq x$.*

Then P has the fixed point property.

Proof. Let $f : P \to P$ be an order preserving map. Note that condition (b) of this corollory is equivalent to saying that M is a maximal antichain. Hence by Proposition 2.6, there is a subset $S = \{x_1, x_2, ..., x_k\}$ of M satisfying $x_{i+1} \leq f(x_i)$, $i = 1, ..., k-1$ and $x_1 \leq f(x_k)$. Then as $f(S^{\Delta}) \subseteq S^{\Delta}$, by (a) there is an element $x \in S^{\Delta}$ such that $f(x) = x$. □

Corollary 2.8. *Let P be a chain complete poset with m minimal elements, satisfying condition (b) of Corollary 2.7. Then P has the n-fixed point property where $n = lcm(1, 2, 3, ..., m)$.*

Proof. Let x be an element of the set S obtained in Proposition 2.6. Clearly, $x \leq f^k(x)$ so that $x \leq f^n(x)$. Thus the corollary follows from Theorem 1 in [1]. □

The following two theorems characterize the maximal chain completeness for posets. Proof of the first theorem follows from Theorem 3.4.7 in [7] and Theorem 1 in [1]

Theorem 2.9. *Let P be a poset. Then every maximal chain in P is a complete lattice if and only if every order preserving map $f : P \to P$ satisfying $x \sim f(x)$ for all $x \in f(P)$, has a fixed point.*

Theorem 2.10. *Let P be a poset of finite width. Then P has the n-fixed point property for some natural number n if and only if every maximal chain in P is a complete lattice.*

Proof. Let P be a poset of width k. Necessity follows from Proposition 2.3.

To prove the converse, suppose that all maximal chains in P are complete. Let $f : P \to P$ be an order preserving map. Then for an element $x \in P$, the set $\{x, f(x), ..., f^k(x)\}$ cannot be an antichain. Thus $x \sim f^{k!}(x)$ for every element $x \in f^{k!}(P)$. Hence by Theorem 2.9, it follows that $f^{k!}(y) = y$ for some $y \in P$. Hence P has the $k!$-fixed point property. □

Corollary 2.11. *Chain complete posets of width 2 have the 2-fixed point property.*

Remark 2.12. In the proof of Theorem 2.10, $k!$ can be replaced by $lcm(1, 2, ..., k)$.

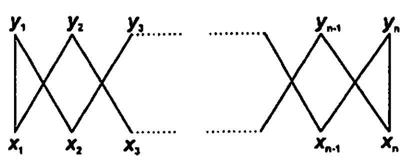

FIGURE 2

Denote by F_n, the set of all posets with the n-fixed point property. Let $\Gamma = \{F_n; n \in N\}$ and $P = (N, |)$, where $|$ denotes the usual divides relation. If m is a multiple of n (ie $n|m$), then $F_n \subseteq F_m$. On the other hand, if $n \not| m$, then the $2n$-crown shown in Figure 2 has the n-fixed point property, but does not have the m-fixed point property. Thus the map $f : P \to \Gamma$, defined by $f(n) = F_n$, is an isomorphism. In this connection we have the following open problem.

Probem. Do the m-fixed point property and n-fixed point property imply the $gcd(m, n)$-fixed point property?

Remark 2.13. The poset represented by Figure 3 has neither the 2-fixed point property nor the 3-fixed point property, but it has the 6-fixed point property.

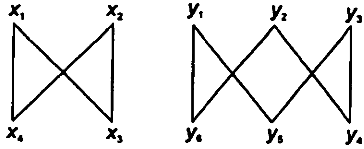

FIGURE 3

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