A Note on the Adjacent Vertex Distinguishing Total Chromatic Number of Some Cubic Graphs

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Abstract

An adjacent vertex distinguishing total coloring of a graph G is a proper total coloring of G such that no two adjacent vertices are incident to the same set of colors. The minimum number of colors needed for such a coloring is denoted by $\chi_{at}(G)$. In this note, we prove that $\chi_{at}(G) = 5$ for some cubic graphs.

Keywords: Adjacent vertex distinguishing total coloring, Flower snark, Goldberg snark, Cubic Halin graph

1 Introduction

Let $G = (V, E)$ be a simple graph. The maximum degree of G is written as $\Delta(G)$, or Δ for short. A proper k-total coloring of G is a mapping $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that no two adjacent vertices, no two incident edges, and no vertex and incident edge receive the same color. The total chromatic number $\chi_t(G)$ of G is the smallest integer k for which G admits a k-total coloring. For a vertex $v \in V(G)$, we set $C_f(v) = \{f(v)\} \cup \{f(uv) | uv \in E(G)\}$ and $\overline{C}_f(v) = \{1, 2, \ldots, k\} \setminus C_f(v)$. The coloring f is called an adjacent vertex distinguishing total coloring (avd-total coloring) if $C_f(u) \neq C_f(v)$ for any pair of adjacent vertices u and v. The adjacent vertex distinguishing total chromatic number, denoted by $\chi_{at}(G)$, is the least k such that G has a k-avd-total coloring. Obviously, $\chi_{at}(G) \geq \chi_t(G)$.

Zhang et al.[9] first introduced the notion of avd-total coloring. They determined the exact values of $\chi_{at}(G)$ for several classes of graphs, including paths, cycles, complete graphs, complete bipartite graphs and trees. In addition, they put forward the following challenging conjecture.

Conjecture 1.1 *[9] Let G be a connected graph with at least two vertices. Then $\chi_{at}(G) \le \Delta + 3$.*

This conjecture was confirmed by Chen [4] and Wang [6], independently, for graphs with $\Delta \le 3$. Later, Hulgan [5] presented a much shorter proof for this result. The adjacent vertex distinguishing total chromatic number for K_4-minor free graphs was characterized completely in Wang et al.[8]. Recently, a similar characterization for plane graphs with $\Delta \ge 14$ was given in Wang [7].

The following two results were obtained in Zhang et al.[9].

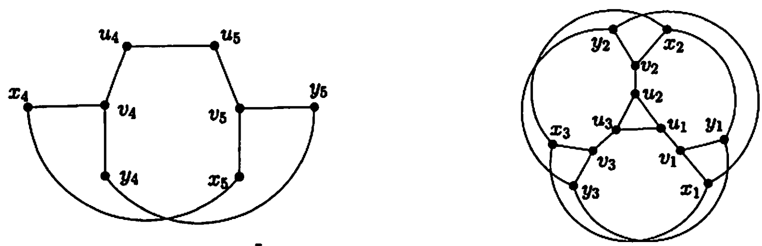
Lemma 1.1 *If G is a graph with two adjacent vertices of maximum degree, then $\chi_{at}(G) \ge \Delta + 2$.*

Lemma 1.2 *Let K_n be a complete graph on n vertices. Then*

$$\chi_{at}(K_n) = \begin{cases} n+1, & n \equiv 0 \ (mod\ 2); \\ n+2, & n \equiv 1 \ (mod\ 2). \end{cases}$$

A graph is cubic if the degree of each vertex is 3. Snarks are cubic bridgeless graphs with chromatic index 4 which had their origin in the search of counterexample to the Four Color Theorem. The Petersen graph is the smallest and earliest known snark. Two infinite families of snarks, Flower and Goldberg snark, were discovered by Isaacs [3] and Goldberg [2], respectively. Total coloring for these two families of snarks has been discussed in [1].

A Halin graph is a plane graph $G = T \cup C$ constructed as follows. Let T be a tree with no vertex of degree 2, and at least one vertex of degree 3 or more. Let C be a cycle connecting all leaves of T in such a way that C forms the boundary of the unbounded face. C is called the adjoint cycle of G.



(a) Link graph L_5 (b) Graph F_3

Figure 1: Link graph L_5 and graph F_3

In this note, we show that $\chi_{at}(G) = 5$ for Flower and Goldberg snarks, and cubic Halin graphs. Note that $\chi_{at}(G) \geq 5$ holds for any cubic graph by Lemma 1.1. So to prove our result, it suffices to provide a 5-avd-total coloring for these cubic graphs.

2 5-avd-total coloring of snarks

Following an argument similar to that used in [1] to prove the total chromatic number of snarks, we determine the avd-total coloring chromatic number of Flower and Goldberg snarks in this section. Graphs in these two families have a common property that they can be built from a suitable glueing of some basic graphs.

2.1 Flower snarks

For this family, we define the basic graph B_i as the graph with vertex set $V(B_i) = \{u_i, v_i, x_i, y_i\}$ and edge set $E(B_i) = \{u_iv_i, x_iv_i, y_iv_i\}$. Define the link graph L_i as the union of B_{i-1}, B_i and the graph spanned by $E_{(i-1)i}$, where $E_{ij} = \{u_iu_j, x_ix_j, y_iy_j\}$. In other words, $V(L_i) = V(B_{i-1}) \cup V(B_i)$ and $E(L_i) = E(B_{i-1}) \cup E(B_i) \cup E_{(i-1)i}$. Figure 1(a) shows L_5.

The first Flower snark, F_3, is formed by graphs B_1, B_2, B_3, and the graph spanned by $E_{23} \cup E_{31} \cup \{u_1u_2, x_1y_2, y_1x_2\}$(see Figure 1(b)). For each odd $i \geq 5$, let F_i be constructed from graphs F_{i-2} and L_i as follows:

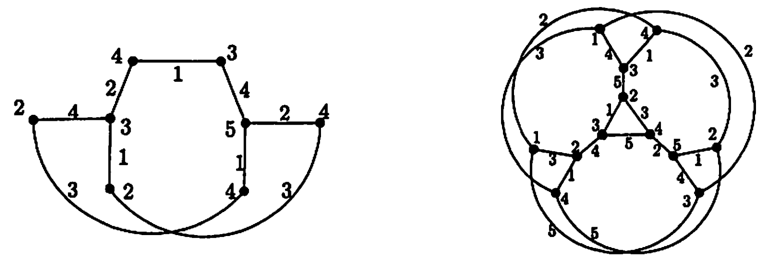


Figure 2: A coloring for graphs L_i and F_3, respectively

$V(F_i) = V(F_{i-2}) \cup V(L_i)$, and $E(F_i) = (E(F_{i-2}) \setminus E_{i-2}^{out}) \cup E(L_i) \cup E_i^{in}$, where $E_{i-2}^{out} = E_{(i-2)1}$, and $E_i^{in} = E_{(i-2)(i-1)} \cup E_{i1}$.

Theorem 2.1 *Let F_i be a Flower snark with odd $i \geq 3$. Then $\chi_{at}(F_i) = 5$.*

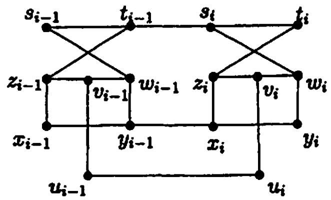
Proof. We prove it by showing that each F_i allows a 5-avd-total coloring such that all edges of E_i^{out} receive the same color 5. It is proceeded by induction based on the recursive procedure described above. We begin with a 5-avd-total coloring of L_i and F_3, as depicted in Figure 2. Notice that edges of E_3^{out} have the same color 5.

Now assume that i is odd and $i \geq 5$. By induction, F_{i-2} has a 5-avd-total coloring such that the edges of E_{i-2}^{out} have the same color 5. Let us construct a 5-avd-total coloring f of F_i in the following ways. Assign color 5 to the edges of E_i^{in}. Elements of $(V(F_i) \cup E(F_i)) \setminus E_i^{in}$ have the same color as their corresponding parts in F_{i-2} or L_i.

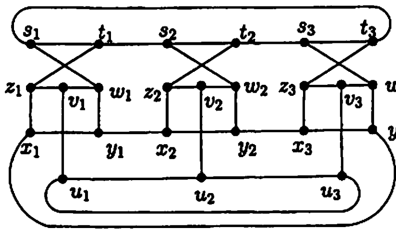
We complete the proof by showing that f is proper. First, it is easy to check that the ends of edges in E_i^{in} have distinct colors. And no edge in L_i has color 5. So f is a proper 5-total coloring. Furthermore, observe that $\overline{C}_f(x_1) = \{1\}$, $\overline{C}_f(y_1) = \{4\}$, $\overline{C}_f(u_1) = \{1\}$ and $\overline{C}_f(u_{i-2}) = \{2\}$, $\overline{C}_f(u_{i-1}) = \{3\}$, $\overline{C}_f(u_i) = \{2\}$. For F_5, $\overline{C}_f(x_3) = \{4\}$, $\overline{C}_f(y_3) = \{2\}$, $\overline{C}_f(x_4) = \{1\}$, $\overline{C}_f(y_4) = \{4\}$, $\overline{C}_f(x_5) = \{2\}$, $\overline{C}_f(y_5) = \{1\}$. When $i \geq 7$, $\overline{C}_f(x_{i-2}) = \{2\}$, $\overline{C}_f(y_{i-2}) = \{1\}$, $\overline{C}_f(x_{i-1}) = \{1\}$, $\overline{C}_f(y_{i-1}) = \{4\}$, $\overline{C}_f(x_i) = \{2\}$, $\overline{C}_f(y_i) = \{1\}$. Therefore, f is adjacent vertex distinguishing. This ends the proof. ∎

2.2 Goldberg snark

We now consider the second family of snarks, Goldberg snarks. For this family, we define the basic graph B_i as the graph with vertex set $V(B_i) =$



(a) Link graph L_i



(b) Graph G_3

Figure 3: Link graph L_i and graph G_3

$\{u_i, v_i, x_i, y_i, z_i, w_i, s_i, t_i\}$ and edge set $E(B_i) = \{u_iv_i, x_iy_i, x_iz_i, y_iw_i, z_iv_i, z_it_i, v_iw_i, w_is_i, s_it_i\}$. The link graph L_i is the union of B_{i-1}, B_i and the graph spanned by $E_{(i-1)i}$, where $E_{ij} = \{t_is_j, y_ix_j, u_iu_j\}$. That is, $V(L_i) = V(B_{i-1}) \cup V(B_i)$ and $E(L_i) = E(B_{i-1}) \cup E(B_i) \cup E_{(i-1)i}$, as shown in Figure 3(a).

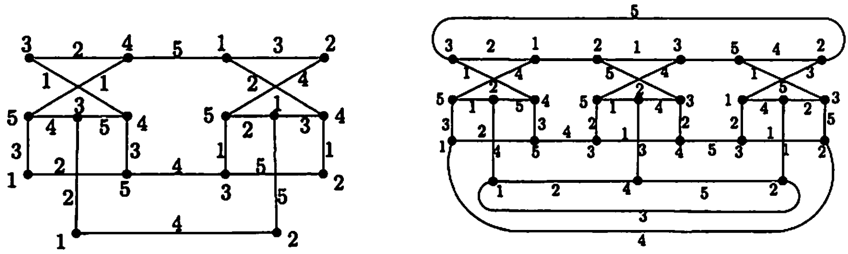
The first Goldberg snark, G_3, is defined as the union of B_1, B_2, B_3, and the graph spanned by $E_{12} \cup E_{23} \cup E_{31}$. For each odd $i \geq 5$, let G_i be constructed from graphs G_{i-2} and L_i as follows: $V(G_i) = V(G_{i-2}) \cup V(L_i)$, and $E(G_i) = (E(G_{i-2}) \setminus E_i^{out}) \cup E(L_i) \cup E_i^{in}$, where $E_i^{out} = E_{(i-2)1}$, and $E_i^{in} = E_{(i-2)(i-1)} \cup E_{i1}$.

Theorem 2.2 *Let G_i be a Goldberg snark with odd $i \geq 3$. Then $\chi_{at}(G_i) = 5$.*

Proof. Our proof also proceeds by induction. A 5-avd-total coloring of G_3 and L_i is described as in Figure 4 (a) and (b), respectively. For each odd $i \geq 5$, a 5-avd-total coloring f of G_i is obtained from the coloring of G_{i-2} and L_i as follows:

$f(t_{i-2}s_{i-1}) = f(t_is_1) = 5;$
$f(y_{i-2}x_{i-1}) = f(y_ix_1) = 4;$
$f(u_{i-2}u_{i-1}) = f(u_iu_1) = 3.$

Now we show that f is proper. First, note that edges of E_i^{in} do not join vertices with the same color. By induction, we have $f(t_{i-2}s_1) = 5$, $f(y_{i-2}x_1) = 4$, and $f(u_{i-2}u_1) = 3$ in G_{i-2}. Therefore, for graph G_i spanned by $E(G_i) \setminus E_i^{in}$, color 5 does not appear on vertices t_{i-2}, s_1, color 4 does not appear on vertices y_{i-2}, x_1, and color 3 does not appear on vertices u_{i-2}, u_1. So f is a 5-total coloring.



(a) (b)

Figure 4: a coloring for graphs L_i and G_3, respectively

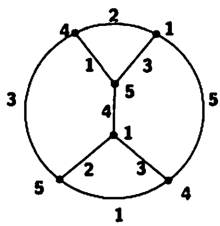


Figure 5: A 5-avd-total coloring of G

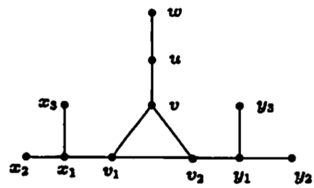


Figure 6: Around the end of a longest path

It remains to check that f is adjacent vertex distinguishing. For each odd $i \geq 5$, $\overline{C}_f(s_1) = \{4\}$, $\overline{C}_f(x_1) = \{5\}$, $\overline{C}_f(u_1) = \{5\}$, $\overline{C}_f(t_i) = \{1\}$, $\overline{C}_f(y_i) = \{3\}$, $\overline{C}_f(u_i) = \{1\}$, $\overline{C}_f(s_{i-1}) = \{4\}$, $\overline{C}_f(x_{i-1}) = \{5\}$, $\overline{C}_f(u_{i-1}) = \{5\}$.

For G_5, $\overline{C}_f(t_3) = \{1\}$, $\overline{C}_f(y_3) = \{3\}$, $\overline{C}_f(u_4) = \{4\}$. When $i \geq 7$, $\overline{C}_f(t_{i-2}) = \{1\}$, $\overline{C}_f(y_{i-2}) = \{3\}$, $\overline{C}_f(u_{i-2}) = \{1\}$. This ends the proof. ∎

3 5-avd-total coloring of cubic Halin graphs

Theorem 3.1 *If G is a cubic Halin graph, then $\chi_{at}(G) = 5$.*

Proof. Our proof proceeds by induction on the length m of the adjoint cycle C. For $m = 3$, G is a complete graph on 4 vertices. So the result follows from Lemma 1.2. For $m = 4$, the unique cubic Halin graph together with its avd-total 5-coloring is shown in Figure 5.

Now assume that $m \geq 5$. In the following inductive steps, we shall use two basic operations to reduce a cubic Halin graph G to another cubic

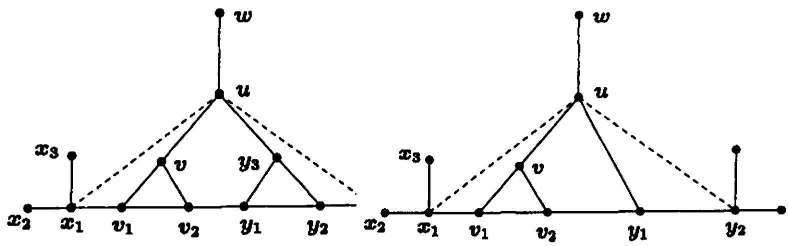


Figure 7: Two cases of induction step

Halin graph G' such that the length of the adjoint cycle of G' is shorter than that of G. Then by the inductive hypothesis, we have $\chi_{at}(G') = 5$.

Let $P = u_0, u_1, \ldots, u_l$, $l \geq 4$, be a longest path in T. Due to the maximality of P, all neighbors of u_1, except u_2, are leaves of T. We rename the first four vertices of P by setting $w = u_3$, $u = u_2$, $v = u_1$ and $v_1 = u_0$. Denote by v_2 the other neighbor of v on C, as depicted in Figure 6.

Due to $deg(v) = 3$, there is a path Q from u to x_1 or y_1, with $P \cap Q = \{u\}$. By symmetry, we may assume that Q is a path from u to y_1. Since P is a longest path in T, the length of Q is at most two. We consider the following two cases:

Case 1. The length of Q is two.

In this case, $uy_3 \in E(T)$ and $y_2y_3 \in E(T)$. Now let G' be the graph obtained from G by deleting v, v_1, v_2, y_1, y_2, y_3 and adding two new edges ux_1 and uz (see the left graph in Figure 7). By the induction hypothesis, there is a 5-avd-total coloring f of G'. Without loss of generality, we may assume that $f(ux_1) = 1$, $f(uw) = 2$, $f(uz) = 3$ and $f(u) = 4$. So $\overline{C}_f(u) = \{5\}$; $\overline{C}_f(x_1) \neq 5$ and $f(x_1) \neq 4$; $\overline{C}_f(z) \neq 5$ and $f(z) \neq 4$. Next we shall extend f to the remaining edges and vertices of G to get a 5-avd-total coloring of G by setting

$f(uv) = f(x_1v_1) = f(y_1y_2) = 1,$

$f(vv_1) = f(y_2y_3) = 2,$

$f(uy_3) = f(v_1v_2) = f(y_2z) = 3,$

$f(v_2y_1) = 4,$

$f(vv_2) = f(y_1y_3) = 5,$

$f(v_2) = f(y_3) = 1$, $f(y_1) = 2$, $f(v) = 3$, $f(v_1) = f(y_2) = 4.$

Obviously, f is a proper total 5-coloring of G. And since $\overline{C}_f(v) = \overline{C}_f(y_3) = 4$, $\overline{C}_f(v_1) = \overline{C}_f(y_2) = 5$, $\overline{C}_f(v_2) = 2$ and $\overline{C}_f(y_1) = 3$, f is also a 5-avd-total coloring.

Case 2. The length of Q is one.

In this case, $u = y_3$ and hence $uy_1 \in E(T)$. Let G' be the graph obtained from G by deleting v, v_1, v_2, y_1, and adding two new edges ux_1 and uy_2 (see the right graph in Figure 7). By the induction hypothesis, there is a 5-avd-total coloring f of G'. Without loss of generality, we may assume that $f(ux_1) = 1$, $f(uw) = 2$, $f(uy_2) = 3$ and $f(u) = 4$. So $\overline{C}_f(u) = \{5\}$; $\overline{C}_f(x_1) \neq 5$ and $f(x_1) \neq 4$; $\overline{C}_f(y_2) \neq 5$ and $f(y_2) \neq 4$. Now we extend f to the remaining edges and vertices of G to get a 5-avd-total coloring of G. If $\overline{C}_f(y_2) = \{2\}$ or $\overline{C}_f(y_2) \neq \{2\}$ and $f(y_2) \neq 5$, first we let $f(uy_1) = f(x_1v_1) = 1, f(uv) = f(y_1y_2) = 3$. Next we distinguish two subcases.

(i) If $\overline{C}_f(y_2) = \{2\}$, then let $f(vv_1) = 2, f(v_1v_2) = 3, f(v_2v) = 4, f(v_2y_1) = 5$, and $f(v_2) = 1, f(y_1) = 2, f(v_1) = 4, f(v) = 5$. Since $f(y_2) \neq 2$, it is easy to check that f is a proper total 5-coloring of G. Furthermore, since $\overline{C}_f(v) = \{1\}$, $\overline{C}_f(v_1) = \{5\}$, $\overline{C}_f(v_2) = \{2\}$ and $\overline{C}_f(y_1) = \{4\}$, f is also a 5-avd-total coloring.

(ii) If $\overline{C}_f(y_2) \neq \{2\}$ and $f(y_2) \neq 5$, then let $f(vv_1) = 2, f(v_1v_2) = 3, f(v_2y_1) = 4, f(vv_2) = 5$, and $f(v) = 1, f(v_2) = 2, f(v_1) = 4, f(y_1) = 5$. Since $\overline{C}_f(v) = \{4\}, \overline{C}_f(v_1) = \{5\}, \overline{C}_f(v_2) = \{1\}$ and $\overline{C}_f(y_1) = \{2\}$, it is easy to see that f is a 5-avd-total coloring of G.

Finally, we consider the case when $\overline{C}_f(y_2) \neq \{2\}$ and $f(y_2) = 5$. First, let $f(x_1v_1) = 1$, $f(y_1y_2) = 3$ and $f(v_1) = 4$. Since $\overline{C}_f(y_2) \neq \{2\}$ or $\{5\}$, and $f(y_1y_2) = 3$, we have $\overline{C}_f(y_2) = \{1\}$ or $\{4\}$.

(i) If $f(w) \neq 3$, then recolor vertex u by color 3. Let $f(uv) = 4$ and $f(uy_1) = 1$. Note that after the recoloring procedure, we still have $\overline{C}_f(u) = \{5\}$. Define $f(vv_1) = 2, f(v_1v_2) = 3, f(v_2v) = 1$ and $f(v_2y_1) = 5$; $f(v_2) = 2, f(y_1) = 4$ and $f(v) = 5$. Since $\overline{C}_f(v) = \{3\}$, $\overline{C}_f(v_1) = \{5\}$, $\overline{C}_f(v_2) = \{4\}$ and $\overline{C}_f(y_1) = \{2\}$, f is a 5-avd-total coloring.

(ii) If $f(w) = 3$, then let $f(uv) = 3, f(vv_1) = 2, f(v_1v_2) = 3, f(v_2v) = 4$ and $f(v_2y_1) = 5$; $f(v) = 5, f(v_2) = 1$ and $f(y_1) = 2$. Hence $\overline{C}_f(v) = \{1\}$, $\overline{C}_f(v_1) = \{5\}$ and $\overline{C}_f(v_2) = \{2\}$. When $\overline{C}_f(y_2) = \{1\}$, then set $f(uy_1) = 1$; when $\overline{C}_f(y_2) = \{4\}$, then recolor vertex u by color 1 and set $f(uy_1) = 4$. It is easy to check that in both cases we have $\overline{C}_f(u) = \{5\}$ and $\overline{C}_f(y_1) \neq \overline{C}_f(y_2)$, so f is a 5-avd-total coloring.

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