LIST COLORING AND n-MONOPHILIC GRAPHS

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ABSTRACT. In 1990, Kostochka and Sidorenko proposed studying the smallest number of list-colorings of a graph G among all assignments of lists of a given size n to its vertices. We say a graph G is n-monophilic if this number is minimized when identical n-color lists are assigned to all vertices of G. Kostochka and Sidorenko observed that all chordal graphs are n-monophilic for all n. Donner (1992) showed that every graph is n-monophilic for all sufficiently large n. We prove that all cycles are n-monophilic for all n; we give a complete characterization of 2-monophilic graphs (which turns out to be similar to the characterization of 2-choosable graphs given by Erdős, Rubin, and Taylor in 1980); and for every n we construct a graph that is n-choosable but not n-monophilic.

1. INTRODUCTION

Suppose for each vertex v of a graph G we choose a list $L(v)$ of a fixed number n of colors, and then to each v we assign a color chosen randomly from its color list $L(v)$. If our goal is to maximize the probability of getting the same color for at least two adjacent vertices, then it seems intuitively plausible that we should give every vertex of G the same list. But this turns out to be false for some graphs! Graphs which do satisfy this property are called "n-monophilic" (defined more precisely below). It is natural to ask: Which graphs are n-monophilic for a given n? This question has been open at least since 1990.

We work with finite, simple graphs, and use the notation and terminology of Diestel [1]. Given a graph $G = (V, E)$, a **list assignment** (resp. n-list assignment, $n \in \mathbb{N}$) for G is a function that assigns a subset (resp. n-subset) of $\mathbb{N}$ to each vertex $v \in V$, denoted $L(v)$. Given a list assignment L for G, a (proper) **coloring** of G from L is a function $\gamma : G \to \mathbb{N}$ such that for each vertex $v \in V$, $\gamma(v) \in L(v)$, and for any pair of adjacent vertices v and w, $\gamma(v) \neq \gamma(w)$. We denote the number of distinct colorings of G from L by $\mathrm{col}(G, L)$. In the special case where $L(v) = [n] = \{1, \cdots, n\}$ for every $v \in V$, we also write $\mathrm{col}(G, n)$ for $\mathrm{col}(G, L)$. We say G is **n-monophilic** if

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col$(G,n) \leq$ col(G,L) for every n-list assignment L for G. Clearly a graph is n-monophilic iff each connected component of it is n-monophilic. So we restrict attention to connected graphs only.

In 1990, Kostochka and Sidorenko [7] proposed studying the minimum value $f(n)$ attained by col(G,L) over all n-list assignments L for a given graph G. They observed that for chordal graphs (see Section 2 for definition) $f(n)$ equals the chromatic polynomial of G evaluated at n; i.e., chordal graphs are n-monophilic for all n. In 1992 Donner [3] showed that for any fixed graph G, $f(n)$ equals the chromatic polynomial of G for all sufficiently large n; i.e., every graph is n-monophilic for all sufficiently large n. There appears to be no further literature on this subject since then.

A graph G is said to be **n-colorable** if col$(G,n) \geq 1$; and G is said to be **n-choosable** (or *n-list colorable*) if col$(G,L) \geq 1$ for every n-list assignment L for G. The **chromatic number** of G, denoted $\chi(G)$, is the smallest n such that G is n-colorable. The **list chromatic number** of G (also called the *choice number* of G), denoted $\chi_l(G)$ (or ch(G)), is the smallest n such that G is n-choosable. Since χ and χ_l are well-known and have been studied extensively, it is interesting to compare the concept of n-monophilic graphs to them. The following are clear from definitions. For every graph G,

(1) $\chi(G) \leq \chi_l(G)$;
(2) if $n < \chi(G)$, then G is n-monophilic;
(3) if $\chi(G) \leq n < \chi_l(G)$, then G is not n-monophilic.

The interesting region is $\chi_l(G) \leq n$, which contains n-monophilic graphs (e.g., all cycles and all chordal graphs), as well as non-n-monophilic graphs (Section 5).

Deciding whether a graph is n-choosable turns out to be difficult. Even deciding whether a given planar graph is 3-choosable is NP-hard [6]. Thus one might expect the decision problem for n-monophilic graphs to be NP-hard as well; so a "nice characterization" (i.e., one that would lead to a polynomial time decision algorithm) of n-monophilic graphs might not exist. In this paper we prove that all cycles are n-monophilic for all n, and G is not 2-monophilic iff all its cycles are even and it contains at least two cycles whose union is not $K_{2,3}$. This characterization of 2-monophilic graphs is fairly similar to that given by Erdős, Rubin, and Taylor [4]. But, as we show in Section 5, for every $n \geq 2$ there is a graph that is n-choosable but not n-monophilic.

2. Chordal graphs are n-monophilic

A graph is **chordal** if every cycle in it of length greater than 3 has a chord. Kostochka and Sidorenko [7] observed that all chordal graphs are n-monophilic for all n. Because the proof is short, we include it below.

Observe that if H is a subgraph of G, then L restricts in a natural way to give a list assignment for H, and $\mathrm{col}(H, L)$ denotes the number of colorings of H from this restricted list assignment.

Lemma 1. *Let G be an n-monophilic graph, and suppose $v_1, \cdots, v_k$ induce a complete subgraph of G. Let G' be the graph obtained from G by adding a new vertex and connecting it to $v_1, \cdots, v_k$. Then G' is n-monophilic.*

Proof. Let L be an n-list assignment for G'. If $n \leq k$, then $\mathrm{col}(G', n) = 0$ and we are done. So assume $n > k$. Then each coloring of G from L extends to at least $n-k$ distinct colorings of G' from L. Hence $\mathrm{col}(G', L) \geq (n-k)\mathrm{col}(G, L) \geq (n-k)\mathrm{col}(G, n) = \mathrm{col}(G', n)$. □

A graph has a **simplicial elimination ordering** if its vertices can be ordered as $v_1, \cdots, v_k$ such that for each v_i the subgraph induced by $\{v_i\} \cup N(v_i) \cap \{v_1, \cdots, v_{i-1}\}$, where $N(v_i)$ denotes the set of neighboring vertices of v_i, is a complete graph.

Theorem. (Dirac [2]) *A graph is chordal iff it has a simplicial elimination ordering.*

The above lemma and Dirac's Theorem give us:

Corollary. (Kostochka and Sidorenko [7]) *Every chordal graph is n-monophilic for every n.*

Note that trees and complete graphs are chordal and hence are n-monophilic for every n.

3. Cycles are n-monophilic

In this section we show that every m-cycle is n-monophilic for all m, n. We first need some definitions. Let L be a list assignment for a graph G. For $i = 1, \cdots, k$, let v_i be a vertex of G, and c_i a color in $L(v_i)$. Then $\mathrm{col}(G, L, v_1, c_1, \cdots, v_k, c_k)$ denotes the number of colorings of G from L which assign color c_i to v_i, $i = 1, \cdots, k$. We say L is **minimizing** for G if $\mathrm{col}(G, L) \leq \mathrm{col}(G, L')$ for every list assignment L' where $|L'(v)| = |L(v)|$ for every v.

Lemma 2. *Let G_1 and G_2 be disjoint subgraphs of a graph G, with v_i a vertex of G_i, such that $G = G_1 \cup G_2 + v_1v_2$. Let L be a list assignment for G. Then there exists a list assignment L' such that $|L'(v)| = |L(v)|$ for every v, $L'(v_1) \subseteq L'(v_2)$ or $L'(v_2) \subseteq L'(v_1)$, and $\mathrm{col}(G, L') \leq \mathrm{col}(G, L)$. Moreover, the inequality is strict provided there exist $c_1 \in L(v_1) \backslash L(v_2)$ and $c_2 \in L(v_2) \backslash L(v_1)$ with $\mathrm{col}(G_1, L, v_1, c_1) \neq 0$ and $\mathrm{col}(G_2, L, v_2, c_2) \neq 0$.*

Proof. If $L(v_1) \subseteq L(v_2)$ or $L(v_2) \subseteq L(v_1)$, then there is nothing to show. So we can assume there exist colors $c_1 \in L(v_1) \backslash L(v_2)$ and $c_2 \in L(v_2) \backslash L(v_1)$.

Let L' be the list assignment that is identical to L except that in the lists assigned to the vertices of G_2 every c_1 is replaced with c_2 and every c_2 with c_1. Then, for each $c \neq c_1$ in $L(v_1)$, $\mathrm{col}(G, L', v_1, c) = \mathrm{col}(G, L, v_1, c)$ (since $c \neq c_2$, as $c_2 \notin L(v_1)$). Furthermore,

(1)

$$\begin{aligned}\mathrm{col}(G, L', v_1, c_1) &= \mathrm{col}(G, L, v_1, c_1) - \mathrm{col}(G, L, v_1, c_1, v_2, c_2) \\ &= \mathrm{col}(G, L, v_1, c_1) - \mathrm{col}(G_1, L, v_1, c_1) \cdot \mathrm{col}(G_2, L, v_2, c_2)\end{aligned}$$

Hence, $\mathrm{col}(G, L', v_1, c_1) < \mathrm{col}(G, L, v_1, c_1)$ if $\mathrm{col}(G_1, L, v_1, c_1)$ and $\mathrm{col}(G_2, L, v_2, c_2$ are both nonzero.

Now, by renaming L' as L and then repeating this process as long as $L(v_1) \not\subseteq L(v_2)$ and $L(v_2) \not\subseteq L(v_1)$, we eventually obtain the desired L'. □

The **length** of a path is the number of edges it contains. For $n \geq 2$, an **$(n, n-1)$-list assignment** for a path of length at least one is a function that assigns n-color lists to the path's interior vertices, if any, and $(n-1)$-color lists to its two terminal vertices. Suppose the interior vertices of the path have identical lists, each of which contains as a subset the $(n-1)$-color list of each of the two terminal vertices. If, in addition, these two $(n-1)$-color lists are identical, we say L is **type A**, and denote $\mathrm{col}(P, L)$ by A_k; otherwise we say L is **type B**, and denote $\mathrm{col}(P, L)$ by B_k. Note that, up to renaming colors, all type A $(n, n-1)$-list assignments for a given path are equivalent, and similarly for type B.

Lemma 3. *Let $n \geq 2$, and let L be an $(n, n-1)$-list assignment for a path P of length $k \geq 2$. Then: (a) $A_k - B_k = (-1)^k$, and $A_k = \frac{n-1}{n}((n-1)^{k+1} + (-1)^k)$; (b) $\mathrm{col}(P, L) \geq \min(A_k, B_k)$; and (c) for $n \geq 3$, L is minimizing only if k is odd and L is type A or k is even and L is type B.*

Proof. Part (a): Let v be a terminal vertex of P, and let w be the vertex adjacent to v. Suppose L is type A. Then, for each color that we choose to assign to v, there remains an $(n-1)$-color list of choices for w, and this list is not the same as the $(n-1)$-color list of the other terminal vertex of P. Thus we get

(2) $$A_k = (n-1)B_{k-1}$$

By a similar (but slightly longer) reasoning, we see that

(3) $$B_k = A_{k-1} + (n-2)B_{k-1}$$

Subtracting (3) from (2) gives

(4) $$A_k - B_k = (-1)(A_{k-1} - B_{k-1})$$

Now, by direct calculation, $A_1 = (n-1)(n-2)$, and $B_1 = (n-2)^2 + (n-1)$. It follows that $A_1 - B_1 = -1$, which together with (4) inductively yield

(5) $$A_k - B_k = (-1)^k$$

Finally, combining (2) with (5) gives $A_k = (n-1)(A_{k-1} + (-1)^k)$. It follows by induction from the base case $A_1 = (n-1)(n-2)$ that $A_k = \frac{n-1}{n}((n-1)^{k+1} + (-1)^k)$.

Part (b): This follows immediately from Lemma 2 and the definition of A_k and B_k.

Part (c): Assume $k \geq 2$, since the case $k = 1$ is trivial. Suppose, toward contradiction, that v_1, v_2 are adjacent vertices of P such that $L(v_1) \not\subseteq L(v_2)$ and $L(v_2) \not\subseteq L(v_1)$. Then, in the proof of Lemma 2, the term $\mathrm{col}(G_1, L, v_1, c_1) \cdot \mathrm{col}(G_2, L, v_2, c_2)$ being subtracted in equation (1) is positive since each G_i is now a path and $n \geq 3$. This would imply that L is not minimizing. Hence, if L is minimizing, it must be type A or type B. The result now follows from equation (5). □

Remark: For $n = 2$ a minimizing list need not be type A or type B; examples are easy to construct.

Lemma 4. *Let L and L' be distinct list assignments for a path P such that for every vertex $v \in P$ we have $L(v) \subseteq L'(v)$ and $|L'(v)| \geq 2$. Then* $\mathrm{col}(P, L) < \mathrm{col}(P, L')$.

Proof. Since $L(v) \subseteq L'(v)$ for every v, every coloring of P from L is also a coloring of P from L'. And since L and L' are distinct, for some vertex w there is a color $c \in L'(w) \backslash L(w)$. By hypothesis, $|L'(v)| \geq 2$ for every v; hence $\mathrm{col}(P, L', w, c) \geq 1$, i.e., there is at least one coloring of P from L' that is not a coloring of P from L. The result follows. □

Let L be a list assignment for a graph G, and let $v_1, \cdots, v_k$ be vertices in G, where $k < |G|$. Let $c_i \in L(v_i)$. We define the list assignment $L_{c_1,\cdots,c_k}$ **induced** by $L, c_1, \cdots, c_k$ on the graph $H = G - \{v_1, \cdots, v_k\}$ by: for every vertex $v \in H$, $L_{c_1,\cdots,c_k}(v) = L(v) \backslash \{c_i : v_i \in N(v)\}$ where $N(v)$ denotes the set of vertices in G adjacent to v. Then clearly $\mathrm{col}(G, L, v_1, c_1, \cdots, v_k, c_k) = \mathrm{col}(H, L_{c_1,\cdots,c_k})$.

Theorem 1. *Every cycle is n-monophilic for all $n \geq 2$.*

Proof. Let C be a cycle of length $k \geq 3$. Suppose we assign the color list $[n]$ to every vertex of C. Then, by Lemma 3, for each $c \in [n]$, $\mathrm{col}(C, n, v, c) = A_{k-2}$. Therefore $\mathrm{col}(C, n) = nA_{k-2}$. Let L be an n-list assignment that does not assign identical lists to all vertices of C. We will show $\mathrm{col}(C, L) \geq nA_{k-2}$. Since L does not assign identical lists to all vertices of C, there are adjacent vertices v and w such that $L(v) \neq L(w)$. Let P be the path $C - v$. We have two cases.

Case 1: k is odd. For each $c \in L(v)$, let L_c be the list assignment induced on P, and let L'_c be an $(n, n-1)$-list assignment on P obtained from L_c by making, if necessary, the color lists of the endpoints of P smaller.

Then, since P has odd length $k-2$, by Lemma 3, for each $c \in L(v)$, $\mathrm{col}(P, L_c) \geq \mathrm{col}(P, L'_c) \geq A_{k-2}$. Thus $\mathrm{col}(C, L) \geq nA_{k-2}$, as desired.

Case 2: k is even. First suppose $n \geq 3$ (we will treat the case $n = 2$ separately). If for every $c \in L(v)$ we have $\mathrm{col}(C, L, v, c) \geq A_{k-2}$, then we are done. So assume for some $c_0 \in L(v)$ we have $\mathrm{col}(C, L, v, c_0) < A_{k-2}$. Then $\mathrm{col}(P, L_{c_0}) < A_{k-2}$ since $\mathrm{col}(P, L_{c_0}) = \mathrm{col}(C, L, v, c_0)$. As P has even length $k-2$, it follows from Lemma 3 part (c) and Lemma 4 that L_{c_0} must be a type B $(n, n-1)$-list assignment for P and $\mathrm{col}(P, L_{c_0}) = B_{k-2} = A_{k-2}-1$. Let u be the vertex adjacent to v in $C-w$. Then c_0 is in both L_u and L_w, and not in $L_{c_0}(u) \cup L_{c_0}(w) = L_{c_0}(x) = L(x)$, where x is any vertex in $P - \{u, w\}$. Therefore, for each $c \neq c_0 \in L(v)$, the induced list assignment L_c on P is not type B because c_0 is in $L_c(u)$ and $L_c(w)$ but not in $L_c(x) = L(x)$. Hence $\mathrm{col}(C, L, v, c) > B_{k-2}$, i.e., $\mathrm{col}(C, L, v, c) \geq A_{k-2}$.

Now, as $L(v) \neq L(w)$, there exists an element $d \in L(v) \backslash L(w)$. We show as follows that $\mathrm{col}(C, L, v, d) \geq A_{k-2}+1$. Note that $L_d(w) = L(w)$ contains n colors. Let L'_d be an $(n, n-1)$-list assignment for P obtained from L_d by removing one element other than c_0 from $L_d(w)$, and also one element from $L_d(u)$ if $|L(u)| = n$. Since $c_0 \in L'_d(w)$, L'_d is not a type B list assignment for P. So, by Lemma 3, $\mathrm{col}(P, L'_d) \geq B_{k-2} + 1 = A_{k-2}$. Hence, by Lemma 4, $\mathrm{col}(C, L, v, d) = \mathrm{col}(P, L_d) > \mathrm{col}(P, L'_d) \geq A_{k-2}$, as desired.

Thus we get

$$\begin{aligned}\mathrm{col}(C, L) &= \mathrm{col}(C, L, v, c_0) + \mathrm{col}(C, L, v, d) + \sum_{c \in L(v) \backslash \{c_0, d\}} \mathrm{col}(C, L, v, c) \\ &\geq B_{k-2} + (A_{k-2} + 1) + (n-2)A_{k-2} \\ &= nA_{k-2}\end{aligned}$$

as desired.

Now suppose $n = 2$. Clearly $\mathrm{col}(C, 2) = 2$. Let Q be the path obtained by removing the edge vw (but not its vertices) from C. We will show there are at least two colorings of Q from L that extend to colorings of C. First, we need a definition. Let x and y be any two vertices in a graph G with a given list assignment M. We say that $c \in M(x)$ **forces** $d \in M(y)$ if $\mathrm{col}(G, M, x, c, y, d) \geq 1$ and for every $d' \neq d \in M(y)$, $\mathrm{col}(G, M, x, c, y, d') = 0$.

Denote the vertices of Q by $v_0, \cdots, v_k$, where $v_0 = v$, $v_k = w$, and v_i is adjacent to v_{i+1} for $i = 0, \cdots, k-1$. Suppose, toward contradiction, that each color in $L(v)$ forces a color in $L(w)$. Then each color in $L(v_i)$ must be in $L(v_{i+1})$. Hence $L(v_i) = L(v_{i+1})$. But $L(v) \neq L(w)$. So at least one of the colors in $L(v)$ forces no color in $L(w)$. Let $L(v) = \{\alpha, \beta\}$ and $L(w) = \{\gamma, \delta\}$. Then, without loss of generality, α forces neither γ nor δ. Therefore $\mathrm{col}(Q, L, v, \alpha, w, \gamma)$ and $\mathrm{col}(Q, L, v, \alpha, w, \delta)$ are both nonzero, since $\mathrm{col}(Q, L, v, \alpha) \geq 1$. Now, if α is different from both γ and δ, then any coloring of Q that assigns α to v extends to a coloring of C, and we're done. On the other hand, suppose α is not different from both γ and δ. Then, without loss of generality, $\alpha = \gamma$. So any coloring of Q with α assigned to v and δ to w extends to a coloring of C. Also, $\beta \neq \gamma$ since $\beta \neq \alpha$. And $\beta \neq \delta$ since $\{\alpha, \beta\} \neq \{\gamma, \delta\}$. So any coloring of Q with β assigned to v also extends to a coloring of C. As $\mathrm{col}(Q, L, v, \beta) \geq 1$, we are done again. □

Note that although cycles are 2-monophilic, every even cycle has a minimizing 2-list assignment that does not assign the same list to every vertex: assign the list $\{1, 2\}$ to two adjacent vertices, and the list $\{2, 3\}$ to all the remaining vertices.

4. A CHARACTERIZATION OF 2-MONOPHILIC GRAPHS

The **core** of a connected graph G is the subgraph of G obtained by repeatedly deleting vertices of degree 1 until every remaining vertex has degree at least 2.

Lemma 5. *A connected graph is n-monophilic iff its core is n-monophilic.*

Proof. This is proved easily using Lemma 2 and induction on the number of vertices in the graph. □

Let $\theta_{a,b,c}$ denote the graph consisting of two vertices connected by three paths of lengths a, b, c with mutually disjoint interiors. In particular, $\theta_{2,2,2}$ is the complete bipartite graph $K_{2,3}$. In the paper by Erdős, Rubin, and Taylor [4], we find the following result by Rubin:

Theorem. (A. L. Rubin) *A connected graph is 2-choosable iff its core is a single vertex, an even cycle, or $\theta_{2,2,2m}$ for some $m \geq 1$.*

We use this to prove that a connected graph is 2-monophilic iff its core is a single vertex, is an even cycle, is $K_{2,3}$, or contains an odd cycle.

Lemma 6. $K_{2,3}$ *is 2-monophilic.*

Proof. In Figure 1 the five vertices of $K_{2,3}$ have been labeled as u, v, w, x, y. Since $K_{2,3}$ has no odd cycles, $\mathrm{col}(K_{2,3}, 2) = 2$. Let L be a 2-list assignment for $K_{2,3}$. We will show that $\mathrm{col}(K_{2,3}, L) \geq 2$. We consider three cases, depending on the number of colors that $L(x)$ and $L(y)$ share.

Case 1. $|L(x) \cap L(y)| = 2$. Then there are two ways to assign the same color to x and y; and for each way, there is at least one way to color each of u, v, and w. Hence $\mathrm{col}(K_{2,3}, L) \geq 2$.

Case 2. $|L(x) \cap L(y)| = 1$. Without loss of generality, $L(x) = \{1, 2\}$ and $L(y) = \{1, 3\}$. If at least one of the vertices u, v, w, does not contain color 1 in its list, then there are at least two distinct colorings of $K_{2,3}$ with color 1 assigned to both x and y. On the other hand, if all three vertices u, v, w contain color 1 in their lists, then we can obtain one coloring by assigning

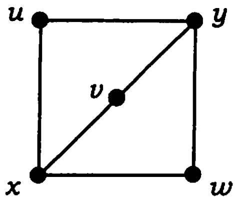

FIGURE 1. $K_{2,3}$

color 2 to x, 3 to y, and 1 to u, v, w, and another coloring by assigning color 1 to both x and y, and using the second color in each of the lists for u, v, w.

Case 3. $|L(x) \cap L(y)| = 0$. Without loss of generality, assume $L(x) = \{1, 2\}$, $L(y) = \{3, 4\}$. Then there are four ways to color the pair x, y. If at least two of these extend to a coloring of $K_{2,3}$, we are done. Otherwise, without loss of generality, $L(u) = \{1, 3\}$, $L(v) = \{1, 4\}$, and $L(w) = \{2, 3\}$. Then $(u, v, w, x, y) = (1, 1, 3, 2, 4)$ and $(u, v, w, x, y) = (3, 1, 3, 2, 4)$ are two distinct colorings of $K_{2,3}$. □

Theorem 2. *A connected graph is 2-monophilic iff its core is a single vertex, is a cycle, is $K_{2,3}$, or contains an odd cycle.*

Equivalently: A graph is *not* 2-monophilic iff all its cycles are even and it contains at least two cycles whose union is not $K_{2,3}$.

Proof. Clearly a single vertex and a graph that contains an odd cycle are both 2-monophilic. Also, by Theorem 1 and Lemma 6, all cycles and $K_{2,3}$ are 2-monophilic. Using Lemma 5, this gives us one direction of the theorem.

To prove the converse, let G be a 2-monophilic graph. If G is not 2-colorable, then it must contain an odd cycle, and we are done. So assume $\chi(G)$ is 1 or 2. Then G is also 2-choosable since it is 2-monophilic. So, by Rubin's theorem above, it is enough to show that for $m \geq 2$, $\theta_{2,2,2m}$ is not 2-monophilic.

Figure 2 shows a 2-list assignment L for the case when $m = 2$. When $m > 2$, we add an even number of vertices to the interior of the edge uv in Figure 2 and assign to each new vertex the list $\{1, 2\}$. It is then easy to check that for $m \geq 2$, $\text{col}(\theta_{2,2,2m}, L) = 1 < 2 = \text{col}(\theta_{2,2,2m}, 2)$, as desired. □

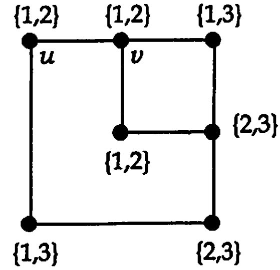

FIGURE 2. $\mathrm{col}(\theta_{2,2,4}, L) = 1$.

5. EXAMPLES OF n-CHOOSABLE, NON-n-MONOPHILIC GRAPHS

Given the close similarity between Theorem 2 and Rubin's theorem above, it is natural to wonder how similar or different the notions of n-choosable and n-monophilic are. In this section, for each $n \geq 2$ we construct a graph H_n that is n-choosable but not n-monophilic. To make the notation simpler, we work with H_{n+1} with $n \geq 1$ instead of H_n with $n \geq 2$.

First, consider the complete bipartite graph K_{n,n^n}. Fix $n \geq 1$, and denote the vertices of K_{n,n^n} by $a_1, \cdots, a_n, b_1, \cdots, b_{n^n}$. Let L_0 be an n-list assignment for K_{n,n^n} such that: for all $i \neq j$, $L_0(a_i) \cap L_0(a_j) = \emptyset$; for all $k \neq l$, $L_0(b_k) \neq L_0(b_l)$; and each $L_0(a_i)$ shares exactly one element with each $L_0(b_k)$. Then there are n^n distinct ways to assign a color to each of $a_1, \cdots, a_n$, and each of them will preclude assigning a color to b_k for some k. It follows that $\mathrm{col}(K_{n,n^n}, L_0) = 0$.

Let L_0' be an n-list assignment for K_{n,n^n} that is the same as L_0 except that its colors are renamed so that colors $1, \cdots, n$ do not appear in any of its lists. For each $j \in [n]$, let L_j be the $(n+1)$-list assignment for K_{n,n^n} given by $L_j(v) = L_0'(v) \cup \{j\}$ for every vertex $v \in K_{n,n^n}$. Let $x = \mathrm{col}(K_{n,n^n}, L_j)$; clearly x is nonzero and independent of j.

Let $\{G_{i,j} : i, j \in [n]\}$ be a set of n^2 disjoint copies of K_{n,n^n}. Let p be the smallest integer such that $n^p > x^{n^2}$. Let $K_{n,p}$ be a complete bipartite graph with vertices $v_1, \cdots, v_n, w_1, \cdots, w_p$. We connect each v_i to all vertices of $G_{i,1}, \cdots, G_{i,n}$. This describes the graph H_{n+1}.

Lemma 7. *For all $n \geq 1$, the graph H_{n+1} is not $(n+1)$-monophilic.*

Proof. Define an $(n+1)$-list assignment L for H_{n+1} as follows. For all $k \in [p]$, $L(w_k) = \{n+1, n+2, \cdots, 2n+1\}$; for each $i \in [n]$, $L(v_i) = [n] \cup \{n+i\}$; and on each $G_{i,j}$, $L = L_j$.

Let γ be a coloring of H_{n+1} from L. Since $\mathrm{col}(K_{n,n^n}, L_0) = 0$, for each $i, j \in [n]$, γ must assign color j to at least one vertex of $G_{i,j}$. Hence for all

$i \in [n]$, $\gamma(v_i) = n + i$; and for all $k \in [p]$, $\gamma(w_k) = 2n + 1$. It follows that $\mathrm{col}(H_{n+1}, L) = x^{n^2}$.

On the other hand, $\mathrm{col}(H_{n+1}, n+1) \geq n^p$: there are n^p ways to color $w_1, \cdots, w_p$ from just $[n]$; then assign color $n+1$ to every v_i; and finally color every $G_{i,j}$ using colors 1 and 2. Hence $\mathrm{col}(H_{n+1}, L) < \mathrm{col}(H_{n+1}, n+1)$, as desired. □

So it remains to show that H_{n+1} is $(n+1)$-choosable. We do this in the three following lemmas. We say two list assignments L and L' for a graph G are **equivalent** if one can be obtained from the other by renaming colors and vertices, i.e., there is a bijection $f : \mathbb{N} \to \mathbb{N}$ and an automorphism $\phi : G \to G$ such that for every vertex $v \in G$, $L'(v) = f(L(\phi(v)))$.

Lemma 8. *Let L be a list assignment for K_{n,n^n} such that for every vertex $v \in K_{n,n^n}$, $|L(v)| \geq n$. If $\mathrm{col}(K_{n,n^n}, L) = 0$, then L is equivalent to L_0.*

Proof. Denote the two vertex-partitions of K_{n,n^n} by A and B, with $|A| = n$ and $|B| = n^n$. Suppose for some $a_i \neq a_j$ in A, $L(a_i) \cap L(a_j) \neq \emptyset$. If we assign the same color to a_i and a_j, and to each $a' \neq a_i, a_j$ we assign a color from $L(a')$, then for every $b \in B$, $L(b)$ contains at least one color that was not assigned to any $a \in A$. Hence $\mathrm{col}(K_{n,n^n}, L) > 0$, which is a contradiction. So for all $a_i \neq a_j$ in A, $L(a_i) \cap L(a_j) = \emptyset$.

Now, suppose we have colored all $a \in A$. Since K_{n,n^n} is not colorable from L, there must exist some $b \in B$ whose color list is exactly the n colors we have chosen for the vertices in A. Since there are only n^n vertices in B, if there were more than n^n ways to color A, then $\mathrm{col}(K_{n,n^n}, L)$ would not be zero. So each $L(a)$ must contain exactly n colors, and there are exactly n^n ways to color A. It follows that every $L(b)$ must contain exactly one color from $L(a)$ for each $a \in A$, and no other colors; Furthermore, distinct vertices in B must have distinct lists. This proves L is equivalent to L_0. □

Lemma 9. *Let v denote the vertex in the one-element partition of the complete tripartite graph $K_{n,n^n,1}$. Let L be an $(n+1)$-list assignment for $K_{n,n^n,1}$ such that $L(v) = [n+1]$. Suppose for some $j \in [n]$, $\mathrm{col}(K_{n,n^n,1}, L, v, j) = 0$. Then L is equivalent to L_j; and for all $i \neq j$, $\mathrm{col}(K_{n,n^n,1}, L, v, i) > 0$.*

Proof. Let L' be the list assignment for K_{n,n^n} obtained by deleting color j from every list in the restriction of L to K_{n,n^n}. Then $\mathrm{col}(K_{n,n^n}, L') = 0$. So, by Lemma 8, L' is equivalent to L_0, and hence L is equivalent to L_j. If for some $i \neq j$, $\mathrm{col}(K_{n,n^n,1}, L, v, i) = 0$, then it would follow that both i and j are in every list of L, which contradicts the fact that L' is equivalent to L_0. □

Lemma 10. *For all $n \geq 1$, H_{n+1} is $(n+1)$-choosable.*

Proof. Let L be an $(n+1)$-list assignment for H_{n+1}. For each $i, j \in [n]$, let $G'_{i,j}$ be the subgraph of H_{n+1} induced by $G_{i,j}$ and v_i. By Lemma 9, there is at most one color c in $L(v_i)$ such that $\mathrm{col}(G'_{i,j}, L, v_i, c) = 0$. Since $|L(v_i)| = n+1$, there exists $c_i \in L(v_i)$ such that for every $j \in [n]$, $\mathrm{col}(G'_{i,j}, L, v_i, c_i) \neq 0$. Furthermore, since each w_k has only n neighbors, $L(w_k)\backslash\{c_1, \cdots, c_n\}$ is non-empty. Hence $\mathrm{col}(H_{n+1}, L) \neq 0$. □

6. Questions

In this section we offer (and try to motivate) two questions. The Dinitz Conjecture, proved by Galvin [5], states that the line graph of the complete bipartite $K_{n,n}$ is n-choosable. The List Coloring Conjecture (which is open as of this writing), generalizes the Dinitz Conjecture: for every graph G, $\chi_l(L(G)) = \chi(L(G))$, where $L(G)$ denotes the line graph of G.

Note that the line graph of $K_{n,n}$ is isomorphic to the product $K_n \times K_n$, where the product $G \times H$ is defined by $V(G \times H) = V(G) \times V(H)$, with two vertices (g, h) and (g', h') in $G \times H$ declared to be adjacent if $g = g'$ and h is adjacent to h' or if $h = h'$ and g is adjacent to g'. Thus, another way to generalize the Dinitz Conjecture is:

Question 1. *Is the product of two n-monophilic graphs n-monophilic?*

For $n = 2$ the answer to this question is *No*: Letting P_i denote the path of length i, it follows from Theorem 2 that $P_2 \times P_3$ is not 2-monophilic, while by Theorem 3 every P_i is 2-monophilic. However, it is possible that the $n = 2$ case is special and for $n \geq 3$ the answer is *Yes*.

Donner's result [3] allows us to define the **monophilic number** of G, denoted $\chi_m(G)$, in two possible natural ways:

(1) the smallest n for which G is n-colorable and n-monophilic, or
(2) the smallest n such that G is n'-monophilic for all $n' \geq n$.

We do not know whether or not these two definitions are equivalent; it depends on the answer to the following:

Question 2. *If a graph is n-colorable and n-monophilic, is it necessarily $(n+1)$-monophilic?*

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