# Antimagic labelings of cycle powers

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**Abstract**

An antimagic labeling of a graph with $n$ vertices and $m$ edges is a bijection from the set of edges to the integers $1, 2, ..., m$ such that all $n$ vertex sums are pairwise distinct. For a cycle $C_n$ of length $n$, the $k^{th}$ *power of* $C_n$, denoted by $C_n^k$, is the supergraph formed by adding an edge between all pairs of vertices of $C_n$ with distance at most $k$. Antimagic labelings for $C_n^k$ are given where $k = 2, 3, 4$.

## 1 Introduction

In this paper, all graphs are finite, undirected, and simple. Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. Suppose the edges of $G$ are labeled using distinct values from $\{1, 2, .., m\}$. For each vertex $v$, define its *vertex sum* be the sum of the labels of the edges incident on $v$. A labeling is an *antimagic labeling* of $G$ if all $n$ vertex sums are pairwise distinct. If a graph has an antimagic labeling, then the graph is *antimagic*. For a vertex $v$, denote its vertex sum by $S_v$.

In 1990, Hartsfield and Ringel [3] introduced the notion of antimagic labelings and antimagic graphs. They conjectured that every connected graph, other than $K_2$, is antimagic. In 2004, Alon et al. [1] validated this conjecture for graphs having minimum degree $\Omega(\log n)$. They also showed that graphs with maximum degree at least $n - 2$ are antimagic, as well as complete $k$-partite graphs, for any $k \geq 2$. In 2005, Hefetz [4] showed that a graph with $3^k$ vertices admitting a $K_3$-factor is antimagic. Also in 2005, Wang [6] showed that the Cartesian product of a finite number of cycles is antimagic. In addition, Wang showed that the Cartesian product of an

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antimagic regular graph and a cycle is antimagic. In 2008, Wang and Hsiao [7] showed that toroidal grids are antimagic.

Suppose $C_n = (V, E)$ is a cycle of length $n$ and $k$ is a positive integer. The $k^{th}$ *power of* $C_n$, denoted by $C_n^k$, is the supergraph of $C_n$ formed by adding an edge between all pairs of vertices of $C_n$ with distance at most $k$. In 2010, Lee, Lin, and Tsai [5] showed that if $n$ is odd, then $C_n^2$ is antimagic. Other results can be found in the dynamic survey by Gallian [2].

In this report, We extended the work of Lee, Lin and Tsai [5] by giving an alternate proof of their result on $C_n^2$, where $n$ is odd. We also showed that, for $n$ even, $C_n^2$ is antimagic by constructing an antimagic labeling for $C_n^2$. Then, We extended the antimagic labelings for $C_n^2$ to obtain antimagic labelings for $C_n^3$, whenever $n \geq 6$. Finally, We showed that the antimagic labelings for $C_n^3$, where $n$ is odd, extend to antimagic labelings for $C_n^4$.

# 2 The Graph $C_n^2$

In this section, We will show that $C_n^2$ is antimagic for all $n \geq 4$. Note that when $n = 3$, $C_n^2 = C_n$. We begin by providing an antimagic labeling of $C_n^2$ that differs from the one given in [5].

**Theorem 2.1** *([5]) If $n > 3$ is an odd integer, then $C_n^2$ is antimagic.*

**Proof** : The vertices of $C_n^2$ will be $V = \{0, 1, 2, ..., n-1\}$. The edges of $C_n^2$ will be denoted by $E$. We note that $C_n^2$ has $2n$ edges. Define a bijection $L : E \to \{1, 2, ..., 2n\}$ that labels the edges of the graph as follows:

$$L(\{i,j\}) = \begin{cases} i+1 & : \ 0 \leq i \leq n-2 \text{ and } j = i+1 \\ n & : \ i = n-1 \text{ and } j = 0 \\ n+1 & : \ i = n-1, j = 1 \\ 2n & : \ i = n-2, j = 0 \\ n+i+2 & : \ 0 \leq i \leq n-3 \text{ and } j = i+2 \end{cases}$$

We claim that the labeling $L$ is an antimagic labeling of $C_n^2$. Observe that $S_1 = 1+2+(n+1)+(n+3) = 2n+7$ and $S_2 = 2+3+(n+2)+(n+4) = 2n+11$, which is 4 greater than $S_1$. In fact, it is easy to verify that for $1 \leq i \leq n-3$, $S_{i+1} = S_i + 4$. Since $S_1$ is odd, then so is every $S_i$, for $1 \leq i \leq n-2$. In addition, they are pairwise distinct. The vertex $n-1$ has vertex sum $S_{n-1} = (n-1) + n + (n+1) + (2n-1) = 5n - 1$ which is even. Finally, vertex 0 has vertex sum $S_0 = 1 + n + 2n + (n+2) = 4n + 3$, which is odd. All that remains is to show $S_0$ does not appear in the set of vertex sums $\{S_1, S_2, .., S_{n-2}\}$. To see this, note that if $S_0$ is the same as the vertex sum of some vertex in $\{1, 2, 3, ..., n-2\}$, then $S_0 - S_1$ must be divisible by 4. But this difference is $4n+3-(2n+7) = 2n-4 = 2(n-2)$. As $n$ is odd, then $n-2$ is odd. Therefore $2(n-2)$ is not divisible by 4 which implies $S_0 \notin \{S_1, S_2, ..., S_{n-2}\}$. Therefore, all the vertex sums of this labeling are pairwise distinct. □

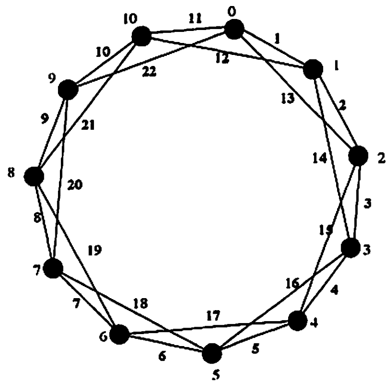


Figure 1: antimagic labeling of $C_{11}^2$

Figure 1 shows the antimagic labeling of $C_{11}^2$ using the labeling given in the proof of Theorem 2.1. Consider the graph $C_n^2$, where the number of vertices is even. We now describe a construction for an antimagic labeling of $C_n^2$ which can be extended to an antimagic labeling for $C_n^3$.

**Theorem 2.2** *If $n > 6$ is an even integer, then $C_n^2$ is antimagic.*

**Proof** : The vertices of $C_n^2$ will be $V = \{0, 1, 2, ..., n-1\}$. Let $E$ denote the edges of the graph. Define a bijection $L : E \to \{1, 2, ..., 2n\}$ that labels the edges of the graph as follows:

$$L(\{i,j\}) = \begin{cases} 2 & : \quad i = 0, j = 1 \\ 1 & : \quad i = 1, j = 2 \\ n-1 & : \quad i = n-3, j = n-2 \\ n-2 & : \quad i = n-2, j = n-1 \\ n & : \quad i = n-1, j = 0 \\ i+1 & : \quad j = i+1 \text{ and } i \notin \{0, 1, n-3, n-2, n-1\} \\ n+1 & : \quad i = n-1, j = 1 \\ 2n & : \quad i = n-2, j = 0 \\ n+i+2 & : \quad 0 \le i \le n-3 \text{ and } j = i+2 \end{cases}$$

By definition of the labeling $L$, $S_0 = 2 + n + 2n + (n+2) = 4n + 4$, $S_1 = 1+2+(n+1)+(n+3) = 2n+7$, $S_2 = 1+3+(n+2)+(n+4) = 2n+10$, $S_3 = 3+4+(n+3)+(n+5) = 2n+15$. It can be verified that $S_{i+1} = S_i + 4$

for $3 \leq i \leq n-5$. Since $S_3$ is odd, $S_i$ is odd for $3 \leq i \leq n-4$. In addition, they are pairwise distinct. Also, $S_{n-3} = 6n-8$, $S_{n-2} = 6n-5 = S_{n-4}+8$ and $S_{n-1} = 5n-2$. Note that $S_1, S_{n-2}$ are both odd. In fact $S_1 = S_3 - 8$ and $S_{n-2} = S_{n-4} + 8$. This implies $S_1$ and $S_2$ are distinct and do not belong in the set of vertex sums $\{S_3, S_4, ..., S_{n-4}\}$. By the labeling $L$, $S_2 < S_0 < S_{n-1} < S_{n-3}$ and they are all even. Therefore all the vertex sums are distinct. □

Figure 2 shows the antimagic labeling of $C_{11}^2$ using the labeling given in the proof of Theorem 2.2. Theorems 2.1 and 2.2 give antimagic labelings of $C_n^2$ for all $n$, except when $n = 4, 6$. Figures 3 and 4 shows that $C_4^2$ and $C_6^2$ are antimagic, respectively. This along with Theorems 2.1 and 2.2 gives the following result.

**Corollary 2.3** *For every $n \geq 4$, $C_n^2$ is antimagic.*

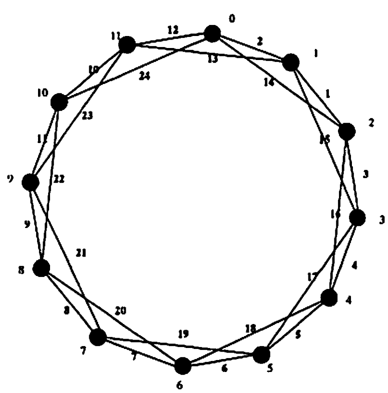

Figure 2: Antimagic labeling of $C_{12}^2$

# 3 The Graph $C_n^3$

In the previous section, we constructed an antimagic labeling of $C_n^2$, for every $n \geq 4$. In this section, we will extend those constructions to give antimagic labelings for $C_n^3$. We will consider the two cases of $n$ odd and $n$ even separately.

**Theorem 3.1** *If $n \geq 7$ is an odd integer, then $C_n^3$ has an antimagic labeling.*

**Proof :** Recall that that the labeling $L$, which was used to prove Theorem 2.1, has the following properties. We will use $S_i^L$ to denote the vertex sum of vertex $i$ under the labeling $L$, of $C_n^2$.

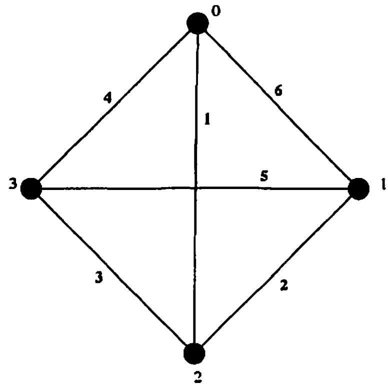


Figure 3: antimagic labeling of $C_4^2$

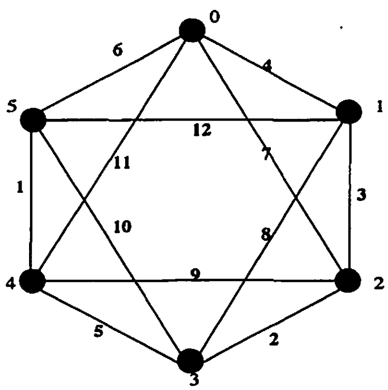


Figure 4: antimagic labeling of $C_6^2$

1. $S_0^L = 4n + 3$,
2. $S_1^L = 2n + 7$,$S_2^L = 2n + 11$,$S_3^L = 2n + 15$
3. $S_{i+1}^L = S_i^L + 4$ for $1 \le i \le n - 3$, and $S_{n-1}^L = 5n - 1$.

In addition, recall that every vertex sum $S_i^L$ is odd except for $S_{n-1}^L$, which is even. We now show how to extend the antimagic labeling $L$ for $C_n^2$ to an antimagic labeling $M$ for $C_n^3$ such that $M|C_n^2 = L$. For each edge $e \in C_n^2$, assign $M(e) = L(e)$. For the edge $e = \{i, i + 3\}$ where $0 \le i < n - 3$, assign $M(e) = 2n + i + 1$. For the edge $e = \{n - 3, 0\}$, we assign $M(e) = 3n-2$. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n-1$. Finally, for the edge $e = \{n - 1, 2\}$, we assign $M(e) = 3n$. This gives a labeling $M$ for $C_n^3$, which extends the labeling $L$. We now show that it is an antimagic labeling of $C_n^3$.

Consider the vertices $0, 1, 2, n-1$. They have vertex sums $S_0 = (4n+3)+(2n+1)+(3n-2) = 9n+2$, $S_1 = (2n+7)+(2n+2)+(3n-1) = 7n+8$, $S_2 = (2n+11)+(2n+3)+(3n) = 7n+14$, and $S_{n-1} = (5n-1)+(3n-3)+(3n) = 11n-4$. Since $n$ is odd, these four vertex sums are odd. Since $n \geq 7$, these four vertex sums are distinct. As $M(e) = 2n+i+1$ for edges of the form $e = \{i, i+3\}$, where $0 \leq i < n-3$ and $S_{i+1}^L = S_i^L + 4$, for $3 \leq i \leq n-3$, then $S_{i+1} = S_i + 6$, for $3 \leq i \leq n-3$. Therefore, it suffices to show that $S_3$ is even. But $S_3 = (2n+15)+(2n+1)+(2n+4) = 6n+20$, which is even.

□

Figure 5 shows the labeling of the edges of $C_n^3 \setminus C_n^2$ as given in the proof of Theorem 3.1.

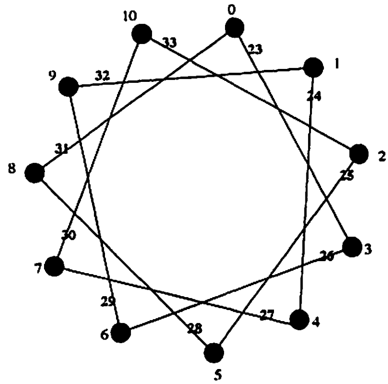


Figure 5: antimagic labeling of edges of $C_{11}^3 \setminus C_{11}^2$

We now consider the case where $n$ is even. Again, we will extend the antimagic labeling $L$ stated in the proof of Theorem 2.2.

**Theorem 3.2** *If $n > 6$ is a even number that is not a multiple of 6, then $C_n^3$ has an antimagic labeling.*

**Proof** : Consider the labeling $L$ used in the proof of Theorem 2.2. Recall that it has the following properties, where we use $S_i^L$ to denote the vertex sum of vertex $i$ under the labeling $L$.

1. $S_0^L = 4n+4$, $S_1^L = 2n+7$, $S_2^L = 2n+10$, $S_3^L = 2n+15$,
2. $S_{i+1}^L = S_i + 4$ for $3 \leq i \leq n-5$,
3. $S_{n-3}^L = 6n-8$, $S_{n-2}^L = S_{n-4} + 8$, and $S_{n-1}^L = 5n-2$.

We now show how to extend the antimagic labeling $L$ for $C_n^2$, as given by the construction in the proof of Theorem 2.2, to an antimagic labeling $M$ for $C_n^3$ such that $M|C_n^2 = L$. For each edge $e \in C_n^2$, assign $M(e) = L(e)$. For the edge $e = \{i, i+3\}$, where $0 \leq 0 \leq i < n-3$, we assign $M(e) = 2n+i+1$. For the edge $e = \{n-3, 0\}$, we assign $M(e) = 3n-2$. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n-1$. Finally, for the edge $e = \{n-1, 2\}$, we assign $M(e) = 3n$. This gives a labeling $M$ for $C_n^3$. We now show that it is an antimagic labeling of $C_n^3$.

By the definition of $M$, $S_0 = (4n+4) + (2n+1) + (3n-2) = 9n+3$, $S_2 = (2n+10) + (2n+3) + (3n) = 7n+13$, $S_{n-3} = (6n-8) + (3n-2) + (3n-5) = 12n-15$ and $S_{n-1} = (5n-2) + (3n-3) + (3n) = 11n-5$. These numbers are odd and pairwise distinct because of the assumptions on $n$ is even and $n > 6$. Also, $S_1 = (2n+7) + (2n+2) + (3n-1) = 7n+8$ and $S_3 = (2n+15) + (2n+1) + (2n+4) = 6n+20$, which are both even and are distinct, as $n \neq 12$. As $M(e) = 2n+i+1$ for edges of the form $e = \{i, i+3\}$, where $0 \leq i < n-3$, and $S_{i+1}^L = S_i^L + 4$, for $3 \leq i \leq n-5$, it follows that $S_{i+1} = S_i + 6$, for $3 \leq i \leq n-5$. Therefore, $S_i$ is even, for $3 \leq i \leq n-4$, and they are pairwise distinct. Thus, $S_{n-2}$, which equals $S_{n-4} + 12$, is also even as $S_{n-2} = 12n - 10$. So it remains to show that $S_1$ is not in the set $\{S_3, S_4, ..., S_{n-4}, S_{n-2}\}$. Consider $S_1 - S_3 = n - 12$. If the vertex sum $S_1$ appears again as one of $\{S_3, S_4, ..., S_{n-4}, S_{n-2}\}$, then $n-12$ must is be a multiple of 6. But $n-12$ is a multiple of 6 if and only if $n$ is a multiple of 6. As we assume $n$ is not a multiple of 6, then the vertex sum $S_1$ occurs only once. Therefore all the vertex sums for the labeling $M$ are distinct. □

As it turns out, a slight modification to labeling $M$ of Theorem 3.2 gives an antimagic labeling for $C_n^3$ where $n > 6$ is even and a multiple of 6.

**Theorem 3.3** *If $n > 6$ is a even number that is a multiple of 6, then $C_n^3$ has an antimagic labeling.*

**Proof** : In the labeling $M$ given in the proof of Theorem 3.2, make the following two modifications.

1. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n$, and

2. for the edge $e = \{n-1, 2\}$, we assign $M(e) = 3n-1$.

With this modification, we have $S_0 = 9n+3$ (odd), $S_1 = 7n+9$ (odd), $S_2 = 7n+12$ (even),$S_{n-3} = 12n-15$ (odd), $S_{n-2} = 12n-9$ (odd), $S_{n-1} = 11n-6$ (even). The vertex sums $S_i$, for $3 \leq i \leq n-4$ have the same values as in the proof of theorem 3.2 and therefore are all even and pairwise distinct. The values $S_0, S_1, S_{n-3}$, and $S_{n-2}$ are all odd and distinct. All that remains to show is that $S_2$ and $S_{n-1}$ are not the vertex sums of some other vertex. Clearly $S_2 \neq S_{n-1}$. To show that $S_2$ and $S_{n-1}$ do not appear in $\{S_3, S_4, ..., S_{n-4}\}$, it suffices to show that $S_2 - S_3$ and $S_{n-4} - S_{n-1}$ are not divisible by 6. If $S_2 - S_3 = n - 8$ is divisible by 6, then $n$ must be of the form $n = 6k + 2$. As we assume that $n$ is a multiple of 6, $n - 8$ cannot be divisible by 6. Similarly, if $S_{n-4} - S_{n-1} = n - 16$ is divisible by 6, then $n$ must be of the form $6k + 4$. As we assume that $n$ is a multiple of 6, $n - 16$ cannot be divisible by 6. Thus, all the vertex sums are distinct, and $M$ is a antimagic labeling for $n > 6$ and a multiple of 6. □

Figures 6 and 7 gives the antimagic labelings of the edges of $C_{12}^3 \setminus C_{12}^2$ and $C_{16}^3 \setminus C_{16}^2$ respectively. Figure 8 gives an antimagic labeling for $C_6^3$. Theorems 3.1 , 3.2 and 3.3 along with Figure 8 implies that $C_6^3$ is antimagic, for all $n \geq 6$.

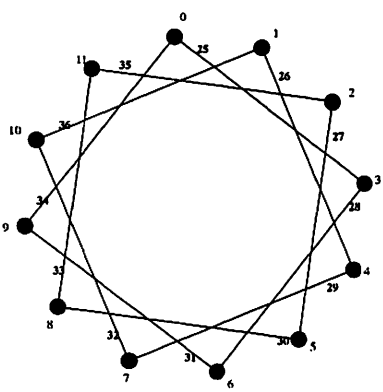


Figure 6: antimagic labeling of edges of $C_{12}^3 \setminus C_{12}^2$

**Corollary 3.4** : *For $n \geq 6$, $C_n^3$ is antimagic.*

# 4 The graph $C_n^4$

In this section, We will prove that $C_n^4$ has an antimagic labeling. We will do this by extending the labeling given in Section 2 for $C_n^3$.

**Theorem 4.1** *If $n \geq 7$ is an odd integer, then $C_n^4$ has an antimagic labeling.*

**Proof** We will show how to extend the labeling $M$ for $C_n^3$, as given in the proof of Theorem 3.1, to an antimagic labeling $N$ for $C_n^4$ such that $N|C_n^3 = M$. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge

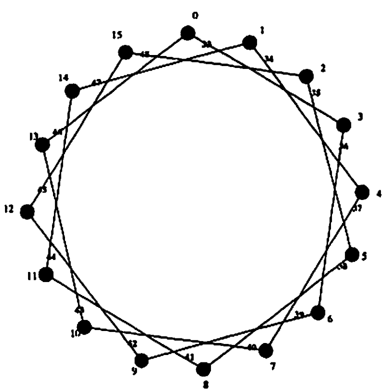


Figure 7: antimagic labeling of edges of $C_{16}^3 \setminus C_{16}^2$

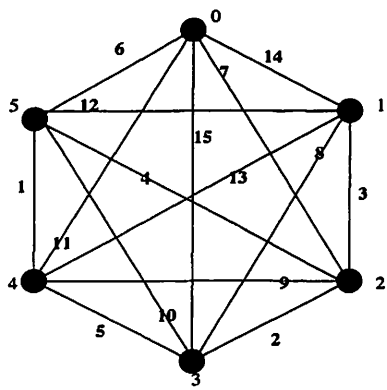


Figure 8: antimagic labeling of $C_6^3$

$e = \{i, i+4\}$, where $0 \leq i < n-4$, assign $N(e) = 3n+i+1$. For the edge $e = \{n-4, 0\}$, assign $N(e) = 4n-3$. For the edge $e = \{n-3, 1\}$, assign $N(e) = 4n-2$. For the edge $e = \{n-2, 2\}$, assign $M(e) = 4n-1$. Finally, for the edge $e = \{n-1, 3\}$, we assign $N(e) = 4n$. We claim that $N$ is an antimagic labeling of $C_n^4$.

Based on the labeling $N$, $S_0 = 16n$, $S_1 = 14n+8$, $S_2 = 14n+16$, $S_3 = 13n+24$, $S_4 = 12n+32$, $S_{i+1} = S_i + 8$, for $4 \leq i \leq n-3$, and $S_{n-1} = 19n-8$. It is easy to see that $S_0, S_1, S_2, S_4$ are even. As $n$ is odd, the vertex sums $S_0, S_1, S_2$ and $S_4$ are pairwise distinct. As $S_4$ is even, so is $S_i$, for $4 \leq i \leq n-2$ and these vertex sums are distinct. As $S_3$ and $S_{n-1}$ are odd, they are distinct from all the other vertex sums. They are also different from each other. It remains to show that the vertex sums $S_0, S_1, S_2$

are not one of the vertex sums $S_4, S_5, ..., S_{n-2}$. For $S_0$, $S_0 - S_4 = 4(n-8)$ is divisible by 8 if and only if $n$ is even. As $n$ is odd, the vertex sum $S_0$ is unique. For $S_1$, $S_1 - S_4 = 2(n-12)$ is divisible by 8 implies $n$ is even. So $S_1$ is unique also. For $S_2$, $S_2 - S_4 = 2(n-8)$ is divisible by 8 implies $n$ is even. So $S_2$ is also unique. Therefore, all the vertex sums are distinct and $N$ is an antimagic labeling of $C_n^4$.

□

**Theorem 4.2** *Let $n \geq 8, n \neq 12, 14$ be an even integer. Then $C_n^4$ is antimagic.*

**Proof** We begin be handling the special case where $n = 8$. To show that $C_8^4$ is antimagic, start with the labeling $M$ for $C_n^3$, as given in the proof of Theorem 3.2. Now label the edge $\{0,4\}$ with 15, the edge $\{1,5\}$ with 16, the edge $\{2,6\}$ with 18, and the edge $\{3,7\}$ with 17. It is easy to show that this is an antimagic labeling for $C_8^4$.

We now suppose that $n > 8$. We will show how to extend the labeling $M$ for $C_n^3$, as given in the proof of Theorem 3.3, to an antimagic labeling $N$ for $C_n^4$ such that $N|C_n^3 = M$. Note that when $n$ is not a multiple of 6, the labeling $M$ may not be an antimagic labeling of $C_n^3$. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i+4\}$, where $0 \leq i < n-4$, assign $N(e) = 3n+i+1$. For the edge $e = \{n-4, 0\}$, assign $N(e) = 4n-3$. For the edge $e = \{n-3, 1\}$, assign $N(e) = 4n-2$. For the edge $e = \{n-2, 2\}$, assign $M(e) = 4n$. Finally, for the edge $e = \{n-1, 3\}$, we assign $N(e) = 4n-1$. We claim that $N$ is an antimagic labeling of $C_n^4$.

Based on the labeling $N$, $S_0 = 16n+1$, $S_1 = 14n+9$, $S_2 = 14n+15$, $S_3 = 13n+23$, $S_4 = 12n+32$, $S_{i+1} = S_i + 8$, for $4 \leq i \leq n-5$, $S_{n-3} = 20n-23$, $S_{n-2} = 20n-14$ and $S_{n-1} = 19n-11$. It is easy to see that $S_4$ is even, and therefore $S_4, S_5, S_6, ..., S_{n-4}$ are all even and distinct. In since $S_1 = S_3$ only when $n = 14$ and $S_{n-3} = S_{n-1}$ only when $n = 12$, $S_0, S_1, S_2, S_3, S_{n-3}, S_{n-1}$ are all odd and pairwise distinct. It remains to show that $S_{n-2}$ is not in the set $S = \{S_4, S_5, S_6, ..., S_{n-4}\}$. This is true, since $S_{n-2} - S_{n-4} = (20n-14) - (20n-32) = 18 > 0$. Therefore, all the vertex sums are distinct. □

At this point, we could handle the remaining cases $n = 12, 14$ separately. Instead, we give another general construction that will deal with these two cases.

**Theorem 4.3** *Let $n > 8$ be an even integer of the form $8k, 8k+4$ or $8k+6$. Then $C_n^4$ is antimagic.*

**Proof** We will show how the extend the labeling $M$ for $C_n^3$, as given in the proof of Theorem 3.2, to an antimagic labeling $N$ for $C_n^4$ such that $N|C_n^3 = M$. Note that when $n$ is a multiple of 6, the labeling $M$ may not be an antimagic labeling of $C_n^3$. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i+4\}$, where $0 \leq i < n-4$, assign $N(e) = 3n+i+1$. For the edge $e = \{n-4, 0\}$, assign $N(e) = 4n-3$. For the edge $e = \{n-3, 1\}$, assign $N(e) = 4n$. For the edge $e = \{n-2, 2\}$, assign $M(e) = 4n-1$. Finally, for the edge $e = \{n-1, 3\}$, we assign $N(e) = 4n-2$. We claim that $N$ is an antimagic labeling of $C_n^4$.

Based on the labeling labeling $N$, $S_0 = 16n+1$, $S_1 = 14n+10$, $S_2 = 14n+15$, $S_3 = 13n+22$, $S_4 = 12n+32$, $S_{i+1} = S_i + 8$, for $4 \leq i \leq n-5$, $S_{n-3} = 20n-21$, $S_{n-2} = S_{n-4} + 16$ and $S_{n-1} = 19n-11$. It is easy to see that $S_4$ is even, and therefore $S_5, S_6, ..., S_{n-4}, S_{n-2}$ are all even and pairwise distinct. In addition, $S_0, S_2, S_{n-3}$ and $S_{n-1}$ are all odd and pairwise distinct, as $n > 8$. It remains to show that $S_1$ and $S_3$, which are both even and distinct from each other, cannot be in the set $S = \{S_4, S_5, S_6, ..., S_{n-4}, S_{n-2}\}$. To see this, suppose $S_1$ is in the set $S$. Then $S_1 - S_4 = 14n + 10 - (12n + 32) = 2(n-11)$ must be divisible by 8. But since $n$ is even, this is not possible. Therefore, $S_1$ is not in the set $S$. Now, suppose $S_3$ is in the set $S$. Then, $S_3 - S_4 = 13n + 22 - (12n + 32) = n - 10$ must be divisible by 8. But $n - 10$ is divisible by 8 if and only if $n$ is of the form $8k + 2$. Since we assumed $n$ is not of this form, $S_3$ cannot be in the set $S$. Therefore, all the vertex sums are distinct.

□

# References


[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, R. Yuster, Dense graphs are antimagic, J. of Graph Theory 47(4) (2004), 297-309.

[2] J. A. Gallian, Graph Labeling, Electronic Journal of Combinatorics, DS6 (2010).

[3] N. Hartsfield and G. Ringel, Pearls in Graph Theory, Academic Press Inc., Boston, (1990), 108-109.

[4] D. Hefetz, Antimagic graphs via the combinatorial nullstellensatz, J. of Graph Theory 50(4) (2005), 263-272.

[5] M. Lee, C. Lin, W. Tsai, On antimagic labeling for power of cycles, Ars Combinatoria 98(2011), 161-165.

[6] T. Wang, Toroidal graphs are antimagic, Lecture Notes in Computer Science (LNCS) 3595 (2005), 671-679.

[7] T. Wang and C. Hsiao, On antimagic labeling for graph products, Discrete Mathematics 308(16) (2008), 3624-3633.