Modular Multiplicative Graphs

Walter O. Krawec
University at Albany
1400 Washington Ave.
Albany NY, 12222
wk247837@albany.edu
(518)442-4619

February 12, 2011

Abstract

In this paper, a new type of labeled graphs, called modular multiplicative graphs, is introduced and studied. Specifically we show that every graph is a subgraph of a modular multiplicative graph. Later we introduce k-modular multiplicative graphs and prove that certain families of paths and cycles admit such a label.

We conclude with several open problems and areas of future possible research including a note on harmonious graph labels.

1 Introduction

In this paper, a new type of labeled graphs, called modular multiplicative graphs, is introduced and studied. We show that every graph is an induced subgraph of a modular multiplicative graph with prime number of edges. Further, we introduce k-modular multiplicative graphs and provide labeling schemes for certain cycles and paths and conclude with some open problems. For a detailed survey of graph labels the reader is encouraged to read [2].

Definition 1.1. We define a *tree* in the usual way, i.e. a connected acyclic graph. For notation we denote the set of trees on n edges to be $\mathcal{T}_n$.

Furthermore, let $V(G)$ denote the set of vertices of some graph G and $E(G)$ to be the set of edges of G. Elements in $E(G)$ are of the form $(v_i v_j)$ representing an undirected edge between vertices v_i and v_j ($v_i, v_j \in V(G)$). Also all graphs considered in this paper are finite without multiple edges.

2 Modular Multiplicative Labels

Definition 2.1. Let G be a graph on n edges. We say G is a *modular multiplicative graph* if there exists a function $f : V(G) \to \mathbb{Z}/n\mathbb{Z}$ (called a *vertex labeling function*) such that:

1. f is injective (if $G \in \mathcal{T}_n$ then we permit one repeated vertex label)
2. The function $F : E(G) \to \mathbb{Z}/n\mathbb{Z}$ given by $F(uv) = f(u)f(v) \bmod n$ for all $(uv) \in E(G)$ is bijective (we call this function the *multiplicative edge label function induced by* f or simply the *induced edge label function* if the context is clear).

We denote the set of all multiplicative graphs with n edges by $\mathcal{M}_n$. Modular multiplicative labeling requires a graph on n edges to have distinct edge labels in $\mathbb{Z}/n\mathbb{Z}$ whereas a strongly multiplicative graph introduced in [1] requires distinct edge labels in $\mathbb{Z}$. Obviously if $G \in \mathcal{M}_n$ then it is also strongly multiplicative.

Definition 2.2. Let G be a graph on p edges and n vertices with $E(G) = \{e_1, e_2, \ldots, e_p\}$. The *edge label polynomial of* G is defined to be the multivariate polynomial $E_G \in \mathbb{Z}[x_0, x_1, \ldots, x_n]$ given by:

$$E_G(x_0, x_1, \ldots, x_n) = \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} (\hat{e_i} - \hat{e_j}) \tag{1}$$

Where given $e_i = (v_{\alpha_i} v_{\beta_i})$ then $\hat{e_i} = x_{\alpha_i} x_{\beta_i}$

Example 2.1. Consider the path on three edges P_3. Then we have:

$$E(P_3) = \{(v_0v_1), (v_1v_2), (v_2v_3)\}$$

And the edge label polynomial is:

$$\begin{aligned} E_{P_3}(x_0, x_1, x_2, x_3) &= (x_0x_1 - x_1x_2)(x_0x_1 - x_2x_3)(x_1x_2 - x_2x_3) \\ &= x_1x_2(x_0 - x_2)(x_0x_1 - x_2x_3)(x_1 - x_3) \end{aligned} \tag{2}$$

Theorem 2.1. Let G be a graph on $n+1$ vertices with prime number of edges p. Then $G \in \mathcal{M}_p$ iff there exists $\{x_0, x_1, \cdots, x_n\} \subset \mathbb{Z}/p\mathbb{Z}$ such that:

1. Each x_i is distinct (with one exception if $G \in \mathcal{T}_p$)
2. $E_G(x_0, x_1, \cdots, x_n) \not\equiv 0 \bmod p$ where E_G is the edge label polynomial of G

Furthermore, the vertex label function $f : V(T) \to \mathbb{Z}/p\mathbb{Z}$ is given by $f(v_i) = x_i$.

Proof. We first show that $G \in \mathcal{M}_p$ implies that there exists $\{x_0, x_1, \cdots, x_n\}$ that satisfies the theorem. Since G is a modular multiplicative graph there exists a function $f : V(G) \to \mathbb{Z}/p\mathbb{Z}$ as in Definition 2.1. Let $x_i = f(v_i)$. Clearly each x_i is distinct in $\mathbb{Z}/p\mathbb{Z}$ (with one exception if $G \in \mathcal{M}_p$). We show $E_G(x_0, x_1, \cdots, x_n) \not\equiv 0 \bmod p$.

Assume for contradiction that $E_G(x_0, x_1, \cdots, x_n) \equiv 0 \bmod p$. Since p is prime this implies that there exists edges of G call them $e_i = (v_{\alpha_i} v_{\beta_i})$ and $e_j = (v_{\alpha_j} v_{\beta_j})$ with $i < j$ such that $\hat{e_i} - \hat{e_j} \equiv 0 \bmod p$. Of course this implies that $x_{\alpha_i} x_{\beta_i} \equiv x_{\alpha_j} x_{\beta_j} \Rightarrow f(v_{\alpha_i}) f(v_{\beta_i}) \equiv f(v_{\alpha_j}) f(v_{\beta_j}) \bmod p$ which contradicts the fact that $G \in \mathcal{M}_p$.

The other direction is similar. □

Hence determining whether or not a graph G with prime edges admits a modular multiplicative label is equivalent to finding solutions to the edge label polynomial given in equation 1. Of course the problem may seem more complicated now, however we may use this to easily prove certain facts about modular multiplicative graphs. We show that every graph is a subgraph of a modular multiplicative graph. Also, for every graph $G \in \mathcal{M}_p$ (with p a prime number) and $k = 2, 3, \ldots$ there exists $H \in \mathcal{M}_{p^k}$ and $K \in \mathcal{M}_{kp}$ such that G is a subgraph of H and K.

Lemma 2.1. Given a graph G on $n+1$ vertices then one can always find distinct $\{x_0, x_1, \cdots, x_n\} \subset \mathbb{N} \cup \{0\}$ such that $E_G(x_0, x_1, \ldots, x_n) \neq 0$

Proof. Begin by setting $x_0 = 1$. Assume we have found appropriate numbers for $x_0, x_1, \cdots, x_{i-1}$. Collect all terms of E_G containing x_i in them (there are of course finitely many of them). Then it is obvious there must exist an unused natural number n such that if $x_i = n$ each of these factors is not equal to 0.

Continue this process to find numbers for each $x_0, x_1, \cdots, x_n$. It is clear then that $E_G(x_0, x_1, \ldots, x_n) \neq 0$. □

Lemma 2.2. Given a graph G on n edges along with an injective function $f : V(G) \to \mathbb{Z}/m\mathbb{Z}$ for some $m \geq n$, such that $f(u)f(v) \not\equiv f(x)f(y) \bmod m$ for every distinct edge (uv) and (xy) in G, we may construct a graph $H \in \mathcal{M}_m$ with vertex labeling function $g : V(H) \to \mathbb{Z}/m\mathbb{Z}$ such that:

1. G is a subgraph of H.
2. $g(v) = f(v)$ for every $v \in V(G)$.

Proof. Clearly if $m = n$ then we're done. So assume $m > n$. There are several ways to construct graph H. For one such construction, start with $H = G$ and let $g \equiv f$. Let $\widetilde{E}$ be the set of unused edge weights and $\widetilde{V} = \mathbb{Z}/m\mathbb{Z} -$ image(f) be the set of unused vertex labels. For every $z \in \widetilde{V}$ add a vertex v_z to H and let $g(v_z) = z$.

Now pick $e \in \widetilde{E}$. Then there must exist $u, v \in V(H)$ (not necessarily distinct) such that $g(u)g(v) \equiv e \mod m$ (such vertices always exist as for every $z \in \mathbb{Z}/m\mathbb{Z}$ there exists $v_z \in V(H)$ with $g(v_z) = z$). So add the edge (uv) to $E(H)$. Remove e from $\widetilde{E}$ and repeat until $\widetilde{E} = \emptyset$.

It is obvious that this process always results in a modular multiplicative graph. □

Theorem 2.2. Given graph G on $n+1$ vertices and distinct $\{x_0, x_1, \cdots, x_n\} \subset \mathbb{Z}$ such that $E_G(x_0, x_1, \cdots, x_n) \neq 0$, let:

$$c = E_G(x_0, x_1, \cdots, x_n) \prod_{i=0}^{n-1} \prod_{j=i+1}^{n} (x_i - x_j)$$

then for every prime $p \geq |E(G)|$ that does not divide c we have that there exists a graph $H \in \mathcal{M}_p$ with vertex labeling function f such that:

1. G is a subgraph of H
2. $f(v_i) = x_i$ for every $v_i \in V(G)$.

Proof. Let c be as above and p a prime number not dividing c. Define $f : V(G) \to \mathbb{Z}/p\mathbb{Z}$ given by $f(v_i) = x_i$. This function is clearly injective. If it weren't, then there would exist distinct vertices $v_i, v_j \in V(G)$ with $i < j$ such that $f(v_i) \equiv f(v_j) \mod p$. But this would imply that $x_i - x_j \equiv 0 \mod p$ which cannot be since p does not divide c.

Now since $E_G(x_0, x_1, \cdots, x_n)$ is not divisible by p it is clear that for every distinct edge $(uv), (xy) \in E(G)$, we have $f(u)f(v) \not\equiv f(x)f(y) \mod p$. Apply Lemma 2.2 with $m = p$ to obtain the desired result.

□

Corollary 2.1. For every graph G on n edges there exists a graph $H \in \mathcal{M}_p$ for some prime $p \geq n$ such that G is a subgraph of H.

Proof. Apply Lemma 2.1 and Theorem 2.2 □

While the above corollary tells us every graph is an induced subgraph of a modular multiplicative graph, it provides us with no information on the size or shape of this supergraph. We show however that if $G \in \mathcal{M}_p$ (p a prime number) then not only is G a subgraph of some larger modular multiplicative graph H but we also have some control over the number of edges in H.

Before we show this, we require a small construction lemma:

Lemma 2.3. Let $G \in \mathcal{M}_n$ with vertex label function f and define $\phi_\alpha(x) = \alpha x$. Then for every $\alpha \in \mathbb{Z}/n\mathbb{Z}$ with $\gcd(\alpha, n) = 1$ we have $\phi_\alpha \circ f$ is a modular multiplicative vertex label function (that is, multiplying by a unit of $\mathbb{Z}/n\mathbb{Z}$ yields a new modular multiplicative label of graph G).

Proof. Assume the contrary. Then there must be at least two edges with the same label. Call these two edges (uv) and (xy) then we have:

$$\begin{array}{rrcll} & (\phi_\alpha \circ f(u))(\phi_\alpha \circ f(v)) & \equiv & (\phi_\alpha \circ f(x))(\phi_\alpha \circ f(y)) & \bmod n \\ \Rightarrow & \alpha f(u)\alpha f(v) & \equiv & \alpha f(x)\alpha f(y) & \\ \Rightarrow & \alpha^2(f(u)f(v)) & \equiv & \alpha^2(f(x)f(y)) & \end{array} \tag{3}$$

Since $\gcd(\alpha, n) = 1$ we have that α^{-1} exists hence the above implies $f(u)f(v) = f(x)f(y)$ but since f is a modular multiplicative vertex label function this cannot be. □

Theorem 2.3. Let p be prime. Given $T \in \mathcal{T}_p \cap \mathcal{M}_p$ with vertex label function f, then for every $k = 2, 3, \ldots$ there exists $G \in \mathcal{M}_{p^k}$ and $H \in \mathcal{M}_{kp}$ such that T is a subtree of G and H.

Proof. Since $T \in \mathcal{T}_p$ there exists distinct vertices $v_i, v_j \in V(T)$ such that $f(v_i) \equiv f(v_j) \bmod p$. Let $\beta \equiv f(v_i) \equiv f(v_j)$. Clearly $\beta \not\equiv 0$ (a vertex label of 0 can only appear once on a tree). Also, since p is prime, there exists a multiplicative inverse $\beta^{-1} \in \mathbb{Z}/p\mathbb{Z}$. Apply Lemma 2.3 with $\alpha = \beta^{-1}$ to change the repeated vertex label to 1.

So without loss of generality we may assume the repeated vertex label is 1. Again, let v_i and v_j represent these repeat vertices (i.e. $f(v_i) \equiv f(v_j) \bmod p$). Furthermore, by reordering the vertices in $V(T)$ we may assume $i = 0$ and $j = 1$.

From Theorem 2.1, there exists $\{x_0, x_1, \cdots, x_p\} \subset \mathbb{Z}/p\mathbb{Z}$ distinct with one exception such that $E_T(x_0, x_1, \cdots, x_p) \not\equiv 0 \bmod p$ and $f(v_a) = x_a$. We may change the value of x_0 to $p+1$ then still $E_T(p+1, x_1, \cdots, x_p) \not\equiv 0 \bmod p$. Of course, neither p^k nor kp divide $E_T(p+1, x_1, \cdots, x_p)$ which implies that the edge weights of T are all distinct modulo p^k and kp.

Let $g : V(T) \to \mathbb{Z}/p^k\mathbb{Z}$ and $h : V(T) \to \mathbb{Z}/kp\mathbb{Z}$ given by:

$$g(v_a) = h(v_a) = \begin{cases} p+1 & \text{if } a = 0 \\ f(v_a) & \text{otherwise} \end{cases}$$

It is obvious that g and h are both injective. Since $E_T(p+1, x_1, \cdots, x_p)$ is not divisible by p^k or kp we have $g(u)g(v) \not\equiv g(x)g(y) \bmod p^k$ and

$h(u)h(v) \not\equiv h(x)h(y) \bmod kp$ for every distinct edge $(uv), (xy) \in E(T)$. Hence we may apply Lemma 2.2 twice (once for g and again for h) to obtain the desired graphs.

□

Theorem 2.4. Let p be prime. Given graph $G \in \mathcal{M}_p$ with vertex label function f, then for every $k = 2, 3, \ldots$ there exists $H \in \mathcal{M}_{p^k}$ and $K \in \mathcal{M}_{kp}$ such that G is a subgraph of H and K.

Proof. The proof is the same as in Theorem 2.3 except there is no need to worry about a repeated vertex. □

3 k-Modular Multiplicative Graphs

Definition 3.1. Let G be a graph on n edges and k an integer greater than or equal to zero. We say G is a *k-modular multiplicative graph* if there exists a function $f : V(G) \to \mathbb{Z}/(n+k)\mathbb{Z}$ such that:

1. f is injective (if $k = 0$ and $G \in \mathcal{T}_n$ then we permit one repeated vertex label)
2. The induced edge label function $F : E(G) \to \mathbb{Z}/(n+k)\mathbb{Z}$ given by $F(uv) = f(u)f(v) \bmod (n+k)$ for all $(uv) \in E(G)$ is injective.

We denote the set of all k-modular multiplicative graphs on n edges by $\mathcal{M}_n(k)$ (hence $\mathcal{M}_n(0) = \mathcal{M}_n$).

Lemma 3.1. Every graph is k-modular multiplicative for some $k \geq 0$.

Proof. Let G be a graph on $n+1$ vertices. Apply Lemma 2.1 to find $\{x_0, x_1, \cdots, x_n\} \subset \mathbb{N} \cup \{0\}$ such that each x_i is distinct and $E_G(x_0, x_1, \cdots, x_n) \neq 0$. Define:

$$c = E_G(x_0, x_1, \cdots, x_n) \prod_{i=0}^{n-1} \prod_{j=i+1}^{n} (x_i - x_j)$$

and let p be a prime number such that $p \geq |E(G)|$ and p does not divide c. It follows that $G \in \mathcal{M}_{|E(G)|}(p - |E(G)|)$. □

We now find some actual label schemes that result in a k-modular multiplicative graph. Let C_n be a cycle on n edges and P_n be a path on n edges. Then we have the following results:

Theorem 3.1. Let $p = 2n + 1$ be prime. Then $C_n \in \mathcal{M}_n(n+1)$ if there does not exist $\alpha \in \{2, 3, \ldots, n\}$ such that $\alpha^2 + \alpha - 1 \equiv 0 \bmod p$.

Proof. Define

$$\begin{array}{rcl} V(C_n) & = & \{v_0, v_1, v_2, \ldots, v_{n-1}\} \\ E(C_n) & = & \{(v_0v_1), (v_1v_2), \ldots, (v_{n-1}v_0)\} \end{array} \tag{4}$$

Let $f : V(C_n) \to \mathbb{Z}/p\mathbb{Z}$ be the vertex label function given by $f(v_i) = i+2$. Then every edge except $(v_{n-1}v_0)$ is of the form $\alpha(\alpha+1)$, $\alpha = 2, 3, \ldots, n$ while the edge $(v_{n-1}v_0)$ is labeled $2(n+1)$. We first show that all edges save the final one are distinct modulo p.

Assume the contrary. Then there must exist $\alpha \in \{2, 3, \cdots, n-1\}$ and $k \in \{1, 2, \cdots, n-\alpha\}$ such that:

$$\begin{array}{rrcll} & \alpha(\alpha+1) & \equiv & (\alpha+k)(\alpha+k+1) & \mod p \\ \Rightarrow & k(k+2\alpha+1) & \equiv & 0 & \end{array} \tag{5}$$

Since p is prime this implies either $k \equiv 0$ or $k + 2\alpha + 1 \equiv 0$. Since k is not equal to zero by definition the latter must be true. Hence $k \equiv -2\alpha - 1$ which implies $k = (2n+1)\lambda - 2\alpha - 1$ for some $\lambda \in \mathbb{Z}$. Since $k \geq 1$ it is obvious that $\lambda \geq 1$. Hence:

$$\begin{array}{rrcl} & \alpha & < & n \\ \Rightarrow & n-\alpha & < & 2n - 2\alpha \leq (2n+1)\lambda - 2\alpha - 1 = k \end{array} \tag{6}$$

But $k \leq n - \alpha$ so this cannot be.

All that remains to show is the final edge $(v_{n-1}v_0)$ is distinct. If not, then there must exist some $\alpha \in \{2, 3, \cdots, n\}$ such that:

$$\begin{array}{rrcll} & 2(n+1) & \equiv & \alpha(\alpha+1) & \mod p \\ \Rightarrow & \alpha^2 + \alpha - 1 & \equiv & 0 & \end{array} \tag{7}$$

But no such α exists by hypothesis. □

Theorem 3.2. Let $p = 2n+1$ be prime. Then $P_n \in \mathcal{M}_n(n+1)$.

Proof. Let $V(P_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(P_n) = \{(v_0v_1), (v_1v_2), \ldots, (v_{n-1}v_n)\}$. Let $f : V(P_n) \to \mathbb{Z}/p\mathbb{Z}$ be the vertex label function given by:

$$f(v_i) = \begin{cases} 0 & \text{if } i = 0 \\ i+1 & \text{otherwise} \end{cases}$$

Then it follows from the proof of Theorem 3.1 that this results in a $(n+1)$-modular multiplicative label. □

4 Open Problems

The following interesting open problems remain regarding this new label scheme.

1. Is $\mathcal{T}_n \subset \mathcal{M}_n$ for every n? Or at least for every prime n?

2. What other families of graphs are in $\mathcal{M}_p$?

3. Can anything be said about the structure of graphs H and K described in Theorems 2.3 and 2.4?

4. The edge label polynomial (equation 1) can be adapted to other graph labeling schemes. For example, the edge label polynomial for a harmonious graph [3] on $n+1$ vertices and p edges would take the form:

$$E_G(x_0, x_1, \ldots, x_n) = \prod_{i=1}^{p-1} \prod_{j=i+1}^{p} (\hat{e_i} - \hat{e_j}) \tag{8}$$

Where given $e_i = (v_{\alpha_i} v_{\beta_i}) \in E(G)$ then $\hat{e_i} = x_{\alpha_i} + x_{\beta_i}$. What kind of results can be proved with the help of the concept of the edge label polynomial? For example, in [4], Liu and Zhang showed that every graph is a subgraph of a harmonious graph. Using the edge label polynomial for harmonious graphs this same result can be shown algebraically as in section 2 of this paper.

5 Acknowledgement

The author would like to thank the referee for the suggestions which greatly improved this paper.

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