On locating-dominating codes in the infinite king grid

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Abstract

Assume that $G = (V, E)$ is an undirected graph with vertex set V and edge set E. The ball $B_r(v)$ denotes the vertices within graphical distance r from v. A subset $C \subseteq V$ is called an r-locating-dominating code if the sets $I_r(v) = B_r(v) \cap C$ are distinct and non-empty for all $v \in V \setminus C$. A code C is an r-identifying code if the sets $I_r(v)$ are distinct and non-empty for all vertices $v \in V$. We study r-locating-dominating codes in the infinite king grid and in particular show that there is an r-locating-dominating code such that every r-identifying code has larger density. The infinite king grid is the graph with vertex set $\mathbb{Z}^2$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| \leq 1,\ |y_1 - y_2| \leq 1, (x_1, y_1) \neq (x_2, y_2)\}$.

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E. Denote by $d(u, v)$ the distance between two vertices u and v i.e. the number of edges on any shortest path from u to v. The ball with center v and radius r is

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

We call any $C \subseteq V$ a *code*. The vertices of C are called *codewords*. In particular, C is an *r-locating-dominating code* if the sets

$$I_r(v) = B_r(v) \cap C$$

*Research supported by the Academy of Finland under grant 210280.

are non-empty and distinct for all non-codewords $v \in V \setminus C$. If the sets $I_r(v)$ are non-empty and distinct for all vertices $v \in V$, then C is an r-identifying code. In particular, an identifying code is always a locating-dominating code.

The symmetric difference of two sets A and B is denoted by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

The r-locating-dominating code could also be defined by symmetric differences: code C is an r-locating-dominating code if and only if $I_r(v) \Delta I_r(u) \neq \emptyset$ and $I_r(v) \neq \emptyset$ for all non-codewords v and u. If $c \in I_r(v) \Delta I_r(u)$, we say that u and v are *separated* by c.

We study r-locating-dominating codes in the infinite king grid. The infinite king grid is the graph where $V = \mathbb{Z} \times \mathbb{Z}$ and two different vertices $u = (u_x, u_y)$ and $v = (v_x, v_y)$ are adjacent if $|u_x - v_x| \leq 1$ and $|u_y - v_y| \leq 1$. Thus vertices u and v are neighbours if the Euclidean distance between u and v is 1 or $\sqrt{2}$.

The density of $C \subseteq \mathbb{Z}^2$ is

$$D(C) = \limsup_{n \to \infty} \frac{|C \cap B_n((0,0))|}{|B_n((0,0))|},$$

where $|C \cap B_n((0,0))|$ is the number of codewords in the ball $\{(x,y) \mid |x| \leq n, |y| \leq n\}$ and $|B_n((0,0))| = (2n+1)^2$ is the number of all vertices in the ball. We also denote $B_n((0,0)) = Q_n$. We search for the minimum density of locating-dominating codes for given r in the infinite king grid.

Locating-dominating codes were introduced in the late 1980s by Slater [17] and [18] and identifying codes in the late 1990s by Karpovsky, Chakrabarty and Levitin [10]. A motivation of such codes is a safeguard analysis of a facility using sensor networks [17] or a fault diagnosis of a multiprocessor system [10]. Assume that we have a multiprocessor system. Some processors are chosen to perform the task of testing if the processor itself is faulty or if there is a faulty processor within distance r. The chosen processor sends the symbol 2, if the processor itself is faulty; symbol 1, if it itself is not faulty, but there is a faulty processor within distance r; and symbol 0, otherwise. Finally, we get the reports from all the chosen processors and based on the reports alone we can perform the fault diagnosis. Here, processors are vertices and the chosen processors are codewords. If the chosen processors form an r-locating-dominating code, then we can locate the processor which is faulty if we assume that at most one of the processors is faulty. If the chosen processor sends symbol 1 instead of 2 also when the processor itself is faulty, then we can use an r-identifying code to locate the faulty processor.

r	Locating-dominating codes		ID codes	
1	$\frac{1}{5}$	[8]	$\frac{2}{9}$	[3],[4]
2	$\frac{1}{10} \leq D \leq \frac{1}{8}$	[3], Thm 2	$\frac{1}{8}$	[2],[3]
3	$\frac{1}{14} \leq D \leq \frac{2}{25}$	Thms 2 & 5	$\frac{1}{12}$	[2],[3]
≥ 4	$\frac{1}{4r+2} \leq D \leq \frac{1}{4r}$ if $2 \mid r$ $\frac{1}{4r+2} \leq D \leq \frac{1}{4r+\frac{2}{r+1}}$ if $2 \nmid r$	[2], Thms 2 & 5	$\frac{1}{4r}$	[2]

Table 1: The known lower and upper bounds for densities of locating-dominating and identifying codes (ID codes) in the infinite king grid.

Although the difference between the definitions of locating-dominating codes and identifying codes is quite small, we show in this paper that there exists an r-locating-dominating code with density D_r for any odd r such that there exist no r-identifying codes with the same or smaller density. Results when $r = 1$ have already been shown in [8]. We also prove two lower bounds for r-locating-dominating codes when $r > 1$. The proof of the better bound is long and quite similar to corresponding proof for r-identifying codes (Theorem 3 of [2]). Therefore, we only add some details to the proof and it is almost impossible to understand our proof if one does not know the proof of Theorem 3 of [2]. The complete proof would nevertheless be a duplicate in many respects of the proof for r-identifying codes, so it is reasonable to present only the differences in this paper. When reading our proof of Theorem 2, the reader should have a copy of [2] at hand. The proof of the weaker bound is short and easy to understand.

Furthermore, we observe in this paper that the bounds of r-locating-dominating codes when $r \geq 2$ are also valid for so-called *open neighbourhood r-locating-dominating codes* i.e. *r-OLD codes.* A code is an r-OLD code if sets $I_r(v) \setminus \{v\}$ are non-empty and distinct for all $v \in V$. OLD codes were considered in [9] and [16] and they can be used in fault diagnosis if the chosen processor sends symbol 0 if the processor itself is faulty. This corresponds for example to the case where the processor is unable to send an alarm if the processor itself is faulty.

Table 1 summarizes what is known about the density of r-locating-dominating codes and r-identifying codes in the infinite king grid. Here, the upper bound means that there exists a locating-dominating or an identifying code with that density and the lower bound means that density of every locating-dominating or identifying code is at least the value given in the table.

Locating-dominating codes, identifying codes and other closely related classes of codes in the infinite king grid and other graphs have also been studied in [1], [3]–[7] and [11]–[15]. See also the web bibliography [19].

2 Lower bounds

Theorem 1. *The density of an r-locating-dominating code is at least* $\frac{1}{4r+4}$.

Proof. In this proof, we use a standard technique for identifying codes. A more detailed presentation of the technique can be found in [1], for instance.

Let C be an r-locating-dominating code. Then

$$A_r(x,y) = (B_r(x,y) \Delta B_r(x,y+1)) \cup \{(x,y),(x,y+1)\}$$

contains at least one codeword for all (x,y). The claim follows from the fact $|A_r(x,y)| = 4r+4$. Indeed, a codeword can belong to only $4r+4$ such sets. □

Theorem 2. *The density of an r-locating-dominating code is at least* $\frac{1}{4r+2}$.

Proof. *(Sketch)* The claim is proved in [8], when $r = 1$. Therefore we can assume that C is an r-locating-dominating code and $r \geq 2$. Next, we denote

$$\begin{aligned}
C_o(x,y) &= \{(x-r,y-r),(x-r,y+r+1),\\
&\qquad (x+r+1,y-r),(x+r+1,y+r+1)\},\\
C_e(x,y) &= \{(x,y),(x,y+1),(x+1,y),(x+1,y+1)\},\\
S_v(x,y) &= \{(a,b)\,|\, a \in \{x-r,x+r+1\} \text{ and } y-r < b \leq y+r\},\\
S_h(x,y) &= \{(a,b)\,|\, x-r < a \leq x+r \text{ and } b \in \{y-r,y+r+1\}\},\\
L(x,y) &= C_o(x,y) \cup C_e(x,y) \cup S_v(x,y) \cup S_h(x,y).
\end{aligned}$$

We call C_o the set of *corners*, C_e the *center*, and S_v and S_h *vertical and horizontal sides* of set $L(x,y)$. Moreover, we observe that

$$L(x,y) = \bigcup_{u,v \in C_e(x,y)} (B_r(u) \,\Delta\, B_r(v)) \cup C_e(x,y).$$

Then $L(x,y)$ has to separate vertices in the center of $L(x,y)$.

Paper [2] shows that the sides of

$$\begin{aligned}
K(x,y) &:= K_r((x,y),(x+1,y),(x+1,y+1),(x,y+1))\\
&= \bigcup_{u,v \in C_e(x,y)} (I_r(u) \,\Delta\, I_r(v))
\end{aligned}$$

contain on average at least two codewords for all identifying codes. Now, we add a few rules to this proof and show that the sides and the center together contain at least two codewords on average for all locating-dominating codes.

If S_v contains codewords u and v, then these vertices separate the codewords of the center in the same way. Therefore, we say that v is *useless* for $L(x,y)$ if the y-coordinate of v is greater than the y-coordinate of u or if the y-coordinates of v and u are the same and the x-coordinate of v is greater than the x-coordinate of u. In the same way, if $u, v \in S_h(x,y)$, then v is useless for $L(x,y)$ if the x-coordinate of v is greater than the x-coordinate of u or if the x-coordinates of v and u are the same and the y-coordinate of v is greater than the y-coordinate of u. This is how useless codewords are defined for $K(x,y)$ in [2]. But in this paper, we give one more rule when a codeword is useless.

Codeword $v \in C_e(x,y)$ is useless for $L(x,y)$ if other non-useless codewords in $L(x,y)$ separate vertices in the center of $L(x,y)$. When we define whether the codewords in the center are useless or not we go through then in the following order:

$$(x,y) \lhd (x,y+1) \lhd (x+1,y) \lhd (x+1,y+1).$$

If possible, we mark useless codewords for the associates of $L(x,y)$ in the same way as they are marked for the associates of $K(x,y)$ in paper [2]. However, it is possible, that the sides of $K(x,y)$ (or $L(x,y)$) do not contain any codeword for locating-dominating codes if $K(x,y) \in \mathcal{E}_2''$ i.e two opposite corners of $K(x,y)$ are codewords and the other two are not. Then at least one vertex in the center of $L(x,y)$ has to be a codeword. See Figure 1. In this case, we can not mark useless codewords as for identifying codes. Therefore we need new rules.

Assume that (x,y), $(x-r,y+r+1)$ and $(x+r+1,y-r)$ are codewords and there are no other codewords on the sides and in the corners (or else we can mark a useless codeword as in [2]). First, if the center also contains other codewords than (x,y), then all except one of the codewords in the center are useless and we can mark one of them for $L(x,y)$.

Second, we assume that (x,y) is the only codeword in the center of $L(x,y)$. Now,

$$\begin{aligned} &B_r((x-1,y)) \Delta B_r((x+1,y+1)) \\ &\subseteq S_v(x-1,y) \cup \{(x-r-1,y-r)\} \cup S_v(x,y) \cup S_h(x,y) \\ &\quad \cup \{(x-r,y-r),(x+r+1,y+r+1)\} \end{aligned} \tag{1}$$

contains at least one codeword or $(x-1,y)$ or $(x+1,y+1)$ is a codeword for all r-locating-dominating codes (cf. Figure 1). The last three sets of (1) and $\{(x+1,y+1)\}$ do not contain any codewords by our assumption. If

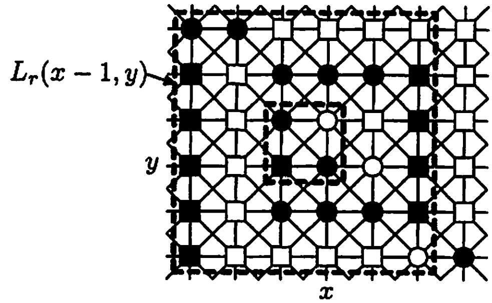

Figure 1: Black dots are codewords and white dots are non-codewords. These dots constitute $L_r(x,y)$. Gray dots can be codewords or non-codewords. The squares constitute the set $B_r((x-1,y))\Delta B_r((x+1,y+1)) \cup \{(x-1,y),(x+1,y+1)\}$. Therefore, at least one of the squares must be a codeword.

$c \in S_v(x-1,y)$ is a codeword, then c and $(x-r, y+r+1) \in S_h(x-1,y)$ separates the codeword in $C_e(x-1,y)$ and so (x,y) is useless for $L(x-1,y)$ and it can be marked for $L(x-1,y)$.

If $S_v(x-1,y)$ does not contain codewords and $(x-r-1, y-r)$ is a codeword, then $(x-r, y+r+1)$ separates (x,y) from $(x-1,y+1)$ and $(x,y+1)$ and $(x-r-1,y-r)$ separates (x,y) from $(x-1,y)$. Thus (x,y) is useless for $L(x-1,y)$ and it can be marked for $L(x-1,y)$. Otherwise, $(x-1,y)$ must be a codeword, then it is useless and it is marked for $L(x-1,y)$, since $(x-r,y+r+1)$ and (x,y) separate $(x,y-1)$ from other vertices in the center of $L(x-1,y)$ and $(x-1,y)$ comes before (x,y) in our ordering.

The rotations of case $\mathcal{E}_2''$ are treated in the same way. In particular, we nevertheless mark useless codewords in the center only by left and right associates. Now, codewords can not be marked twice, because $r \geq 2$. Indeed, by the new rules we can mark codewords only from centers. Moreover, the useless codeword in the center of $L(x,y)$ can be marked only if there are no codewords in the sides of $L(x,y)$ or if there is exactly one codeword on the horizontal sides of $L(x,y)$ and it is at distance one from a corner.

Thus, equations (4) and (5) of [2] are also valid for r-locating-dominating codes when

$$\mathcal{S} = \{(L(x,y),c) \mid L(x,y) \in \mathcal{E}, c \in C \cap Q_n, c \text{ marked for } L(x,y)\},$$

but equation (6) is now

$$(8r+8) \cdot |C \cap Q_n| \geq 2|Q_n| + p_1 + p_2 + p_3 + 2p_4 + |\mathcal{S}| - 2hn$$

since $|L(x,y)| = 8r+8$. Moreover,

$$p_1 + p_2 + p_3 + 2p_4 + |S| \geq 4 \cdot |C \cap Q_n| - 10kn - 8n$$

as in [2]. Then

$$D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|} \geq \limsup_{n\to\infty} \left(\frac{1}{4r+2} - \frac{(10k+2h+8)n}{8r|Q_n|} \right) = \frac{1}{4r+2}.$$

□

Remark 3. An r-OLD code is also an r-locating-dominating code. Therefore the lower bounds for r-locating-dominating codes are also valid for r-OLD codes.

3 Upper bounds

Theorem 4. *There is an r-locating-dominating code with density $\frac{1}{4r}$ for all r.*

Proof. The code

$$C = \{ (x,y) \mid x \equiv y \pmod{2r},\ x \equiv 0 \pmod 2 \}$$

is an r-identifying code with density $\frac{1}{4r}$ for all r by [2]. The claim follows since r-identifying code is automatically an r-locating-dominating code. □

Theorem 5. *There exists an r-locating-dominating code with density $\frac{1}{4r+\frac{2}{r+1}}$ for all odd r.*

Proof. We show that the code

$$C = \left\{ \left(2k, (2r+2)k + 2rl + \left\lceil \frac{l}{r+1} \right\rceil \right) \middle| k, l \in \mathbb{Z} \right\}$$

is an r-locating-dominating code for all odd r. The density of C is

$$D(C) = \frac{r+1}{2(2r(r+1)+1)} = \frac{1}{4r+\frac{2}{r+1}}.$$

Indeed, when we look at any even-numbered column, then always $r+1$ of any $(r+1)2r+1$ consecutive vertices belong to the code. Moreover, there are no codewords in the odd-numbered columns. A part of C when $r=3$ is shown in Figure 2.

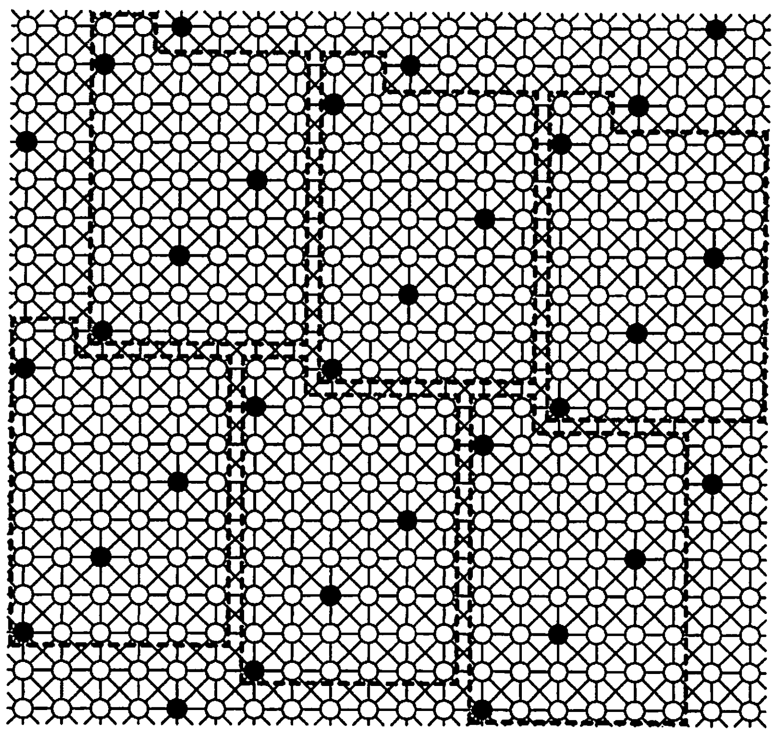

Figure 2: A 3-locating-dominating code. Black dots are codewords and white dots are non-codewords.

First, we make a small remark. Assume that $u = (x, y_u)$ and $v = (x, y_v)$ are two arbitrary vertices in the same column and $y_u < y_v$. If $I_r(u) \neq I_r(v)$ and $w = (x, y_w)$ is an arbitrary vertex where $y_w < y_u$ (or $y_w > y_v$), then also $I_r(w) \neq I_r(v)$ (or $I_r(w) \neq I_r(u)$, resp.). Indeed, $\Delta(I_r(u), I_r(v)) \subseteq \Delta(I_r(w), I_r(v))$ if $y_v - y_w \leq 2r$ or $I_r(w) \cap I_r(v) = \emptyset$, if $y_v - y_w > 2r$.

Now, we observe that at least one of $2r + 1$ consecutive vertices is a codeword in every even column. Then $I_r(v)$ is always non-empty and it contains a codeword from each even column that intersects $B_r(v)$. Thus, we see in which column v is. Indeed, if two vertices u and v are in different columns, then balls $B_r(u)$ and $B_r(v)$ can not intersect in exactly the same even columns.

Therefore, it is enough to show that $I_r(u) \neq I_r(v)$ for all non-codewords $u = (x, y_u)$ and $v = (x, y_v)$ in the same column.

Since the code consists of tiles with size $2r \times (2r+2) \cup 2 \times 1$ (cf. Figure

separated vertices	codeword	x	t
$(x,-1+2t)$, $(x,2t)$	$(-r-1+2t, r+2t)$	$0,\ldots,-1+2t$	$1,\ldots,\frac{r-1}{2}$
	$(r-1+2t, -r-1+2t)$	$2t,\ldots,2r-1$	$0,\ldots,\frac{r-1}{2}$
$(x,2t)$, $(x,1+2t)$	$(-r-1+2t, -r+2t)$	$0,\ldots,-1+2t$	$1,\ldots,\frac{r-1}{2}$
	$(r+1+2t, r+1+2t)$	$1+2t,\ldots,2r-1$	$0,\ldots,\frac{r-1}{2}$
$(x,r+2t)$, $(x,r+1+2t)$	$(2t,2t)$	$0,\ldots,r+2t$	$0,\ldots,\frac{r-1}{2}$
	$(2r+2+2t, 2r+1+2t)$	$r+2+2t,\ldots,2r-1$	$0,\ldots,\frac{r-3}{2}$
$(x,r-1+2t)$, $(x,r+2t)$	$(2t,2r+2t)$	$0,\ldots,r+2t$	$1,\ldots,\frac{r-1}{2}$
	$(2r+2t,-1+2t)$	$r+2t,\ldots,2r-1$	$1,\ldots,\frac{r-1}{2}$
$(x,2r)$, $(x,2r+1)$	$(r+1,3r+1)$	$1,\ldots,2r-1$	
$(x,2r+1)$, $(x,2r+2)$	$(-r+1,3r+2)$	$0,1$	

Table 2: The codeword is in the symmetric difference of the separated vertices. Here x and t are two integer parameters.

2), then it is enough to prove that each vertex (i,j) in the tile

$$\{(x,y)\,|\,0 \le x < 2r, 0 \le y < 2r+2\} \cup \{(0,2r+2),(1,2r+2)\}$$

is separated from the vertex $(i,j-1)$. Table 2 shows that this is true except for the $r+1$ pairs

$$\{(2t,2t),(2t,2t+1)\}, \quad \text{for } t = 0,1,\ldots,\frac{r-1}{2},$$

and

$$\{(r+2t+1,r+2t),(r+2t+1,r+2t+1)\}, \quad \text{for } t = 0,1,\ldots,\frac{r-3}{2},$$

and

$$\{(0,2r),(0,2r+1)\}.$$

Moreover, exactly one of the vertices in each of these $r+1$ pairs is a codeword, and every codeword belongs to at most one of these pairs.

We now claim that a non-codeword (i,j) is separated from all other non-codewords in the same column. If $(i,j+1)$ is not in the code, then by the previous paragraph, (i,j) is separated from $(i,j+1)$ and therefore from all the non-codewords above (i,j), as we saw earlier. The same is true even if $(i,j+1) \in C$ unless $\{(i,j),(i,j+1)\}$ is one of the exceptional pairs listed above. But then $(i,j+2)$ is not in the code (by the structure of C) and $\{(i,j+1),(i,j+2)\}$ is not an exceptional pair (as each codeword is contained in at most one such pair). But then $(i, j+2)$ and $(i, j+1)$ are separated and by the argument proved earlier the same is true for $(i, j+2)$ and (i, j), and by referring to the same argument a second time, we see that (i, j) is separated from all non-codewords above it. This concludes the proof. □

Remark 6. The codes in the proofs of Theorems 4 and 5 are also r-OLD codes, when $r \geq 2$. Indeed, $I_r(v) \setminus \{v\} = I_r(v)$ for any non-codeword v and by the previous proofs the sets $I_r(v)$ are distinct and non-empty for all non-codewords. Moreover, sets $I_r(c) \setminus \{c\}$ are non-empty for all codewords c. Then, the claim is true if $I_r(c) \setminus \{c\} \neq I_r(v) \setminus \{v\}$ for all $c \in C$ and for all $v \in V$.

Let $c = (a, b)$ be a codeword. Then either $c_1 = (a-2, b-2)$ or $c_1' = (a-2, b-1)$ and either $c_2 = (a+2, b+2)$ or $c_2' = (a+2, b+1)$ are also codewords for all codes in the proofs of Theorems 4 and 5. Now, $c_1, c_1', c_2, c_2' \in B_r(c)$. Then c_1 or c_1' contains in $I_r(c) \setminus \{c\}$ and c_2 or c_2' contains in $I_r(c) \setminus \{c\}$. Furthermore, if c_1 or c_1' is in $B_r(v)$ and c_2 or c_2' is in $B_r(v)$, then c also belongs to $B_r(v)$. Therefore, c is only vertex which has $c \notin I_r(v) \setminus \{v\}$, but $I_r(v) \setminus \{v\} \cap \{c_1, c_1'\} \neq \emptyset$ and $I_r(v) \setminus \{v\} \cap \{c_2, c_2'\} \neq \emptyset$.

Acknowledgement

Thank you to the anonymous referee who has made many grammatical and structural remarks with the help of which it is considerably easier to read this paper.

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