On some results for the $L(2,1)$-labeling on Cartesian sum graphs

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Abstract

An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$, where $d(x,y)$ denotes the distance between vertices x and y in G. The $L(2,1)-$labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)-$labeling with $\max\{f(v) : v \in V(G)\} = k$. We consider Cartesian sums of graphs and derive, both, lower and upper bounds for the $L(2,1)-$labeling number of this class of graphs; we use two approaches to derive the upper bounds for the $L(2,1)-$labeling number and both approaches improve previously known upper bounds. We also present several approximation algorithms for computing $L(2,1)$-labelings for Cartesian sum graphs.

Keywords: channel assignment, $L(2,1)$-labeling, Cartesian sum

1 Introduction

Graph coloring is a well studied and important topic in graph theory. Research on graph coloring can be traced to over one hundred years ago [15] and it finds applications in many areas. The $L(2,1)-$labeling problem is a generalization of the famous vertex coloring problem. An $L(2,1)-$labeling

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of a graph G is an assignment of non-negative integers to all the vertices of G such that: i) integers assigned to adjacent vertices differ by at least 2, and ii) different integers are assigned to vertices at distance 2 from each other. The $L(2,1)-$labeling problem is to find an $L(2,1)-$labeling of a given graph G that minimizes the largest label used; this largest label is called the $L(2,1)-$labeling number of G and it is denoted $\lambda(G)$.

The $L(2,1)$-labeling problem models some frequency assignment problems [8]: Suppose, for example, that it is desired to assign transmitting frequencies to a group of radio transmitters so that any two transmitters that could potentially interfere with one another are assigned sufficiently different frequencies to avoid interference. Griggs and Yeh [7] formulated this problem as the $L(2,1)-$labeling problem by proposing that if the distance between two interfering transmitters is very small they should be assigned frequency channels at least a distance of two apart, but if two interfering transmitters are not too close then they should just be assigned different frequency channels.

Since the $L(2,1)$- labeling problem is a generalization of the vertex coloring problem, it is strongly NP-hard. There are many papers studying the L(2,1)-labeling problem and due to the inherent hardness of the problem, most research considers the problem on particular classes of graphs. Griggs and Yeh [7] first gave an upper bound of $\Delta^2 + 2\Delta$ for the value of $\lambda(G)$ on arbitrary graphs with maximum degree Δ. Later, Chang and Kuo [5] improved the bound to $\Delta^2 + \Delta$. Recently, Král' and Škrekovski [12] reduced the bound to $\Delta^2 + \Delta - 1$. For graphs G of diameter 2, $\lambda(G) \leq \Delta^2$ [7]: This upper bound is attainable by *Moore graphs*; such graphs exist for $\Delta = 2, 3, 7$, and possibly 57 [7]. Griggs and Yeh [7] conjectured that the best bound for $\lambda(G)$ is Δ^2 for graphs with maximum degree $\Delta \geq 2$ [7]. The conjecture is not true for $\Delta = 1$ since, for example, $\Delta(K_2) = 1$ but $\lambda(K_2) = 2$.

In [19], [11] and [20], Shao and Yeh, S. Klavžar and S. Špacapan, and Shao and Zhang proved that the L(2,1)-labeling number of the Cartesian product, the composition, the direct product, the strong product and the

Cartesian sum product of graphs is bounded by the square of the maximum degree. Hence Griggs and Yeh's conjecture holds in the above five cases (with some minor exceptions). Shao, Klavžar, Shiu and Zhang [18] improved the upper bounds obtained in [11] with a more refined analysis of neighborhoods in product graphs than that used in [11].

Approximation algorithms and inapproximability results for the $L(2,1)$-labeling problem on general graphs are rare. In [4], by using an algorithm by McCormick [14], Calamoneri et al. proved that there exists an algorithm for the $L(h,k)$-labelling problem with approximation ratio $h((n-1)^{1/2}+1)$, where n is the number of vertices in the input graph. In [9], Halldorsson improved the above result by proving that the approximation ratio of the first-fit algorithm is $\min\{n^{1/2}+h/k, \Delta\}$ which is the currently best known result. He also proved that it is hard to approximate the $L(h,k)$-labelling problem within a factor of $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$ and $h \in [n^{1/2-\varepsilon}, n]$.

We consider the Cartesian sum of graphs and derive, both, lower and upper bounds for the L(2,1)-labeling number; we use two approaches to derive the upper bounds and both approaches improve previously known bounds. We also present new approximation algorithms for the $L(2,1)$-labelings on Cartesian sum graphs.

Throughout the paper, all graphs are assumed to be simple (i.e. they have no loops or parallel edges).

2 A Labeling Algorithm

Given a graph G, its set of vertices is denoted as $V(G)$ and its set of edges as $E(G)$. The number of vertices in G is denoted $\nu(G)$. A vertex u of G is isolated if its degree is zero. The number of isolated vertices in G is denoted $t(G)$. The maximum degree of G is denoted $\Delta(G)$. If u and v are two adjacent vertices of G, the edge connecting them is denoted as uv. The *Cartesian sum,* $G \bigoplus H$, of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which a vertex (u, v) is adjacent to another vertex (u', v') if and only if either $uu' \in E(G)$, or $vv' \in E(H)$, or both [15] (see Figure 1 for an example).

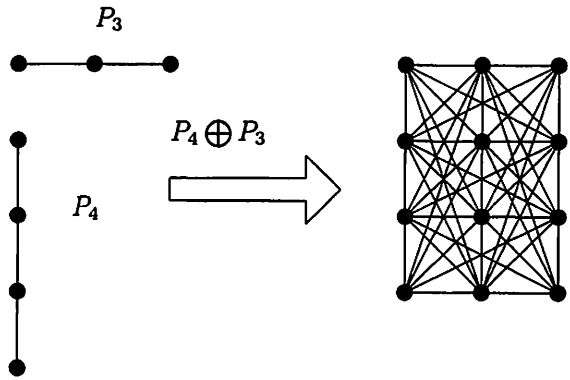

Figure 1. Cartesian sum of graphs P_3 and P_4.

Lemma 2.1 *Let G and H be two graphs. Then $G \bigoplus H$ has a subgraph of diameter two with $(\nu(G) - t(G))(\nu(H) - t(H))$ vertices and it also has a subgraph of diameter three with* $\max\{\nu(G)(\nu(H) - t(H)), \nu(H)(\nu(G) - t(G))\}$ *vertices.*

Proof. Let G' and H' be the subgraphs of G and H obtained by removing all the isolated vertices, respectively. Observe that if G' or H' are empty then the first bound of the Lemma holds trivially, so let us assume that G' and H' are not empty. Let G' and H' consist of connected components $G_1, G_2, ..., G_k$ $(k \geq 1)$ and $H_1, H_2, ..., H_p$ $(p \geq 1)$, respectively. Note that $\nu(G_i) \geq 2$ and $\nu(H_j) \geq 2$ for each connected component $G_i, H_j, i = 1, 2, ..., k, j = 1, 2, ..., p$. Let $(u, v), (u', v')$ be any two nonadjacent vertices of $G \bigoplus H$, where $u \in G_i$, $v' \in H_l$, for $i \in \{1, 2, ..., k\}$ and $l \in \{1, 2, ..., p\}$. Since G_i and H_l are connected, let u'' be a vertex adjacent to u in G_i and let v'' be a vertex adjacent to v' in H_l. By the definition of $G \bigoplus H$, (u, v) and (u'', v'') are adjacent and (u', v') and

(u'', v'') are also adjacent. Hence (u, v) and (u', v') are at distance two in $G \bigoplus H$, and so $G \bigoplus H$ has a subgraph $G' \bigoplus H'$ of diameter two with $(\nu(G) - t(G))(\nu(H) - t(H))$ vertices.

We now derive the first part for the second bound of the Lemma. Note that if H' is empty this part of the bound is trivially zero, so we assume that H' is not empty. Let $(u, v), (u', v')$ be two nonadjacent vertices of $G \bigoplus H$, where $u \in G_i, u' \in G_j$ and v, v' are two different vertices in H. Let w, w' be two adjacent vertices in H. Since G_i and G_j are connected, let u'' be a vertex adjacent to u in G_i and let u''' be a vertex adjacent to u' in G_j. By the definition of $G \bigoplus H$, (u, v) and (u'', w) are adjacent, (u''', w') and (u', v') are adjacent, and (u'', w) and (u''', w') are adjacent. Hence (u, v) and (u', v') are at distance three. Then $G' \bigoplus H$ has a subgraph of diameter three that includes G' and all vertices of H; this subgraph has $\nu(H)(\nu(G) - t(G))$ vertices. Similarly, $G \bigoplus H'$ has a subgraph of diameter three with $\nu(G)(\nu(H) - t(H))$ vertices. ■

Corollary 2.2 *Let G and H be two connected graphs. Then $G \bigoplus H$ is of diameter two.*

A subset X of $V(G)$ is called an *i-stable set* (or *i-independent set*), if the distance between any two vertices in X is greater than i. A 1-stable set is a usual independent set. A *maximal* 2-stable subset X of a set Y of vertices is a 2-stable subset of Y that is not a proper subset of any other 2-stable subset of Y. Chang and Kuo [5] proposed the following algorithm to compute an $L(2,1)$-labeling for a given graph.

Algorithm Label(G)

Input: A graph G.

Output: The value of the maximum label in an $L(2,1)$-labeling computed for G.

Initialization: Set $X_{-1} = \emptyset$; $V = V(G)$; $i = 0$.

Iteration:

1. Determine Y_i and X_i.
   - $Y_i = \{x \in V : x$ is unlabeled and $d(x, y) \geq 2$ for all $y \in X_{i-1}\}$.
   - Compute X_i, a maximal 2-stable subset of Y_i.
2. Label the vertices in X_i (if there are any) with label i.
3. $V \leftarrow V \backslash X_i$.
4. If $V \neq \emptyset$ then set $i \leftarrow i + 1$ and go to Step 1.
5. Record the current value i as k (which is the maximum label). Stop.

Note that the value returned by the algorithm is an upper bound for $\lambda(G)$. We would like to find a bound for the largest label used by the algorithm in terms of the maximum degree $\Delta(G)$ of G, analogous to existing bounds for the chromatic number $\chi(G)$ in terms of $\Delta(G)$.

Let x be a vertex with the largest label k assigned by algorithm Label. We consider the following sets of labels:

$I_1 = \{i : 0 \leq i \leq k-1$ and $d(x, y) = 1$ for some $y \in X_i\}$. This is the set of labels of the neighbors of x.

$I_2 = \{i : 0 \leq i \leq k-1$ and $d(x, y) \leq 2$ for some $y \in X_i\}$. These are the labels of the vertices at distance at most 2 from x.

$I_3 = \{i : 0 \leq i \leq k-1$ and $d(x, y) \geq 3$ for all $y \in X_i\}$. These are the labels not used by vertices at distance at most 2 from x.

It is clear that $|I_2| + |I_3| = k$. For any $i \in I_3$, $x \notin Y_i$ since otherwise $X_i \cup \{x\}$ would be a 2-stable subset of Y_i, which contradicts the choice of X_i. That is, $d(x, y) = 1$ for some vertex y in X_{i-1}; i.e., $i - 1 \in I_1$. Since for every $i \in I_3$, $i - 1 \in I_1$ then, $|I_3| \leq |I_1|$. Hence

$\lambda(G) \leq k = |I_2| + |I_3| \leq |I_2| + |I_1|$ (1).

In order to estimate $\lambda(G)$, in the next sections we will bound $|I_1| + |I_2|$ in terms of $\Delta(G)$.

3 Lower and upper bounds on the $L(2,1)$-labelings of Cartesian sum graphs

Theorem 3.1 *For any two graphs G and H, $\lambda(G \bigoplus H) \geq (\nu(G) - t(G))(\nu(H) - t(H)) - 1$.*

Proof. By Lemma 2.1, $G \bigoplus H$ has a subgraph of diameter two with $(\nu(G)-t(G))(\nu(H)-t(H))$ vertices. Since in a $L(2,1)$-labeling of a diameter two graph all the vertices must have different labels, then $\lambda(G \bigoplus H) \geq (\nu(G) - t(G))(\nu(H) - t(H)) - 1$. ■

We now compute an upper bound for $\lambda(G \bigoplus H)$.

Theorem 3.2 *For any two graphs G and H, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1$.*

Proof. Note that $G \bigoplus H$ has $t(G)t(H)$ isolated vertices. Thus, the number of vertices within distance two from any vertex x, is at most $\nu(G)\nu(H) - t(G)t(H) - 1$. Therefore, by equation (1), $\lambda(G \bigoplus H) \leq |I_2| + |I_1| \leq \nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1$. ■

In [20] it is proved that $\lambda(G \bigoplus H) \leq D' = (\Delta(G \bigoplus H))^2 - \nu(G)(\Delta(G)-1)\Delta(H) - \nu(H)(\Delta(H) - 1)\Delta(G) - (\Delta(G) + \Delta(H))\Delta(G)\Delta(H) - \Delta(G) - \Delta(H) + 1$. Let $D = \nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1$, be the bound from Theorem 3.2. We now compare the bounds D' and D.

Note that $\Delta(G \bigoplus H) = \nu(G)\Delta(H) + \nu(H)\Delta(G) - \Delta(G)\Delta(H) \geq 2(\nu(G)\nu(H)\Delta(G)\Delta(H))^{1/2} - \Delta(G)\Delta(H) = (\Delta(G)\Delta(H))^{1/2}(2(\nu(G)\nu(H))^{1/2} - (\Delta(G)\Delta(H))^{1/2}) = (\Delta(G)\Delta(H))^{1/2}((\nu(G)\nu(H))^{1/2} + (\nu(G)\nu(H))^{1/2} - (\Delta(G)\Delta(H))^{1/2}) \geq (\Delta(G)\Delta(H))^{1/2}((\nu(G)\nu(H))^{1/2} + (\Delta(G)\Delta(H) + \Delta(G) + \Delta(H) + 1)^{1/2} - (\Delta(G)\Delta(H))^{1/2}) > (\Delta(G)\Delta(H))^{1/2}(\nu(G)\nu(H))^{1/2}$, the second inequality follows from $\nu(G) \geq \Delta(G) + 1$ and $\nu(H) \geq \Delta(H) + 1$.

Thus, $(\Delta(G \bigoplus H))^2 > \nu(G)\nu(H)\Delta(G)\Delta(H)$ and so

$D' - D = [(\Delta(G \bigoplus H))^2 - \nu(G)(\Delta(G)-1)\Delta(H) - \nu(H)(\Delta(H)-1)\Delta(G) - (\Delta(G) + \Delta(H))\Delta(G)\Delta(H) - \Delta(G) - \Delta(H) + 1] - [\nu(G)\nu(H) - t(G)t(H) +$

$\Delta(G \bigoplus H) - 1] = [(\Delta(G \bigoplus H))^2 - \nu(G)(\Delta(G) - 1)\Delta(H) - \nu(H)(\Delta(H) - 1)\Delta(G) - (\Delta(G) + \Delta(H))\Delta(G)\Delta(H) - \Delta(G) - \Delta(H) + 1] - [\nu(G)\nu(H) - t(G)t(H) + (\nu(G)\Delta(H) + \nu(H)\Delta(G) - \Delta(G)\Delta(H)) - 1] = (\Delta(G \bigoplus H))^2 - (\nu(G)\nu(H) - t(G)t(H)) - (\nu(G)\Delta(G)\Delta(H) + \nu(H)\Delta(H)\Delta(G) + (\Delta(G) + \Delta(H) - 1)\Delta(G)\Delta(H) + \Delta(G) + \Delta(H) - 2) > \nu(G)\nu(H)\Delta(G)\Delta(H) - (\nu(G)\nu(H) - t(G)t(H)) - (\nu(G)\Delta(G)\Delta(H) + \nu(H)\Delta(H)\Delta(G) + (\Delta(G) + \Delta(H) - 1)\Delta(G)\Delta(H) + \Delta(G) + \Delta(H) - 2)$.

Noting again that $\nu(G) \geq \Delta(G) + 1$ and $\nu(H) \geq \Delta(H) + 1$, we conclude that $D' - D = \Theta(\nu(G)\nu(H)\Delta(G)\Delta(H))$. So, our bound is asymptotically better than in [20].

4 Algorithm BlockLabel

In this section, we present a different algorithm for computing an $L(2,1)$-labeling for the Cartesian sum of two graphs that is better than the algorithm presented in the previous section.

In the vertex coloring problem the goal is to color the vertices of a given graph G with the minimum possible number of colors so that adjacent vertices have different colors. The minimum number of colors needed to color the vertices of a graph G is called the chromatic number of G, denoted $\chi(G)$. Consider two graphs G, H and optimum colorings χ_G, χ_H for them. Without loss of generality, let the colors assigned to the vertices of G and H be $1, ..., \chi(G)$ and $1, ..., \chi(H)$ respectively; moreover, let all the isolated vertices in G and H be assigned color 1. We partition the vertices of $G \bigoplus H$ into blocks, as follows. All vertices (u, v) of $G \bigoplus H$ where u has color i and v has color j are placed in block B_{ij}. Let B be the set of all these blocks. We use the following algorithm for labeling $G \bigoplus H$.

Algorithm BlockLabel(B)

Input: Set B of blocks as described above.

Output: The maximum label used in an $L(2,1)$-labeling for the vertices in B.

1. Sort the blocks in B in any order.
2. $l \leftarrow 0$.
3. For each block $B_{ij} \in B$ do {
4. If $i = 1$ and $j = 1$ then {
5. For each vertex $u \in B_{11}$ do {
6. If u is isolated in $G \bigoplus H$ then Assign u label 0.
7. otherwise Assign u label l and then set $l \leftarrow l + 1$.

}

}

8. otherwise
9. For each vertex $u \in B_{ij}$ do Assign u label l and then set $l \leftarrow l + 1$.
10. $l \leftarrow l + 1$//skip a label.

}

11. Return $l - 1$.

Theorem 4.1 *Let G and H be two graphs. Then one of the following holds.*

a. If both G and H are not complete graphs or odd cycles, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \Delta(G)\Delta(H) - 2$;

b. If both G and H are odd cycles, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 7$;

c. If both G and H are complete graphs, then $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$;

d. If one of G and H is not a complete graph or odd cycle, but the other is an odd cycle, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\Delta(G) - 2$ or $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\Delta(H) - 2$;

e. If one of G and H is not a complete graph or odd cycle, but the other is a complete graph, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + \Delta(G)\nu(H) - 2$ or $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + \Delta(H)\nu(G) - 2$;

f. If one of G and H is a complete graph and the other is an odd cycle, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\nu(G) - 2$ or $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\nu(H) - 2$.

Proof. We first show that algorithm BlockLabel produces an $L(2,1)$-labeling for $G \bigoplus H$. Let us consider the non-isolated vertices in some block

$B_{ij} \in B$. For any two vertices (u, v) and (u', v') in B_{ij}, since u and u' have the same color i in χ_G, then u and u' are at distance at least two in G; similarly, v and v' are at distance at least two in H. By the definition of $G \bigoplus H$, (u, v) and (u', v') are at distance at least two in $G \bigoplus H$. Thus, all the vertices in block $B_{i,j}$ can be labelled consecutively.

Now let us consider the vertices in two different blocks. For any two vertices (u, v) and (u', v') from two different blocks of B, there is the possibility that they are adjacent in $G \bigoplus H$. Note that since in algorithm BlockLabel at least one label has been skipped between the labelling of (u, v) and (u', v'), then the labels of these vertices differ by at least 2 and so the above labelling scheme is feasible.

The number of labels used to label $G \bigoplus H$ is equal to the number $\nu(G)\nu(H) - t(G)t(H)$ of non-isolated vertices plus the number of labels skipped in step 10 of the algorithm. Notice that the number of labels skipped is equal to the number of blocks in B minus 1.

Since the number of blocks in B is at most $\chi(G)\chi(H)$, then $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2$. We can now combine this result with Brooks Theorem to prove (a)- (e).

(a). If both G and H are not complete graphs or odd cycles, then $\chi(G) \leq \Delta(G)$ and $\chi(H) \leq \Delta(H)$. And the conclusion follows.

(b). If both G and H are odd cycles, then $t(G) = t(H) = 0$ and $\chi(G) = \chi(H) = 3$. Thus, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2 \leq \nu(G)\nu(H) + 7$.

(c). If both G and H are complete graphs , then $G \bigoplus H$ is a complete graph. Thus, $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$.

(d). If G is not a complete graph or odd cycle, and H is an odd cycle, then $t(H) = 0$, $\chi(G) \leq \Delta(G)$ and $\chi(H) = 3$. So, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2 \leq \nu(G)\nu(H) + 3\Delta(G) - 2$. The other case is similar.

(e). If G is not a complete graph or odd cycle, and H is a complete graph, then $t(H) = 0$, $\chi(H) = \nu(H)$ and $\chi(G) \leq \Delta(G)$. So, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2 \leq \nu(G)\nu(H) + \Delta(G)\nu(H) - 2$. The other case is similar.

(f). If G is a complete graph and H is an odd cycle, then $t(G) = t(H) = 0$, $\chi(G) = \nu(G)$ and $\chi(H) = 3$. So, $\lambda(G \bigoplus H) \leq \nu(G)\nu(H) + 3\nu(G) - 2$. The other case is similar. ∎

We now compare the bounds in Theorem 3.2 and Theorem 4.1. Note that $[\nu(G)\nu(H) - t(G)t(H) + \Delta(G \bigoplus H) - 1] - [\nu(G)\nu(H) - t(G)t(H) + \chi(G)\chi(H) - 2] \geq \Delta(G)\nu(H) + \Delta(H)\nu(G) - \Delta(G)\Delta(H) - (\Delta(G)+1)(\Delta(H)+1) = \Delta(G)(\nu(H) - \Delta(H)) + \Delta(H)(\nu(G) - \Delta(G)) - \Delta(G) - \Delta(H) - 1 = \Delta(G)(\nu(H) - \Delta(H) - 1) + \Delta(H)(\nu(G) - \Delta(G) - 1) - 1 \geq \Delta(G) + \Delta(H) - 1$. Then, the bound in Theorem 4.1 is better than that in Theorem 3.2.

The following Corollaries follow from Theorems 3.1 and 3.3.

Corollary 4.2 *Let $G \bigoplus H$ be the cartesian sum of any two graphs G and H and let both G and H have non-isolated vertices. Then one of the following holds.*

a. If both G and H are not complete graphs or odd cycles, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) - t(G)t(H) + \Delta(G)\Delta(H) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

b. If both G and H are odd cycles, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 7)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

c. If both G and H are complete graphs, then there is an exact algorithm to $L(2,1)$-label $G \bigoplus H$ with $\lambda(G \bigoplus H) = 2\nu(G)\nu(H) - 2$;

d. If one of G and H is not a complete graph or odd cycle, but the other is an odd cycle, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + 3\Delta(G) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$ or $(\nu(G)\nu(H) + 3\Delta(H) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

e. If one of G and H is not a complete graph or odd cycle, but the other is a complete graph, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio $(\nu(G)\nu(H) + \Delta(G)\nu(H) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$ or $(\nu(G)\nu(H) + \Delta(H)\nu(G) - 2)/((\nu(G) - t(G))(\nu(H) - t(H)) - 1)$;

f. If one of G and H is a complete graph and the other is an odd cycle, then there is an algorithm to $L(2,1)$-label $G \bigoplus H$ with approximation ratio

$(\nu(G)\nu(H)+3\nu(G)-2)/((\nu(G)-t(G))(\nu(H)-t(H))-1)$ *or* $(\nu(G)\nu(H)+3\nu(H)-2)/((\nu(G)-t(G))(\nu(H)-t(H))-1)$.

Corollary 4.3 *Let* $G\bigoplus H$ *be the cartesian sum of two connected graphs* G *and* H. *Then one of the following holds.*

a. If both G *and* H *are not complete graphs or odd cycles, then there is an algorithm to* $L(2,1)$*-label* $G\bigoplus H$ *with approximation ratio less than 2;*

b. If both G *and* H *are odd cycles, then there is an algorithm to* $L(2,1)$*-label* $G\bigoplus H$ *with approximation ratio* $(\nu(G)\nu(H)+7)/((\nu(G)\nu(H)-1)$;

c. If both G *and* H *are complete graphs, then there is an exact algorithm to* $L(2,1)$*-label* $G\bigoplus H$ *with* $\lambda(G\bigoplus H)=2\nu(G)\nu(H)-2$;

d. If one of G *and* H *is not a complete graph or odd cycle, but the other is an odd cycle, then there is an algorithm to* $L(2,1)$*-label* $G\bigoplus H$ *with approximation ratio* $(\nu(G)\nu(H)+3\Delta(G)-2)/((\nu(G)\nu(H)-1)$ *or* $(\nu(G)\nu(H)+3\Delta(H)-2)/((\nu(G)\nu(H)-1)$;

e. If one of G *and* H *is not a complete graph or odd cycle, but the other is a complete graph, then there is an algorithm to* $L(2,1)$*-label* $G\bigoplus H$ *with approximation ratio* $(\nu(G)\nu(H)+\Delta(G)\nu(H)-2)/((\nu(G)\nu(H)-1)$ *or* $(\nu(G)\nu(H)+\Delta(H)\nu(G)-2)/((\nu(G)\nu(H)-1)$;

f. If one of G *and* H *is a complete graph and the other is an odd cycle, then there is an algorithm to* $L(2,1)$*-label* $G\bigoplus H$ *with approximation ratio* $(\nu(G)\nu(H)+3\nu(G)-2)/((\nu(G)\nu(H)-1)$ *or* $(\nu(G)\nu(H)+3\nu(H)-2)/((\nu(G)\nu(H)-1)$.

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